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Optimal Uniform Finite Difference Scheme For The Scalar Singularly Perturbed Riccati Equation

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Abstract

A finite difference scheme of order one is presented for the scalar singularly Perturbed Riccati equation

$$\varepsilon u'(x) = c(x)\, u^2(x) + d(x) u(x) + e(x),\ x > 0,\ u(0) = \varphi$$

with a small parameter ε multiplying the first derivative. The scheme is derived from Euler's Backward Rule. And the scheme satisfies error estimate of the form

$$|\, u(x_i) - u_i \,| \leq C \min(h, \varepsilon),$$

where C is independent of i, h and ε. Here h is the mesh size and x_i is any mesh point. The scheme presented in this paper is new and it is different from the uniform schemes of order one available in the literature. Finally numerical experiments are presented.

Running Head : **Fitted finite difference scheme**

1. INTRODUCTION

Consider the scalar Riccati equation on the interval $\Omega = (0, \infty)$

$$\varepsilon u'(x) = c(x)\, u^2(x) + d(x)\, u(x) + e(x),\ x \in \Omega \quad (1a)$$

$$u(0) = \varphi \quad (1b)$$

where $\varepsilon > o$ is a small parameter and c , d and e are smooth functions on $\bar{\Omega}$

In addition we assume that

$$d^2(x) - 4\,c(x)\,e(x) \geq \alpha > o\,,\; x \in \bar{\Omega} \qquad (2)$$

Equation(1a,b) has wide applications in many areas of science, such as Chemical Kinetics [8], Mathematical physics [4], for example in the propagation of auxillary symmetric waves [7], the input wave impedance to an induction device [5], quadratic periodic optimization, in the design of solar heating system [11],etc.

The equation (1a,b) can be written in the form [2]

$$N\,u(x) = \varepsilon\, u'(x) - c(x)[u(x) - a(x)]\,[u(x) - b(x)] = 0, \quad x \in \bar{\Omega} \qquad (3a)$$

$$u(o) = \varphi \qquad (3b)$$

The condition (2) is sufficient to guarantee that operator N has a maximum principle and the solution u(x) of expression (3a,b) is unique and bounded.

The problem (3a,b) is a singularly perturbed equation with an initial layer at $x = 0$ whose width is of order ε [2,9,10]. As ε goes to zero , the equation (3a,b) reduces to the form

$$c(x)[u_0(x) - a(x)]\,[u_0(x) - b(x)] = 0, \quad x \in \bar{\Omega} \qquad (4)$$

The solution of (4) is $u_0(x) = a(x)$ and $u_0(x) = b(x)$. That is the critical points are clearly a(x) and b(x).We may define the corresponding reduced problem $u_0(x)$ by

$$u_0(x) = a(x),\; x \in \bar{\Omega} \qquad (5)$$

which is the stable critical point ,throughout this paper. In general, numerical solution of (3a,b) using the classical Euler' s forward rule will not yield satisfactory result. A modified form of the Euler's forward Rule is presented in [2,9,10] which gives satisfactory result but not for small values of ε. Again, Euler' s Backward Rule will not give uniform result and it is not approximating the initial layer for small values of ε though the scheme is implicit in nature.

We introduce a uniform mesh of width h on $\bar{\Omega}$ with mesh points $x_i = ih$ .We solve problem (3a,b) by finite difference methods of the form

$$N^h u_i \equiv \varepsilon\, \sigma_i(\rho) D_+ u_i - c_{i+1}[u_{i+1} - a_{i+1}]\,[u_{i+1} - b_{i+1}] = 0 \qquad (6a)$$

$$u_0 = \varphi \qquad (6b)$$

where $c_{i+1} = c(x_{i+1})$, $a_{i+1} = a(x_{i+1})$, $b_{i+1} = b(x_{i+1})$ and the fitting factor $\sigma_i(\rho)$ are specified later. The scheme of this paper is chosen in such a way that it must solve exactly the reduced problem (5) as ε goes to zero, because the schemes which solve exactly the reduced problem (5) are expected to work well for large values of x. If the solution u_i of the scheme (6a,b) satisfies the reduced equation (5) exactly at the interior points , as ε goes to zero then we call such finite difference scheme with this property as optimal.

