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PRIMARY AND SEMIPRIMARY Γ-IDEALS IN Γ -SEMIGROUPS

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ABSTRACT

In this paper a Γ -semigroup S with identity, if $\sqrt{A} = M$ for some Γ -ideal of S, where M is the unique maximal Γ-ideal of S, then A is a primary Γ -ideal. It is proved that if S is a Γ -semigroup with identity, then for any natural number *n*, $(M\Gamma)^{n-1}M$ is primary Γ -ideal of S, where M is the unique maximal Γ-ideal of S. Further it is proved that in quasi commutative -semigroup S, a Γ-ideal A of S is left primary iff A is right primary. It is proved that in semipseudo symmetric semiprimary Γ-semigroup, globally idempotent principal Γ-ideals form a chain under set inclusion. In a semisimple Γ-semigroup S it is proved that the conditions every ideal of S is prime, S is a primary Γ-semigroup, S is a left primary Γ-semigroup, S is a right primary Γ-semigroup, S is a semiprimary Γ-semigroup, prime Γ-ideals of S form a chain, principal Γ-ideals of S form a chain, Γ-ideals of S form a chain are equivalent.

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KEY WORDS: left primary Γ-ideal, right primary Γ-ideal, primary Γ-ideal, left primary Γ-semigroup, right primary Γ-semigroup and primary Γ-semigroup, semiprimary Γ-ideal.

1. INTRODUCTION :

Γ- semigroup was introduced by Sen and Saha [13] as a generalization of semigroup. Anjaneyulu. A [1], [2] and [3] initiated the study of pseudo symmetric ideals, radicals, semipseudo symmetric ideals in semigroups and N(A)-semigroups and primary and semiprimary ideals in semigroups. Giri and wazalwar [6] intiated the study of prime radicals in semigroups. Madhusudhana Rao, Anjaneyulu and Gangadhara Rao [7], [8], [9] , [10] and [11]initiated the study of pseudo symmetric Γ-ideals, prime Γ-radicals and semipseudo symmetric Γ-ideals in Γ-semigroups , N(A)-semigroups and pseudo integral Γ-senugroup. In

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this paper we introduce the notions of primary and semiprimary Γ-ideals in Γ-semigroup and characterize primary Γ-ideals and semiprimary Γ-ideals in Γ-semigroup.

2. PRELIMINARIES :

DEFINITION 2.1: Let S and Γ be any two non-empty sets. S is called a ***Γ-semigroup*** if there exist a mapping from $S\times\Gamma\times S$ to S which maps $(a, \alpha, b) \to a\alpha b$ satisfying the condition : $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a,b,c \in M$ and $\gamma, \mu \in \Gamma$.

NOTE 2.2: Let S be a Γ-semigroup. If A and B are subsets of S, we shall denote the set $\{ a\alpha b : a\in A, b\in B \text{ and } \alpha\in\Gamma \}$ by AΓB.

DEFINITION 2.3: An element ***a*** of a Γ-semigroup S is said to be a ***left identity*** of S provided $a\alpha s = s$ for all $s \in S$ and $\alpha \in \Gamma$.
DEFINITION 2.4: An element '***a***' of a Γ-semigroup S is said to be a ***right identity*** of S provided $s\alpha a = s$ for all $s \in S$ and $\alpha \in \Gamma$.

DEFINITION 2.5: An element '***a***' of a Γ-semigroup S is said to be a ***two sided identity*** or an ***identity*** provided it is both a left identity and a right identity of S.

NOTATION 2.6: Let S be a Γ-semigroup. If S has an identity, let $S^1 = S$ and if S does not have an identity, let S^1 be the Γ-semigroup S with an identity adjoined usually denoted by the symbol 1.

DEFINITION 2.7: A Γ-semigroup S is said to be ***commutative*** provided $a\gamma b = b\gamma a$ for all $a,b \in S$ and $\gamma \in \Gamma$.

DEFINITION 2.8: A Γ-Semigroup S is said to be ***quasi Commutative*** provided for all a,b ∈S, there exists a natural number n such that, $a\gamma b = (b\gamma)^n a \forall \gamma \in \Gamma$.

DEFINITION 2.9: A Γ- Semigroup S is said to be ***a globally idempotent Γ- semigroup*** provided SΓS = S.

