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Congruence on Bi-Simple inverse ω-Semigroup of Left Quotient

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ABSTRACT

We construct the congruence ρ on Bisimple inverse ω-semigroup Q of left quotients S and show that congruence on S can be carried to its order Q. We also show that there is group congruence on Q and H is a congruence on Q, indeed its maximum congruence. This helps us to deduce some nice properties of this congruence on Q.

AMS Classification No: 20H18

Key Words: Bi-Simple Semigroup, Congruence, Left Orders, Green's Relations.

1.0 Introduction

A lot of studies have been done on Bisimple semigroup of various kinds. [6] constructed Bisimple ω-semigroup, and showed that every Bisimple ω-simple semigroup is isomorphic to a type of the constructed one. This led to the classification of Bisimple semigroup. [5] studied congruences on Bisimple ω-semigroup, and concluded that its congruences are either group or idempotent separating congruences. In particular, the maximum group congruence on it is the Green's relation H. [7] made the same conclusion on Bisimple ω^n-semigroup and used it to study its structure. [5] constructed Bisimple inverse ω-semigroup of left quotient Q and characterized it. [4] gave a structure for a semigroup which is a semilattice of bisimple inverse semigroups and satisfied certain conditions. Our work is to examine if the congruence in S can be extended to Q, the semigroup of its left quotients. To do this, we shall construct an arbitrary congruence on its order S and show that it is also a congruence on its left quotients, Q. It is pertinent to point out here that the Green's relation H is a congruence on bisimple inverse ω-semigroup and we shall show that this congruence can be carried to its left quotients. This will lead us to examining the relationship between an arbitrary congruence and H and also help to study the its properties on Q. Section two contains definitions, reviewing of structure of Q as constructed by [2] and highlighting some known properties that will be very useful in subsequent sections. Section three contains the constructed congruence and some deduced properties. Definitions, explanations of some terms are in [1] ,[2], and [3] and we shall most often avoid them except in cases where they are absolutely necessary.

2.0 Preliminary

In this section we shall give definitions that would be used in the work. Notations would follow [1] and [3]. We shall also look at the structure of bisimple inverse ω-semigroup as constructed [2]. These will provide the essential tools that will enable us understand other sections.

2.1 Definition

(i) The set $E = C_\omega = \{e_1, e_2, e_3, \ldots\}$ of idempotents of a semigroup such that $e_0 > e_1 > e_2 > e_3 > \ldots$ is called ω-chain of idempotents.

(ii) A semigroup S will be called an inverse ω-semigroup if S is an inverse semigroup and also has an ω-chain of idempotents.

(iii) An inverse ω-semigroup S is called bisimple if S is without a zero and Green's D-relation is the universal relation.

2.2 Example

Let B = N x N with multiplication defined by

$$(m, n)(x, y) = (m-n + t, y - x + t), \text{ where } t = \max\{n, x\}.$$

B with the defined multiplication is a semigroup. Idempotents are represented in the form (n, n). So, we have ω-chain of idempotents (0, 0) > (1, 1) > (2, 2) > .... This semigroup is called Bicyclic semigroup.

2.3 Notations

(i) Let $\Psi: Q \rightarrow B$ be onto homomorphism with kernel H, the H-class of Q will be denoted by

$$H_{i,j} = \Psi^{-1}\{(i, j)\}.$$

(ii) aRb if and only if $\Psi(a) = (i, j)$ and $\Psi(b) = (i, k) \in$ □.

(iii) $\varphi : S \rightarrow B$ is a homomorphism such that $\varphi(n) = (n, n)$. Using this definition, we have the notation $\varphi(x) = (r(x), \ell(x))$.

(iv) $\hat{S} = \{a \in S: aH^*a^2\}$ is an ω-chain of right reversible, cancellative subsemigroup S, where $n \in N$.

2.4 The Structure of Bisimple Inverse ω-Semigroup of Left Quotient.

Let us look at the structure of bisimple inverse ω-semigroup of left quotient as constructed by [2]. Notations are strictly that of [2]. Let $E = \{(a, b) \in S \times S : a \in \hat{S}, r(a) = r(b)\}$. On $\sum$, define a relation $\sim$ as follows: $(a, b) \sim (c, d)$ if and only if $r(a) = r(c)$ and there exist $x, y \in S_{r(a)}$ such that $xa = yc$ and $xb = yd$. Then $\sim$ is an equivalent relation.