In this paper the fitting factor will be always be chosen so that the difference scheme is uniform with respect to the small parameter, that is, if u and u_i are the solutions of (3a,b)and (6a,b) respectively, then at each node x_i, there is an error estimate of the form

$$|u(x_i) - u_i| \leq C h^p \quad (7)$$

where C is independent of i, h and ε

Uniformly convergent finite difference schemes for the solutions of singular perturbation problems have been about for some time now. For singularly perturbed initial value problems the traditional approach has been to use techniques involving adaptive mesh refinement. Uniform results for linear and nonlinear singularly perturbed problems have been approached from the point of view of singular perturbations and exponential fitting [2,3,9,10,12,13].Uniform results for nonlinear initial value problems based on exponential fitting appear in [2,10,9,12].Uniformly convergent finite- difference schemes for the problem (3a,b) have been proposed by Carroll [2] and O' Reilly [9,10].

The purpose of this paper is to propose a finite difference scheme for the problem (3a,b) such that the solution of the scheme reflects the asymptotic properties of the solution of (3a,b). We derive error estimates of the form

$$|u(x_i) - u_i| \leq C \min(h, \varepsilon) \quad (8)$$

where C is independent of i, h and ε. Scheme satisfying inequality (8) are clearly uniform of order one and optimal.

The schemes proposed in [2,9,10] are uniform of order one and satisfies the estimate (7) for p=1. And these schemes are not satisfying the estimate (8) and so they are not optimal. All schemes available in the literature are uniform of order one but not optimal. The scheme presented in this paper is derived from Eulers Backward Rule and it is entirely different from all other schemes available in the literature. And the scheme works well for moderate and small values of ε and even for large values of the step size h.

Throughout this paper $\rho = h/\varepsilon$ and C will denote a generic constant independent of i , h and ε

2. FINITE DIFFERENCE SCHEME

In this section a fitted finite difference scheme with a variable fitting factor is proposed. The consistency ,stability and convergence are discussed. The fitting factor will be fixed by deriving necessary condition for uniform convergence as in [3].

The asymptotic expansion for the solution of (3a,b) is [9,10]

$$u(x) = u_0(x) + v_0(x/\varepsilon) + o(\varepsilon) \qquad (9)$$

where $u_0(x) = a(x)$, the stable solution of the reduced problem and

$v_0(x/\varepsilon) = k[a(0) - b(0)]\exp(q_0 x/\varepsilon)/[1-\exp(q_0 x/\varepsilon)]$,

$q_0 = c(0)[a(0) - b(0)]$ and $k = (\varphi - a(0))/(\varphi - b(0))$.

2.1 NECESSARY CONDITION FOR UNIFORM CONVERGENCE

Assume the solution of the scheme (6a,b) converge uniformly in ε ,that is as limit h goes to zero lim u_i = lim $u(x_i)$.Taking limit on both sides of the scheme (6a,b)

we have

$$\lim\{(\sigma_i(\rho)/\rho)[u_{i+1} - u_i] - c_{i+1}[u_{i+1} - a_{i+1}][u_{i+1} - b_{i+1}]\} = 0$$

That is,

$$\lim\{(\sigma_i(\rho)/\rho)[u((i+1)h) - u(ih)] - c_0[u((i+1)h) - a_0][u(i+1)h - b_0]\} = 0$$

substituting the asymptotic expansion for u(x) from (9) we have

$$\lim\{(\sigma_i(\rho)/\rho)[v_0((i+1)\rho) - v_0(i\rho)] - c_0 v_0((i+1)\rho) - a_0][v_0((i+1)\rho) + a_0 - b_0]\} = 0$$

That is ,

$$\lim \sigma_i(\rho) = \sigma(\rho q_0)[1- k \exp(i \rho q_0)] / [1- k \exp((i+1) \rho q_0)] \quad (10)$$

where

$$\sigma(\rho q_0)= \rho q_0 \exp(\rho q_0)] / [1- \exp(\rho q_0)]$$

the condition (10) is the necessary condition for uniform convergence of the scheme (6a,b). In the following a scheme is defined for the problem (3a,b) with a fitting factor satisfying the necessary condition (10) exactly.