DEFINITION 2.10: A Γ- semigroup S is said to be a ***left duo Γ- semigroup*** provided every left Γ- ideal of S is a two sided Γ- ideal of S.

DEFINITION 2.11: A Γ- semigroup S is said to be a ***right duo Γ- semigroup*** provided every right Γ- ideal of S is a two sided Γ- ideal of S.

DEFINITION 2.12: A Γ- semigroup S is said to be a ***duo Γ- semigroup*** provided it is both a left duo Γ- Semigroup and a right duo Γ- Semigroup.

DEFINITION 2.13: A nonempty subset A of a Γ-semigroup S is said to be a ***left Γ- ideal*** provided $S\Gamma A \subseteq A$.

DEFINITION 2.14: A nonempty subset A of a Γ-semigroup S is said to be a ***right Γ-ideal*** provided $A\Gamma S \subseteq A$.

DEFINITION 2.15: A nonempty subset A of a Γ-semigroup S is said to be a ***two sided Γ- ideal*** or simply a ***Γ- ideal*** provided it is both a left and a right Γ– ideal of S.

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DEFINITION 2.16: A Γ- ideal A of a Γ-semigroup S is said to be a ***principal* Γ- *ideal*** provided A is a Γ- ideal generated by single element ***a***. It is denoted by J[a] = <a>.

NOTE 2.17: If S is a Γ-Semigroup and $a \in S$, then
$<a> = \{a\} \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S = S^1\Gamma a\, \Gamma S^1$.

DEFINITION 2.18: A Γ- ideal A of a Γ-semigroup S is said to be a ***completely prime* Γ- *ideal*** provided $x\Gamma y \subseteq A$; $x, y \in S$ implies either $x \in A$ or $y \in A$.

DEFINITION 2.19: A Γ- ideal A of a Γ-semigroup S is said to be a ***prime* Γ- *ideal*** provided X ΓY ⊆ A ; X, Y are Γ- ideal of S, then either X ⊆ A or Y⊆ A.

DEFINITION 2.20: A Γ- ideal A of a Γ-Semigroup S is said to be a ***completely semiprime* Γ- *ideal*** provided $x\Gamma x \subseteq A$; $x \in S$ implies either $x \in A$.

DEFINITION 2.21: A Γ- ideal A of a Γ-Semigroup S is said to be a ***semiprime* Γ- *ideal*** provided $x\Gamma S^1\Gamma x \subseteq A$; $x \in S$ implies $x \in A$.

THEOREM 2.22 [8]: Every prime Γ-ideal of a Γ-semigroup S is a semiprime Γ-ideal of S.

DEFINITION 2.23: A Γ-ideal A in a Γ-Semigroup S is said to be a ***semipseudo symmetric Γ-ideal*** provided for any natural number n, $x \in S$, $(x\Gamma)^{n-1}x \subseteq A \Rightarrow (< x >\Gamma)^{n-1} < x> \subseteq A$.

3. PRIMAY Γ-IDEALS:

DEFINITION 3.1: A Γ-ideal A of a Γ-semigroup S is said to be a ***left primary Γ-ideal*** provided

i) If X, Y are two Γ-ideals of S such that XΓY ⊆ A and Y ⊈ A then X ⊆ √A.
ii) √A is a prime Γ-ideal of S.

DEFINITION 3.2: A Γ-ideal A of a Γ-semigroup S is said to be a ***right primary Γ-ideal*** provided

i) If X, Y are two Γ-ideals of S such that XΓY ⊆ A and X ⊈ A then Y ⊆ √A.
ii) √A is a prime Γ-ideal of S.

DEFINITION 3.3: A Γ-ideal A of a Γ-semigroup S is said to be a ***primary Γ-ideal*** provided A is both a left primary Γ-ideal and a right primary Γ-ideal.

THEOREM 3.4: Let A be a Γ-ideal of a Γ-semigroup S. If X, Y are two Γ-ideals of S such that XΓY ⊆ A and Y ⊈ A then X ⊆ √A if and only if $< x > \Gamma < y > \subseteq A$ and $y \notin A$, $x \in \sqrt{A}$.