Let $\sim$ -equivalence class of the element (a, b) be denoted by $[a, b]$. We shall denote by $Q = \sum/\sim$ the set of $\sim$ -equivalence classes of $\sum$. Given any two elements $[a, b]$, $[c, d]$ in Q, we define multiplication in the following way:

$[a, b][c, d] = [xa, yd]$, where $xb = yc$ and

$\varphi(x) = (r(a) - \ell(b) + \max(\ell(b), r(c)), r(a) - \ell(b) + \max(\ell(b), r(c)),$

$\varphi(y) = (r(a) - \ell(b) + \max(\ell(b), r(c)), \max(\ell(b), r(c)).$

Using the multiplication above, Q is a semigroup. Let $\varsigma : S \rightarrow Q$ be a function defined by $\varsigma(a) = [ax, axa]$, where $\varphi(x) = (r(x), \ell(x))$. This definition makes ς an embedding. Therefore, S is an order in Q. Now, if $a \in \hat{S}$, then $a = [a, a^2]$. Furthermore, if a is in a subgroup of Q, its inverse in this subgroup is $[a^2, a]$. Thus, we have that for each $n \in N$, $G_n = \{[a, b] \in Q: a, b \in S_n\}$ is the group of left quotient of S_n. Moreover, if $\bar{Q} = \{[a, b] \in Q: b \in \hat{S}\} = \bigcup_{n \varepsilon N} G_n$, then $\bar{Q}$ is an ω-chain of groups. We remark here that every element of Q can be written as $a^{-1}b$, where $a, b \in \hat{S}$, a^{-1} is the inverse of a in the subgroup $G_{r(a)}$ of Q and $r(a) = r(b)$ and if $a, x \in$, S, $a \in \hat{S}$ and $r(a) = \ell(x)$, then $xa^{-1}a = x$. Using the fact in Q, if $z = qb'aq$, where $q = a^{-1}b$, $a^{-1} \in G_{r(a)}$, $b = [by, byb]$, $\varphi(yb) = [\ell(b), \ell(b)]$ and $b' = [yb, y]$, we have,

$$z = a^{-1}bb'aa^{-1}b$$

$$= a^{-1}bb'a^{-1}ab$$

$$= a^{-1}bb'b \text{ and } bb'$$

$$= [yb, y]\,[by, byb]$$

$$= [syb, tbyb],$$

where

$$sy = tby$$

and

$$\varphi(s) = (r(yb) - \ell(y) + \max(\ell(y), r(by)), r(yb) - \ell(y) + \max(\ell(y), r(by))$$

$$\varphi(t) = (r(yb) - \ell(y) + \max(\ell(y), r(by)), \max(\ell(y), r(by)).$$

Noting that b'b is the identity and is equal to $(tbyb)^{-1}$ (tbyb), we deduce that,

$z = a^{-1}b(tbyb)^{-1}(tbyb) = a^{-1} = q$. So Q is regular. The set of idempotent E(Q) of Q is {[a, a] : a ∈ Ŝ} and this set formed an ω-chain and they also commute. Thus Q is an inverse semigroup. From the construction above, given idempotents e, f in Q, there is an element q in Q such that $qq^{-1} = e$ and $q^{-1}q = f$. This makes Q bisimple. Therefore, Q is bisimple inverse -semigroup. We remark here that Q is unique up isomorphism. Note the element [a, b] of Q will be written as $a^{-1}b$.

3.0 Congruence Construction

In this section, we shall construct an arbitrary congruence on Q and use it to deduce some of the properties of Q. We shall first show that H is congruence on Q.

Recall that H is congruence on any bisimple inverse ω-semigroup. Thus we move further than this by showing that H is the maximum congruence on Q.

3.1 Proposition

Let Q be a bisimple inverse ω-semigroup of left quotient of S. The Green's relation H is congruence on Q. Furthermore, for any congruence p on Q, $\rho_{max} = H$.

Proof

Recall $a^{-1}bzHc^{-1}d$ implies that $x^{-1}y$, $z^{-1}t$ in Q such that $a^{-1}bx^{-1}y = a^{-1}bz^{-1}t$. That is, taking into consideration multiplication in Q, $(ua)^{-1}vy = (u_1a)^{-1} v_1t$, where $ub = vx, u_1b = v_1z$ and

$\varphi(u) = (r(a) - \ell(b) + \max(\ell(b), r(x)), r(a) - \ell(b) + \max(\ell(b), r(x)))$