The finite difference scheme for (3a,b) is

$$N^h u_i \equiv \varepsilon \sigma_i(\rho)D_+u_i – c_{i+1}[u_{i+1}- a_{i+1}] [u_{i+1}- b_{i+1}] = 0 \quad (11a)$$

$$u_0 = \varphi \quad (11b)$$

where $\sigma_i(\rho) = \sigma(\rho q_0)R_i$ (11c)

$$\sigma(\rho q_0)= \rho q_0 \exp(\rho q_0) / [1- \exp(\rho q_0)] \quad (11d)$$

$$q_0= c(0)[a(0) – b(0)] \quad (11e)$$

$$R_i =[1- k \exp(i \rho q_0)] / [1- k \exp((i+1) \rho q_0)] \quad (11f)$$

$$c_{i+1} = c(x_{i+1}), a_{i+1} = a(x_{i+1}), b_{i+1} = b(x_{i+1}) \quad (11g)$$

and

$$k = (\varphi – a(0)) / (\varphi – b(0)) \quad (11h)$$

for the case $a(x) \neq b(x)$

The scheme (11a-h) is consistent with the problem (3a,b) in the sense that the discrete problem coincides with the problem (3a,b) when h approaches zero. The scheme satisfies the necessary condition (10) for uniform convergence exactly. The scheme (11a –h) models the equation (5) exactly as ε goes to zero,

$$u_{i+1} = a(x_{i+1}).$$

And so one can expect the scheme (11a-h) to work well for large x. The scheme is fitted, because the necessary condition (10) gives minimum requirement on the scheme to model the transient behavior of the problem (3a,b) accurately.

In [2,P34] a scheme of the form

$$N^h u_i \equiv \varepsilon \sigma_i(\rho)D_+u_i +a(x_{i+1}) u_{i+1} = f(x_{i+1}), \ i \geq 0 \quad (12a)$$

$$u_0= \varphi \quad (12b)$$

with a fitting factor

$\sigma(\rho)= \rho\, a(0) \exp(-\rho\, a(0)) / [1- \exp(-\rho\, a(0))]$, $\rho=h/\varepsilon$

is proposed for the linear problem

$N\, u(x) \equiv \varepsilon\, u'(x) + a(x)\, u(x)=f(x)$, $x > 0$ (13a)

$u(0) = \varphi$ (13b)

The problem(13a,b) is the reduced form of the problem(3a,b). And the scheme(12a,b) is the reduced form of the scheme(11a-h) presented in this paper.

2.2 STABILITY RESULT

The scheme (11a-h) is a quadratic equation in which u_{i+1} is unknown in terms of known value u_i. And the stable solution u_{i+1} of the quadratic form of the scheme (11a- h) is

$$u_{i+1} =[P_i + \gamma(P_i^2 – 4Q_i)] / 2 , \quad i \geq 0 , \quad u_0= \varphi$$

where

$$P_i = a_{i+1} + b_{i+1}+ ((\sigma(\rho)/ \rho)/c_{i+1}),$$

$$Q_i = a_{i+1} + b_{i+1}+ ((\sigma(\rho)/ \rho)/c_{i+1})u_i$$

It is observed that the term $[P_i^2 – 4Q_i] \geq 0$ and so there is no restriction on the factor ρ and on the coefficients of the given equation (3a,b). Hence the scheme (11a-h) is uniformly stable for all values of h and ε.

Following the results of Keller [6] ,a stability result is given for numerical stability of the solution of the scheme (11a- h) in the form of a lemma .

Lemma 2.1 [6]

Assume that $d^2(x) – 4c(x)e(x) \geq \alpha \geq 0$ for all $x \in \bar{\Omega}$. Let N^h be the operator defined in (11a- h). If (v_i) and (w_i) be any two mesh functions then , for all $x \in \bar{\Omega}$ and $i \geq 0$

$$| v_i –w_i | \leq | v_0 –w_0 | + \max | N^h v_j – N^h w_j |$$

Following theorem gives the convergence result for the scheme (11a- h). An estimate of the form (8) is obtained in this theorem .