Proof **:** Suppose that A is a Γ-ideal of a Γ-semigroup S and if X, Y are two Γ-ideals of S such that XΓY ⊆ A and Y ⊈ A then X ⊆ √A. Let $x, y \in S$, $y \notin A$. Then $< x > \Gamma < y > \subseteq X\Gamma Y \subseteq A$ and $< y > \not\subseteq A$. Therefore by supposition $< x > \Gamma < y > \subseteq A$ and $< y > \not\subseteq A \Rightarrow < x > \subseteq \sqrt{A}$. Therefore $x \in \sqrt{A}$.

Conversely suppose that $x, y \in S$, $< x > \Gamma < y > \subseteq A$ and $y \notin A$ then $x \in \sqrt{A}$. Let X, Y be two Γ-ideals of S such that $X\Gamma Y \subseteq A$ and $Y \nsubseteq A$. Suppose if possible $X \nsubseteq \sqrt{A}$. Then there exists $x \in X$ such that $x \notin \sqrt{A}$. Since $Y \nsubseteq A$, let $y \in Y$ so that $y \notin A$. Now $< x > \Gamma < y > \subseteq X\Gamma Y \subseteq A$ and $y \notin A \Rightarrow x \in \sqrt{A}$. It is a contradiction. Therefore $X \subseteq \sqrt{A}$. Therefore if X, Y are two Γ-ideals of S such that $X\Gamma Y \subseteq A$ and $Y \nsubseteq A$ then $X \subseteq \sqrt{A}$.

THEOREM 3.5: Let A be a Γ-ideal of a Γ-semigroup S. If X, Y are two Γ-ideals of S such that $X\Gamma Y \subseteq A$ and $X \nsubseteq A$ then $Y \subseteq \sqrt{A}$ if and only if $< x > \Gamma < y > \subseteq A$ and $x \notin A, y \in \sqrt{A}$.

***Proof* :** The proof is similar to the theorem 3.4.

THEOREM 3.6: Let S be a commutative Γ-semigroup and A is a Γ-ideal of S. Then the following conditions are equivalent.

1) A is a primary Γ-ideal.

2) X, Y are two Γ-ideals of S, $X\Gamma Y \subseteq A$ and $Y \nsubseteq A$ then $X \subseteq \sqrt{A}$.

3) $x, y \in S$, $x\Gamma y \subseteq A$, $y \notin A$ then $x \in \sqrt{A}$.

***Proof*:** (1) ⇒ (2) : Suppose that A is a primary Γ-ideal of S. Then A is a left primary Γ-ideal of S. So by definition 3.1, we get X, Y are two Γ-ideals of S, $X\Gamma Y \subseteq A$, $Y \nsubseteq A \Rightarrow X \subseteq \sqrt{A}$.

(2) ⇒ (3): Suppose that X, Y are two **Γ**-ideals of S, $X\Gamma Y \subseteq A$ and $Y \nsubseteq A$ then $X \subseteq \sqrt{A}$. Let $x, y \in S$, $x\Gamma y \subseteq A$, $y \notin A$. $x\Gamma y \subseteq A \Rightarrow < x > \Gamma < y > \subseteq A$ also $y \notin A \Rightarrow < y > \nsubseteq A$.
Now $< x > \Gamma < y > \subseteq A$ and $< y > \nsubseteq A$. Therefore by assumption $< x > \subseteq \sqrt{A}$ $\Rightarrow x \in \sqrt{A}$.
(3) ⇒ (1) : Suppose that $x, y \in S$, $x\Gamma y \subseteq A$, $y \notin A$ then $x \in \sqrt{A}$. Let X, Y are two **Γ**-ideals of S, $X\Gamma Y \subseteq A$ and $Y \nsubseteq A$. $Y \nsubseteq A \Rightarrow$ there exists $y \in Y$ such that $y \notin A$.
Suppose if possible $X \nsubseteq \sqrt{A}$. Then there exists $x \in X$ such that $x \notin \sqrt{A}$.
Now $x\Gamma y \subseteq X\Gamma Y \subseteq A$. Therefore $x\Gamma y \subseteq A$ and $y \notin A$, $x \notin \sqrt{A}$.
It is a contradiction. Therefore $X \subseteq \sqrt{A}$. Let $x, y \in S$, $x\Gamma y \subseteq \sqrt{A}$. Suppose that $y \notin \sqrt{A}$.
Now $x\Gamma y \subseteq \sqrt{A} \Rightarrow (x\Gamma y\Gamma)^{m-1}(x\Gamma y) \subseteq A \Rightarrow (x\Gamma)^{m-1}x\Gamma(y\Gamma)^{m-1}y \subseteq A$.
Since $y \notin \sqrt{A}$, $(y\Gamma)^{m-1}y \nsubseteq A$. Now $(x\Gamma)^{m-1}x\Gamma(y\Gamma)^{m-1}y \subseteq A$, $(y\Gamma)^{m-1}y \nsubseteq A \Rightarrow (x\Gamma)^{m-1}x \subseteq \sqrt{A} \Rightarrow$ $x \in \sqrt{(\sqrt{A})} = \sqrt{A}$. $\sqrt{A}$ is a completely prime Γ-ideal and hence $\sqrt{A}$ is a prime Γ-ideal. Therefore A is a left primary Γ-ideal. Similarly A is a right primary Γ-ideal. Hence A is a primary Γ-ideal.