$\varphi(v) = (r(a) - \ell(b) + \max(\ell(b), r(x)), \max(\ell(b), r(x)))$,

$\varphi(u_1) = (r(a) - \ell(b) + \max(\ell(b), r(z)), r(a) - \ell(b) + \max(\ell(b), r(z)))$

$\varphi(V^1) = (r(a) - \ell(b) + \max(\ell(b), r(z)), \max(\ell(b), r(z)))$ if and only if

$c^{-1}dx^{-1}y = c^{-1}dz^{-1}t$. That is, taking into consideration multiplication in Q,

$(u_2c)^{-1}vy\ (u_3c')^{-1}t$, where $u_2d = vx$, $u_3d = v_3z$ and

$\varphi(u_2) = (r(c) - \ell(d) + \max(\ell(d), r(x)), r(c) - \ell(d) + \max(\ell(d), r(x)))$

$\varphi(v_2) = (r(c) - \ell(d) + \max(\ell(d), r(x)), \max(\ell(d), r(x)))$,

$\varphi(u_3) = (r(c) - \ell(d) + \max(\ell(d), r(z)), r(c) - \ell(d) + \max(\ell(d), r(z)))$

$\varphi(v_3) = (r(c) - \ell(d) + \max(\ell(d), r(z)), \max(\ell(d), r(z)))$ and

$x^{-1}ya^{-1}b = z^{-1}ta^{-1}b$. That is, taking into consideration multiplication in Q,

$(u_4x)^{-1}v_4b = (u_5z)^{-1}v_5b$, where $u_4y = v_4a$, $u_5t = v_5a$ and

$\varphi(u_4) = (r(x) - \ell(y) + \max(\ell(y), r(a)), r(x) - \ell(y) + \max(\ell(y), r(a)))$

$\varphi(v_4) = (r(x) - \ell(y) + \max(\ell(y), r(a)), \max(\ell(y), r(a)))$,

$\varphi(u_5) = (r(z) - \ell(t) + \max(\ell(t), r(a)), r(z) - \ell(t) + \max(\ell(t), r(a)))$

$\varphi(v_5) = (r(z) - \ell(t) + \max(\ell(t), r(a)), \max(\ell(t), r(a)))$ if and only if

$x^{-1}yc^{-1}d = z^{-1}tc^{-1}d$. That is, taking into consideration multiplication in Q,

$(u_6x)^{-1}v_6d\ (u_7z)^{-1}v_7d$, where $u_6y = v_6c$, $u_7t = v_7c$ and

$\varphi(u_6) = (r(x) - \ell(y) + \max(\ell(y), r(c)), r(x) - \ell(y) + \max(\ell(y), r(c)))$

$\varphi(v_6) = (r(x) - \ell(y) + \max(\ell(y), r(c)), \max(\ell(y), r(c)))$,

$\varphi(u_7) = (r(z) - \ell(t) + \max(\ell(t), r(c)), r(z) - \ell(t) + \max(\ell(t), r(c)))$

$\varphi(v_7)\ (r(z) - \ell(t) + \max(\ell(t), r(c)), \max(\ell(t), r(c)))$. We shall now show that the relation *H* is an equivalent relation. We shall avoid repetition of the above remark taking into consideration that it applies where necessary. Let $a^{-1}b \in Q$, we can always find $x^{-1}y$ and $z^{-1}t$ in Q such that $a^{-1}bx^{-1}y = a^{-1}bz^{-1}t$ and $x^{-1}ya^{-1}b = z^{-1}ta^{-1}b$. Here $x^{-1}y = z^{-1}t$. Thus, $a^{1-}bHa^{-1}b$. That is, H is reflexive. To show that is H symmetry, let $a^{-1}b$, $c^{-1}d \in Q$ such that $a^{-1}bHc^{-1}d$. That is, there exists $x^{-1}y$, $z^{-1}t$ in Q such that $a^{-1}bx^{-1}y = a^{-1}bz^{-1}t$ if and only if $c'dx^{-1}y = c^{-1}dz^{-1}t$ and $x^{-1}ya^{-1}b = z^{-1}ta^{-1}b$ if and only if $x^{-1}yc^{-1}d = z^{-1}tc^{-1}d$. This implies that, $c^{-1}dx^{-1}y = c^{-1}dz^{-1}t$ if and only if $a^{-1}bx^{-1}y = a^{-1}bz^{-1}t$ and $x^{-1}yc^{-1}d = z^{-1}$'$tc^{-1}d$ if and only if $x^{-1}ya^{-1}b = z^{-1}ta^{-1}b$. Thus,