Theorem 2.2

Let u and u_i be the solution of problem (3a,b) and (11a –h) respectively Then ,at mesh point x_i , we have the following error estimate,

$$|u(x_i) - u_i| \leq C \min(h, \varepsilon) \quad (14)$$

where C is independent of i, h and ε

Proof

From the stability result of N^h in scheme (11a-h) it suffices to prove that

$$|\tau_i| = |N^h u(x_j) - N^h u_j| \leq C \min(h, \varepsilon),$$

where τ_i is the truncation error of the scheme (11a- h) with respect to the problem (3a,b)

For i=0, $\tau_0 = \varphi - \varphi = 0$

for $i \geq 1$, $\tau_i = N^h u(x_i) - N^h u_i$

$= N^h u(x_i) - 0$

since $Nu(x)=0$, we have $Nu(x_{i+1})=0$ and so

$$\tau_1 = N^h u(x_i) - Nu(x_{i+1})$$

$$= \varepsilon \sigma_i(\rho) D_+ u(x_i) - c(x_{i+1})[u(x_{i+1}) - a(x_{i+1})][u(x_{i+1}) - b(x_{i+1})]$$

$$- \{ \varepsilon u'(x_{i+1}) - c(x_{i+1})[u(x_{i+1}) - a(x_{i+1})])[u(x_{i+1}) - b(x_{i+1})] \}$$

$$= \varepsilon \sigma_i(\rho) D_+ u(x_i) - \varepsilon u'(x_{i+1}).$$

Using asymptotic expansion (9) for the solution of (3a,b)

$$\tau_i = (\sigma_i(\rho)/\rho)[u_0(x_{i+1}) - u_0(x_i)] + (\sigma(\rho)/\rho)[v_0(x_{i+1}/\varepsilon) - v_0(x_i/\varepsilon)]$$

$$- \varepsilon u_0'(x_{i+1}) - \varepsilon v_0'(x_{i+1}/\varepsilon) + o(\varepsilon).$$

But, $\varepsilon v_0'(x/\varepsilon) = c(0)[v_0(x/\varepsilon) + u_0(0) - a(0)][v_0(x/\varepsilon) + u_0(0) - b(0)].$

And

$$(\sigma_i(\rho)/\rho)[v_0(x_{i+1}/\varepsilon) - v_0(x_i/\varepsilon)] = c(0)[v_0(x/\varepsilon) + u_0(0) - a(0)][v_0(x/\varepsilon) + u_0(0) - b(0)].$$

Therefore,

$$\varepsilon v_0'(x_{i+1}/\varepsilon) = (\sigma_i(\rho)/\rho)[v_0(x_{i+1}/\varepsilon) - v_0(x_i/\varepsilon)]$$

and so

$$\tau_i = \varepsilon \sigma_i(\rho) D_+ u_0(x_i) - \varepsilon u_0'(x_{i+1}) + o(\varepsilon)$$

$$= \varepsilon [\sigma_i(\rho) - 1] D_+ u_0(x_i) + \varepsilon [D_+ u_0(x_i) - u'_0(x_{i+1})] + 0(\varepsilon).$$

Again from the scheme (11a-h) we have

$$\varepsilon\,[\sigma_i(\rho) - 1] = \varepsilon\,[\sigma(\rho q_0)R_i - 1]$$
$$= \varepsilon\,[\sigma(\rho q_0) - 1] + \varepsilon\,\sigma(\rho q_0)[R_i - 1]$$
$$= \varepsilon\,[\sigma(\rho q_0) - 1] + \varepsilon\,\sigma(\rho q_0)[\exp(pq_0) - 1]\, v_0(x/\varepsilon)/[a(0) - b(0)].$$

From [2] we have

$$\varepsilon\left|\ \sigma(\rho) - 1\ \right| \le C \min(h, \varepsilon)$$
$$\left|\ \exp(\rho q_0) - 1\ \right| \le C \min(1, \rho)$$
$$\left|\ \sigma(\rho q_0) - 1\ \right| \le C$$

and hence

$$\varepsilon\left|\ \sigma_i(\rho) - 1\ \right| \le C \min(h, \varepsilon).$$

Therefore ,

$$\left|\ \tau_i\ \right| \le C \min(h, \varepsilon) \text{ for all } i \ge 0.$$

Since $\left| D_+u_0(x_i) \right| \le C$ and $\left| D_+u_0(x_i) - u'_0(x_{i+1}) \right| \le ch$

Using stability result ,we have

$$\left| u(x_i) - u_i \right| \le \left| u(0) - u_o \right| + \max\left| N^h u(x_j) - N^h u_j \right|$$
$$\le \left| \varphi - \varphi \right| + \max\left| \tau_i \right|$$
$$\le \max\left| \tau_i \right|$$
$$\le C \min(h, \varepsilon)$$

which is the required estimate .

Hence the scheme (11a-h) is proved to be uniform of order one and optimal.