NOTE 3.7: In an arbitrary Γ-semigroup a left primary Γ-ideal is not necessarily a right primary Γ-ideal.

EXAMPLE 3.8 : Let $S = \{a, b, c\}$ and $\Gamma = \{x, y, z\}$. Define a binary operation **.** in S as shown in the following table.

.	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*a*	*b*	*c*

Define a mapping S X Γ X S ⟶ S by $a\alpha b = ab$, for all $a, b \in$ S and $\alpha \in \Gamma$. It is easy to see that S is a Γ-semigroup. Now consider the Γ-ideal $< a > = S^1\Gamma a\Gamma S^1 = \{a\}$. Let $p\Gamma q \subseteq < a >$, $p \notin < a > \Rightarrow q \in \sqrt{< a >} \Rightarrow (q\Gamma)^{n-1}q \subseteq < a >$ for some $n \in$ N. Since $b\Gamma c \subseteq < a >$, $c \notin < a > \Rightarrow b \in < a >$. Therefore $< a >$ is left primary. If $b \notin < a >$ then $(c\Gamma)^{n-1}c \notin < a >$ for any $n \in$ N $\Rightarrow c \notin \sqrt{< a >}$. Therefore $< a >$ is not right primary.

THEOREM 3.9: Every Γ-ideal A in a Γ-semigroup S is left primary if and only if every Γ-ideal A satisfies condition (i) of definition 3.1.

***Proof* :** If every Γ-ideal A of S is left primary, then clearly every Γ-ideal satisfies condition (i) of definition 3.1. Conversely suppose that every Γ-ideal of S satisfies condition (i) of definition 3.1. Let A be any Γ-ideal of S. Suppose $< x > \Gamma < y > \subseteq \sqrt{A}$. If $y \in \sqrt{A}$ then by our supposition $x \in \sqrt{(\sqrt{A})} = \sqrt{A}$. Therefore $\sqrt{A}$ is a prime Γ-ideal. Hence A is left primary.

THEOREM 3.10: Every Γ-ideal A in a Γ-semigroup S is right primary if and only if every Γ-ideal A satisfies condition (i) of definition 3.2.

***Proof* :** The proof is similar to the proof of theorem 3.9.

DEFINITION 3.11: A Γ-semigroup S is said to be ***left primary*** provided every Γ-ideal of S is a left primary Γ-ideal.

DEFINITION 3.12: A Γ-semigroup S is said to be ***right primary*** provided every Γ-ideal of S is a right primary Γ-ideal.

DEFINITION 3.13: A Γ-semigroup S is said to be ***primary*** provided every Γ-ideal of S is a primary Γ-ideal.

THEOREM 3.14: Let S be a Γ-semigroup with identity and let M be the unique maximal Γ-ideal of S. If $\sqrt{A}$ = M for some Γ-ideal of S. Then A is a primary Γ-ideal.