$c^{-1}d\ H\ a^{-1}b$. So H is symmetry. We now show that H is transitive. Let $a^{-1}b$, $c^{-1}d$, $m^{-1}n$ be elements of Q such that $a^{-1}bHc^{-1}d$ and $c^{-1}dHm^{-1}n$. Then, $\exists\ x^{-1}y$, $z^{-1}t$, $u^{-1}v$, $q^{-1}r$ in Q such that

$$a^{-1}bx^{-1}y = a^{-1}bz^{-1}t$$

if and only if

$$c^{-1}dx^{-1}y = c^{-1}dz^{-1}t$$

and

$$x^{-1}ya^{-1}b = a^{-1}ta^{-1}b$$

if and only if

$$x^{-1}yc^{-1}d = z^{-1}tc^{-1}d.$$

Also,

$$c^{-1}du^{-1}v = c^{-1}dq^{-1}r$$

if and only if

$$m^{-1}nu^{-1}v = m^{-1}nq^{-1}r \text{ and } u^{-1}vc^{-1}d$$

$$= q^{-1}rc^{-1}d \text{ if and only if } u^{-1}vm^{-1}n$$

$$= q^{-1}rm^{-1}n. \text{ Therefore, } m^{-1}nu^{-1}v$$

$$= m^{-1}nq^{-1}r \text{ if and only if } c^{-1}du^{-1}v$$

$$= c^{-1}dq^{-1}r,$$

Using the first part, then,

$$c^{-1}du^{-1}v = c^{-1}dq^{-1}r$$

if and only if

$$a^{-1}bu^{-1}v = a^{-1}bq^{-1}r.$$

That is,

$$a^{-1}bu^{-1}v = a^{-1}bq^{-1}r$$

if and only if

$$m^{-1}nu^{-1}v = m^{-1}nq^{-1}r.$$

Using the same argument, we have,

$$u^{-1}va^{-1}b = cq^{-1}ra^{-1}b$$

if and only if

$$u^{-1}vm^{-1}n = q^{-1}rm^{-1}n.$$

Thus $a^{-1}bHmn$. That is, H is transitive and so H is an equivalent relation. We now show that H is congruence. So, let $a^{-1}bHc^{-1}d$, for $a^{-1}b, c^{-1}d \in Q$.

Now, $a^{-1}bHc^{-1}d$ implies that, $\exists\ x^{-1}y, z^{-1}t \in Q$ such that

$$a^{-1}bx^{-1}y = a^{-1}bz^{-1}t$$

if and only if

$$c^{-1}dx^{-1}y = c^{-1}dz^{-1}t$$

and

$$x^{-1}ya^{-1}b = z^{-1}ta^{-1}b$$

if and only if

$$x^{-1}yc^{-1}d = z^{-1}tc^{-1}d.$$

Let $m^{-1}n \in Q$, then since Q is a semigroup, it is closed, and so,

$$m^{-1}na^{-1}bx^{-1}y = m^{-1}na^{-1}bz^{-1}t$$

if and only if

$$m^{-1}nc^{-1}dx^{-1}y = m^{-1}nc^{-1}dz^{-1}t$$

and

$$x^{-1}ya^{-1}bm^{-1}n = z^{-1}ta^{-1}bm^{-1}n$$

if and only if

$$x^{-1}yc^{-1}dm^{-1}n = z^{-1}tc^{-1}dm^{-1}n.$$

Thus if

$$a^{-1}bHc^{-1}d, \text{ then } m^{-1}na\text{-}bHm^{-1}nc^{-1}d.$$

Similarly, if $a^{-1}bHc^{-1}d$, then $a^{-1}b^{-1}m^{-1}nHc^{-1}dm^{-1}n$. So, H is congruence on Q.

We now prove that $\rho_{max} = H$. To do this, we recall that $a^{-1}b\rho_{max}c^{-1}d$ if and only if $b^{-1}aea^{-1}b\rho d^{-1}cec^{-1}d$, where e is an idempotent in Q and ρ any congruence on Q. This makes $\rho_{max} \subseteq H$. Let $a^{-1}bHc^{-1}d$. Since H is congruence and it is not the minimum congruence, then congruence ρ such that $a^{-1}b\rho c^{-1}d$. Let e an idempotent in Q. By the compatibility property of p, we have, $ea^{-1}b\rho ec^{-1}d$.