3. NUMERICAL EXPERIMENT

This section gives numerical results for three singularly perturbed Riccati equations for large values of x . We compare the finite difference scheme (11a-h) in tables1-3 with a number of integration formulae when applied to the approximate computation of the solution of three sample scalar Riccati equations. For all problems ,we use a common uniform mesh and interval

$h = 0.0625$ and $x \in [0,1]$ respectively and we compute the absolute error

$$e_i^h = \max\left|\ u(x_i) - u_i\ \right| \text{for all } i = 0(1)16.$$

All computations were performed in pascal single precision on a micro vax II computer at Bharathidasan University ,Thirichirapalli – 620024, India.

The sample problems which we consider are as follows :

Problem 1: (source[1])

$\varepsilon u'(x) = - (1/80)u(x)(u(x) - 20), u(0) = 1.$

numerical results are given in Table 1.

Problem 2: (source[1])

$\varepsilon u'(x) = - \cos(x)u(x)(u(x) - 1), u(0) = 0.5.$

numerical results are given in Table 2.

Problem 3: (source[2])

$\varepsilon u'(x) = - 0.8u^2(x) + 0.6\sin(x) + 0.1 , u(0) = 0.25.$

numerical results are given in Table 3.

In Tables 1-3, a comparative study is made. The exact solution of the problems 1 and 2 are known and for the problem 3 exact solution is not known and so the zeroth order asymptotic expansion for the solution of the problem 3 is considered. The schemes compared with the scheme (11a-h) presented in this paper are

1.Euler Forward method

2.Euler Backward method

3.Trapezoidal method

4.Scheme of Carroll [2] and

5.Scheme of O'Reilly [9,10].

It is observed from Tables 1-3 the scheme (11a-h) is better than all other schemes. From Table 4 it is evident that the explicit scheme of Carroll [2] and the explicit scheme of O'Reilly [9,10] will not solve the reduced problem though they are uniform of order one and fitted. From Table 5 it is clear that the scheme (11a-h) solves exactly the reduced problem of the test problem 3, for small values of ε .

The scheme (11a-h) is fitted, uniform of order one and optimal. The scheme (11a-h) works well for small values of ε and even for large values of the mesh size h. Finally, the scheme (11a-h) is better than all other schemes available in the literature.

CONCLUSION

We have presented a finite difference scheme of order one for the scalar singularly perturbed Riccati equation. The classical Euler's Rule will not give good approximation for the solution of the problem (1a ,b) for moderate and small values of ε. To have good approximation for the solution of the problem (1a ,b), finite difference schemes of one are given in [2] and [9,10]. These schemes are uniform of order one not optimal and so one can not use these schemes for large values of x. In this paper a scheme derived from Euler's Backward Rule is given which is uniform of order one optimal and so one can use the scheme (11a-h) for large values of x. The scheme (11a-h) is exact when the coefficients are constants.

From Tables 4 and 5 it is observed that the solution of the scheme (11a –h) solves the solution of the reduced problem of (1a,b) exactly for small values of ε . At the same time other schemes do not have property. This property shows that the scheme (11a-h) reflects the asymptotic properties of the Riccati equation and from Table 5 is clear that the scheme (11a-b) works well for large values of x .

The scheme (11a-h) satisfies the necessary condition for uniform convergence and so the scheme (11a-h) is fitted and it approximate the boundary layer well. The scheme (11a-h) satisfies the error estimate (8) and so the scheme is uniform of order one and optimal. Since the scheme (11a-h) solves exactly the reduced problem of original Riccati equation it is clear that solution of the scheme reflects the asymptotic properties of the solution of the Riccati equation (1a,b).

The finite difference scheme presented in this paper use constant mesh size. To study the local behavior of the solution in the neighborhood ε when ε is small one can use variable mesh size . One can also try for higher order schemes . Investigation is going on to derive higher order methods to solve the problem(3a,b) by the author of this paper.

Finally, the scheme (11a-h) is fitted, uniform of order one and optimal.