***Proof* :** $< x > \Gamma < y > \subseteq$ A and $y \notin$ A. If $x \notin \sqrt{A}$ then $< x > \nsubseteq \sqrt{A}$ = M. Since M is the union of all proper Γ-ideals of S, we have $< x >$ = S and hence $< y > = < x > \Gamma < y > \subseteq$ A. It is a contradiction. Therefore $x \in \sqrt{A}$. Let $< x > \Gamma < y > \subseteq \sqrt{A}$ and $< y > \nsubseteq \sqrt{A}$. Since M is the maximal Γ-ideal, we have $< x >$ = S. Hence $< y > = < x > \Gamma < y > \subseteq \sqrt{A}$. It is a contradiction. Therefore $< x > \subseteq \sqrt{A}$. Similarly if $< x > \nsubseteq \sqrt{A}$, then $< y > \subseteq \sqrt{A}$ and hence $\sqrt{A}$ = M is a prime Γ-ideal. Thus A is left primary. By symmetry it follows that A is right primary. Therefore A is a primary Γ-ideal.

NOTE 3.15: If a Γ-semigroup S has no identity, then the theorem 3.14, is not true, even if the Γ-semigroup S has a unique maximal Γ-ideal. In example 3.8, $\sqrt{< a >}$ = M where M = $\{a, b\}$ is the unique maximal Γ-ideal. But $< a >$ is not a primary Γ-ideal.

THEOREM 3.16: If S is a Γ-semigroup with identity, then for any natural number *n*, $(M\Gamma)^{n-1}M$ is primary Γ-ideal of S where M is the unique maximal Γ-ideal of S.

***Proof* :** Since M is the only prime Γ-ideal containing $(M\Gamma)^{n-1}M$, we have $\sqrt{((M\Gamma)^{n-1}M)}$ = M and hence by theorem 3.14, $(M\Gamma)^{n-1}M$ is a primary Γ-ideal.

NOTE 3.17: If S has no identity then theorem 3.16, is not true. In example 3.8, M = $\{a, b\}$ is the unique maximal Γ-ideal, but MΓM = $\{a\}$ is not primary.

THEOREM 3.18: In quasi commutative Γ-semigroup S, a Γ-ideal A of S is left primary iff right primary.

***Proof* :** Suppose that A is a left primary Γ-ideal. Let $x\Gamma y \subseteq A$ and $x \notin A$. Since S is a quasi commutative Γ-semigroup, we have $x\Gamma y = (y\Gamma)^n x$ for some $n \in N$. So $(y\Gamma)^n x = (y\Gamma)^{n-1} y\Gamma\, x$ and $x \notin A$. Since A is left primary, we have $(y\Gamma)^{n-1} y \subseteq \sqrt{A}$ and since $\sqrt{A}$ is a prime Γ-ideal, $y \in \sqrt{A}$. Therefore A is a right primary Γ-ideal. Similarly we can prove that if A is a right primary Γ-ideal then A is a left primary Γ-ideal.

COROLLARY 3.19: If A is a Γ-ideal of a quasi commutative Γ-semigroup S, then the following are equivalent.

1) **A is primary**
2) **A is left primary**
3) **A is right primary**

4. SEMIPRIMARY Γ-IDEALS:

DEFINITION 4.1: A Γ-ideal A of a Γ-semigroup S is said to be semiprimary provided $\sqrt{A}$ is a prime Γ-ideal of S.

DEFINITION 4.2: A Γ-semigroup S is said to be a semiprimary Γ-semigroup provided every Γ-ideal of S is a semiprimary Γ-ideal.

THEOREM 4.3: Every left primary or right primary Γ-ideal of a Γ-semigroup is a semiprimary Γ-ideal.

THEOREM 4.4: Let S is a Γ-semigroup (not necessarily with identity) and let A be a Γ-ideal of S with $\sqrt{A}$ is a maximal Γ-ideal of S. Then A is a semiprimary Γ-ideal.

***Proof* :** If there is no proper prime Γ-ideal P containing A, then every prime Γ-ideal equal to S. Then the intersection of all prime Γ-ideals = S = $\sqrt{A}$. But $\sqrt{A}$ is maxima Γ-ideal which implies $\sqrt{A}$ must be proper. Therefore there exists a proper prime Γ-ideal P containing A. Now $\sqrt{A} \subseteq P \subseteq S$ and $\sqrt{A}$ is maximal, we have $\sqrt{A} = P$. Therefore $\sqrt{A}$ is a prime Γ-ideal and hence A is a semiprimary Γ-ideal.

THEOREM 4.5: If A is a semiprime Γ-ideal of a Γ-semigroup S, then the following are equivalent.