Also, for $b^{-1}a$, $d^{-1}c$ in Q, we have, $b^{-1}a\rho d^{-1}c$. Thus combining the two, we have, $b^{-1}aea^{-1}b\rho d^{-1}cec^{-1}d$. This implies, $a^{-1}b\rho c^{-1}d$. Therefore if $a^{-1}bHc^{-1}d$, then $a^{-1}b\rho c^{-1}d$. So $H \subseteq \rho_{max}$.

Combining the two, we have $\rho_{max} = H$. This completes the proof.

3.2 Remark

The set of all idempotents in Q will be denoted by E_Q. A congruence ρ on E_Q will be called normal congruence if for any e, f ∈ E_Q, $a^{-1}b \in Q$, eρf implies $b^{-1}aea^{-1}b\rho b^{-1}afa^{-1}b$.

3.3 Lemma

Let be H congruence on E_Q. Then H is a normal congruence.

Proof

By 3.l H is congruence on Q. Suppose we restrict H to E_Q, then H congruence on E_Q. Also, $\rho_{max} = H$, so H is normal on Q.

3.4 Lemma

Let Q be a bisimple inverse co-semigroup of left quotient of S. If a^{-1} is the inverse of a in a subgroup of Q, then $(a\rho)^{-1} = a^{-1}\rho$ is the inverse of aρ in a subgroup of Q/ρ.

Proof

Since a^{-1} is the inverse of a in a subgroup of Q, then $aa^{-1} = e \in Q$, the idempotent. Now,

$$(aa^{-1})\rho = e\rho$$

$$(a\rho)(a^{-1}\rho) = e\rho$$

$$(a\rho)^{-1}(a\rho)\,(a^{-1}\rho) = (a\rho)^{-1}(e\rho)$$

$$(a^{-1}\rho) = (a\rho)^{-1}.$$

3.5 Lemma

Let e be the idempotent in Q, then eρ is the idempotent in Q/ρ.

Proof

$$(e\rho)^2 = (e\rho)\,(e\rho)$$

$$= e^2\rho = e\rho.$$

3.6 Remark

We note that given an element aρ in S/ρ, we can choose an element xρ in S such that $\varphi(x\rho) = (\ell(a\rho), r(a\rho))$. Then $(ax\rho) = (x\rho)\,(a\rho) \in S_{r(a)}$ and $\varphi((axa)\rho) = \varphi(a\rho)$, so $[(ax)\rho\ , (axa)\rho]$

∈ Q. Let us define the map $\theta : S/\rho \to Q/\rho$ by $\theta(a\rho)$ $[(ax)\rho , (axa)\rho]$, where $\varphi(x\rho) = (\ell(a\rho) , r(a\rho))$. θ is an embedding.

3.7 Lemma

Let S be a left order in a bisimple inverse ω-semigroup Q. Suppose a is square cancellable in S, then aρ is square cancellable in S/ρ.

Proof

Suppose a is square cancellable in S, then ∃ x, y in S^1 such that xa^2 ya^2 implies xa = ya and $a^2x = a^2y$ implies ax = ay. Using this definition, we have,

$$(a^2x)\rho = (a^2y)\rho$$
$$a^2\rho x\rho = a^2\rho y\rho$$
$$a\rho x\rho = a\rho y\rho$$
$$(ax)\rho\ (ay)\rho$$

Similarly,

$$(xa^2)\rho = (ya^2\rho)$$
$$(xa)\rho\ (ay)\rho$$

That is, aρ is square cancellable.

3.8 Remark

We remark here that if a, b ∈ S and aRbρ, then aρRbρ. Also, the intersection of S with the H -class is non-empty. We recall that every element of Q can be written $a^{-1}b$ where a, b are elements of S, a is in a subgroup of Q, and aRb in Q.

3.9 Proposition

Let S be a left order in a bisimple inverse ω-semigroup Q. Then, S/ρ is a left order in Q/ρ.

Proof

Now let aρ, bρ, cρ, dρ ∈ S/ρ, where aρ, cρ are square cancellable. Define multiplication in Q as follows:

$$[(a^{-1}\rho)(b\rho)]\ [c^{-1}\rho d\rho] = (a^{-1}b\rho)\ (c^{-1}d\rho) = (xa)^{-1}\rho(yd)\rho,$$ where $(xb)\rho = (yc)\rho$ and

$$\varphi(x\rho)\ (r(a\rho) - \ell(b\rho) + \max(\ell(b\rho) , r(cp)) , r(a\rho) - \ell(b\rho) + \max(\ell(b\rho) , r(c\rho))),$$

φ(xρ) = (r(aρ) - ℓ(bρ) + max(ℓ(bρ) , max(ℓ(bρ) , r(cρ))). With this definition in mind and

using IV.2.4 S/ρ is a left order in Q/ρ.