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Table 1

Method / ε	1	0.1	0.01	0.001
Euler Forward	2.03574E-03	5.74981E-01	5.01635E+00	-----
Euler Backward	2.26462E-03	6.20243E-01	4.66148E+00	1.21181E+00
Trapezoidal	4.60148E-05	3.85036E-02	1.56595E+00	------
scheme of Carroll	1.28150E-04	1.43051E-06	1.90735E-06	7.62939E-06
scheme of O'Reilly	3.37628E-07	4.29153E-06	1.90735E-06	0.0
scheme (11a-h)	8.83341E-05	8.58307E-06	2.86102E-06	5.72205E-06

Table 2

Method / ε	1	0.1	0.01	0.001
Euler Forward	3.88807E-03	3.15889E-02	------	----
Euler Backward	3.95903E-03	2.61525E-02	7.18376E-02	1.21181E+00
Trapezoidal	4.36306E-05	5.05388E-03	-------	------
scheme of Carroll	2.95782E-03	6.36101E-04	7.86781E-06	0.0
scheme of O'Reilly	1.78814E-07	1.19209E-07	5.96046E-08	0.0
scheme (11a-h)	2.75385E-03	1.53601E-04	4.11272E-06	0.0

Table 3

Method / ε	1	0.1	0.01	0.001
Euler Forward	3.10281E-03	9.17339E-02	------	-----
Euler Backward	3.72441E-01	2.89718E-01	5.16458E-01	5.73484E-01
Trapezoidal	3.33508E-01	6.17581E-02	------	------
scheme of Carroll	3.12735E-01	1.07107E-01	6.09879E-02	6.09879E-02
scheme of O'Reilly	2.70707E-01	9.07679E-02	5.95078E-02	6.09879E-02
scheme (11a-h)	2.92601E-01	6.55365E-02	1.27727E-03	5.96046E-08

Table 4

h=0.0625, $\varepsilon = 0.0001$

SCHEME OF CARROL [2] & SCHEME OF O'REILLY [9,10]

x	u_i	$u_0(x_i)$	$u_0(x_i) - u_i$
6.25000E-02	3.53553E-01	4.14541E-01	6.09879E-02
1.25000E-01	4.10055E-01	4.67446E-01	5.73913E-02
1.87500E-01	4.63923E-01	5.14590E-01	5.06664E-02
2.50000E-01	5.12095E-01	5.57273E-01	4.51775E-02
3.12500E-01	5.55442E-01	5.96304E-01	4.08628E-02
3.75000E-01	5.94904E-01	6.32222E-01	3.73176E-02
4.37500E-01	6.31120E-01	6.65400E-01	3.42796E-02
5.00000E-01	6.64517E-01	6.96110E-01	1.15930E-02
5.62500E-01	6.95393E-01	7.24553E-01	2.91598E-02
6.25000E-01	7.23966E-01	7.50881E-01	2.69153E-02
6.87500E-01	7.50399E-01	7.75213E-01	2.48140E-02
7.50000E-01	7.74816E-01	7.97640E-01	2.28238E-02
8.12500E-01	7.97313E-01	8.18234E-01	2.09206E-02
8.75000E-01	8.17966E-01	8.37053E-01	1.90866E-02
9.37500E-01	8.36835E-01	8.54143E-01	1.73080E-02
1.00000E+00	8.53968E-01	8.69542E-01	1.55740E-02

Table 5

h=0.0625, $\varepsilon = 0.0001$

SCHEME (11a-h)

x	u_i	$u_0(x_i)$	$u_0(x_i) - u_i$
6.25000E-02	4.14541E-01	4.14541E-01	2.98023E-08
1.25000E-01	4.67446E-01	4.67446E-01	0.00000E+00
1.87500E-01	5.14590E-01	5.14590E-01	0.00000E+00
2.50000E-01	5.57273E-01	5.57273E-01	5.96046E-08
3.12500E-01	5.96304E-01	5.96304E-01	0.00000E+00
3.75000E-01	6.32222E-01	6.32222E-01	5.96046E-08
4.37500E-01	6.65400E-01	6.65400E-01	0.00000E+00
5.00000E-01	6.96110E-01	6.96110E-01	0.00000E+00
5.62500E-01	7.24553E-01	7.24553E-01	0.00000E+00
6.25000E-01	7.50881E-01	7.50881E-01	0.00000E+00
6.87500E-01	7.75213E-01	7.75213E-01	0.00000E+00
7.50000E-01	7.97640E-01	7.97640E-01	0.00000E+00
8.12500E-01	8.18234E-01	8.18234E-01	0.00000E+00
8.75000E-01	8.37053E-01	8.37053E-01	5.96046E-08
9.37500E-01	8.54143E-01	8.54143E-01	0.00000E+00
1.00000E+00	8.69542E-01	8.69542E-01	5.96046E-08