1) **A is a prime Γ-ideal.**
2) **A is a primary Γ-ideal.**
3) **A is a left primary Γ-ideal.**
4) **A is a right primary Γ-ideal.**
5) **A is a semiprimary Γ-ideal.**

***Proof* :** (1) ⇒ (2) ⇒ (3) ⇒ (5) and (2) ⇒ (4) ⇒ (5) are clear.
(5) ⇒ (1) : Suppose that A is a semiprimary Γ-ideal. Then $\sqrt{A}$ is a prime Γ-ideal. Since A is semiprime, A is the intersection of all prime Γ-ideals of S containing A. Therefore $A = \sqrt{A}$ is a prime Γ-ideal.

THEOREM 4.6: A Γ-semigroup S is semiprimary iff prime Γ-ideals of S form a chain under set inclusion.

***Proof* :** Suppose that S is a semiprimary Γ-semigroup. Let A and B are two prime Γ-ideals of S. Now $\sqrt{(A \cap B)} = \sqrt{A} \cap \sqrt{B} = A \cap B$. Therefore $A \cap B$ is semiprime. By theorem 4.5,

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Since S is a semiprimary Γ-semigroup, it follows that A ∩ B is prime. Suppose that A ⊈ B and B ⊈ A. Then there exists $x \in$ A\B and $y \in$ B\A.
Now < x > Γ < y > ⊆ A ∩ B and x, y ∈ A ∩ B. It is a contradiction. Therefore prime Γ-ideals of S form a chain.
Conversely suppose that prime Γ-ideals of S form a chain under set inclusion. For every Γ-ideal A, √A = ∩P_α, where intersection is over all prime Γ-ideals P_α containing A yields √A = P_α for some α, so that A is a semiprimary Γ-ideal. Therefore S is a semiprimary Γ-semigroup.

THEOREM 4.7: In a semipseudo symmetric semiprimary Γ-semigroup S, globally idempotent principal Γ-ideals form a chain under set inclusion.
Proof **:** Let < a >, < b > are two globally idempotent principal Γ-ideals of S. Since S is semiprimary Γ-semigroup, we have √< a > and √< b> are prime Γ-ideals and by theorem 4.6, either √< a > ⊆ √< b > or √< b > ⊆ √< a >. Assume that √< a > ⊆ √< b >, then a ∈ < a > ⊆ √< a > ⊆ √< b >. If a ∈ < b >, then $(< a > \Gamma)^{n-1} < a > \subseteq < b >$ for some $n \in$ N. Since < a > is a globally idempotent principal Γ-ideal, < a > = < a > Γ < a > ⊆ < b >. Therefore < a > ⊆ < b >. Similarly we can show that if √< b > ⊆ √ < a >,
then < b > ⊆ < a >. Therefore globally idempotent principal Γ-ideals form a chain under set inclusion.

NOTE 4.8: In an arbitrary Γ-semigroup S, Γ-idempotents need not form a chain under natural ordering, even if S is pseudo symmetric primary Γ-semigroup. For example in a left zero Γ-semigorup, Γ-idempotents do not form a chain under natural ordering.

NOTATION 4.9: For any Γ-semigroup S, let E_s denotes the set of all Γ-idempotents of S together with the binary relation denoted by $e \leq f$ if and only if $e = e\alpha f = f\beta e$ for $e, f \in E_s$, $\alpha, \beta \in \Gamma$. i.e, $e \in e\Gamma f = f\Gamma e$.

THEOREM 4.10: Let S is a duo semiprimary Γ-semigroup, then the Γ-idempotents of S form a chain under natural ordering.

Proof **:** Let e and f are two Γ-idempotents of S. Since S is a semiprimary Γ-semigroup, √< e > and √< f > are prime Γ-ideals of S. By theorem 4.6, either √< e > ⊆ √< f > or √< f > ⊆ √< e >. Assume that √< e > ⊆ √< f >. Since e ∈ < e > ⊆ √< e > ⊆ √< f >, we have e ∈ √< f >. Since S is a duo Γ-semigroup, √< f > = $\{x \in S : (x\Gamma)^{n-1}x \subseteq < f >$ for some $n \in N\}$. Thus, since e is a Γ-idempotent, we have e ∈ < f >. Since S is a duo Γ-semigroup, e ∈ < f > = $f\Gamma S = S\Gamma f$ and hence $e = f\alpha s = t\beta f$ for some $s, t \in S$, $\alpha, \beta \in \Gamma$. So $e\alpha f = t\beta f\alpha f = t\beta f = e$ and $f\beta e = f\beta f\alpha s = f\alpha s = e$. Therefore $e \leq f$. Similarly we can prove that if √ < f > ⊆ √ < e > then $f \leq e$. Hence Γ-idempotens of S form a chain under natural ordering.