3.10 The Reversible Congruence

Let a, b ∈ S and ρ a relation on Q such that $a^{-1}b\rho c^{-1}d$ if and only if ∃ x, y ∈ S such that xb = yd. Let x y be such that xb = xb, then $a^{-1}b\rho a^{-1}b$. So p is reflexive. Let then ∃ x, y such that xb = yd. Now yd = xb implies $c^{-1}d$ ρ $a^{-1}b$. So ρ symmetry. We now show that the relation ρ is transitive. Let $a^{-1}b$, $c^{-1}d$, $m^{-1}n$ ∈ Q. Suppose that &‘bρc1d and $c^{-1}d\rho m^{-1}n$, then ∃ x, y, u, v such that xb = yd and ud = vn. So, we have xb = yd = $yu^{-1}ud$ = $yu^{-1}vn$. If $yu^{-1}v$ = t, then, we have, xb tn. Therefore, $a^{-1}b\rho m^{-1}n$. Thus ρ is transitive. That is ρ is an equivalent relation. To conclude the prove, we show that ρ is a congruence. Let $a^{-1}b\rho c^{-1}d$ if and only if ∃ x, y such that xb = yd. Since ρ is reflexive, then for $m^{-1}n$ we have $m^{-1}npm^{-1}n$. Combining the two gives $m^{-1}na^{-1}b\rho m^{-1}nc^{-1}d$.

Similarly, $a^{-1}bm^{-1}n\rho c^{-1}dni^{-1}n$. So, p is congruence on Q. We have just shown that.

3.11 Proposition

Let $c^{-1}b$, $c^{-1}d$ ∈ Q, where a, b, c, d ∈ S and a, c are square cancellable elements. Suppose ρ is a relation such that $a^{-1}b\rho c^{-1}d$ if and only if ∃ x, y such that xb = yd. Then ρ is congruence, which we call reversible congruence.

3.12 Construction of Normal Subgroup.

In this section we shall construct a normal subgroup of Q using our reversible congruence. This section verifies the fact that reversible congruence is always available in a semigroup of left quotient.

3.13 Proposition

Consider the set N_ρ = {$a^{-1}b$ ∈ Q: $a^{-1}b\rho e$}. Then N_ρ is a normal subgroup of a subgroup of Q.

Proof

By definition, the idempotent e ∈ N_ρ and so N_ρ ∅. Therefore given $a^{-1}b$,

$c^{-1}d$ ∈ Q such that $a^{-1}bpe$, $c^{-1}dpe$. Since ρ is a congruence, then $a^{-1}bc^{-1}dpef$. We note that $a^{-1}bc$’d = $a^{-1}u^{-1}vd$, where ub = vc for u, v in S. So, $a^{-1}u^{-1}vdpe$, where ub vc. Thus, $a^{-1}bc^{-1}d$ ∈ N_ρ, that is, N_ρ is closed. Now, let $b^{-1}a$ be the inverse of $a^{-1}b$ in a subgroup of Q. Suppose that $a^{-1}b\rho e$, and then by the compatibility property of ρ, we have $b^{-1}aa^{-1}bpb^{-1}ae$, that is, $b^{-1}apb^{-1}ae$, that is, $b^{-1}bpb^{-1}a$, since e, aa^{-1} are idempotents. Also $b^{-1}b$ is an idempotent, so let $b^{-1}b$ = e. Therefore, $epb^{-1}a$. Since p is congruence, it is symmetric. So, $b^{-1}a\rho e$, that is,

$b^{-1}a \in N_\rho$, that is, $(a^{-1}b)^{-1} = b^{-1}a \in N_\rho$. Therefore, N_ρ is a subgroup. Now, suppose that $a^{-1}b$ is in N_ρ, then, $b^{-1}a$ is also in N_ρ. Then, $a^{-1}b\rho e$ and also for any $m^{-1}n$ in Q, $m^{-1}n\rho e$. Also, since $b^{-1}a$ in N_ρ, then $b^{-1}a\rho e$. Combining all, we have, $a^{-1}bm^{-1}nb^{-1}a\rho ee$, that is, $a^{-1}bm^{-1}nb^{-1}a\rho ee$, that is, $a^{-1}bm^{-1}nb^{-1}a\rho e$. Therefore, $a^{-1}bm^{-1}nb^{-1}a \in N$. Therefore, N_ρ is a normal subgroup of Q.

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