THEOREM 4.18: If a Γ-semigroup S is regular then every principal Γ-ideal of S is generated by a Γ-idempotent.

Proof**:** Suppose S is a regular Γ-semigroup. Let < a > is a principal Γ-ideal of S. Since S is regular, there exists $x \in$ S and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.
Now $(a\alpha x)\beta(a\alpha x) = a\alpha x\beta a\alpha x = (a\alpha x\beta a)\alpha x = a\alpha x$. Let $a\alpha x = e$.
$a = a\alpha x\beta a = e\beta a \in < e > \Rightarrow < a > \subseteq < e >$. Now $e = a\alpha x \in < a > \Rightarrow < e > \subseteq < a >$. Therefore < a > = < e >. Therefore every principal Γ-ideal is generated by a Γ-idempotent.

THEOREM 4.19: In a semisimple Γ-semigroup S, the following are equivalent.

1) Every Γ-ideal of S is a prime Γ-ideal.
2) S is a primary Γ-semigroup.
3) S is a left primary Γ-semigroup.
4) S is a right primary Γ-semigroup.
5) S is a semiprimary Γ-semigroup.
6) Prime Γ-ideals of S form a chain.
7) Γ-ideals of S form a chain.
8) Principal Γ-ideals of S form a chain.
If in addition S is duo, then the above statements are equivalent to
9) Γ-idempotents of S form a chain under natural ordering.

***Proof* :** Since S is semisimple, $< x > = < x > \Gamma < x >$ for all $x \in$ S. Let A be any Γ-ideal of S. $x \in$ S, $< x > \Gamma < x > \subseteq$ A $\Rightarrow < x > \subseteq$ A $\Rightarrow x \in$ A. Therefore A is semiprime. Therefore every Γ-ideal of a semisimple Γ-semigroup is semiprime. By theorem 4.5, (1) to (5) are equivalent. By theorem 4.6, (5) and (6) are equivalent. By theorem 4.7, (5) implies (8). By theorem 4.11, (7) and (8) are equivalent.
(7) ⇒ (6) : Suppose that Γ-ideals of S form a chain. So prime Γ-ideals of S form a chain. Hence the conditions (1) to (8) are equivalent. If S is a duo Γ-semigroup, since S is semisimple, then by theorem 4.17, S regular and hence by theorem 4.18, every principal Γ-ideal is generated by a Γ-idempotent. So (8) and (9) are equivalent.

THEOREM 4.20: In a Γ-semigroup S, the following conditions are equivalent.
1) Every Γ-ideal of S is prime.
2) S is a semisimple primary Γ-semigroup.
3) S is a semisimple semiprimary Γ-semigroup.

***Proof* :** (1) ⇒ (2): Suppose that every Γ-ideal of S is prime. Let A is a Γ-ideal of S. Then A is a prime Γ-ideal of S. By theorem 2.36[3], A is a semiprime Γ-ideal and by theorem 4.5, A is a primary Γ-ideal. Therefore S is a primary Γ-semigroup. Let $a \in$ S. Since every Γ-ideal is prime. Therefore $< a > \Gamma < a > \subseteq < a >$, we have
$< a > \subseteq < a > \Gamma < a >$ and hence $< a > = < a > \Gamma < a >$. So a is semisimple. Therefore S is a semisimple primary Γ-semigroup.
(2) ⇒ (3) : Suppose that S is a semisimple primary Γ-semigroup. Since every primary Γ-semigroup is a semiprimary Γ-semigroup. Therefore S is a semisimple semiprimary Γ-semigroup.
(3) ⇒ (1) : Suppose that S is a semisimple semiprimary Γ-semigroup. By theorem 4.19, every Γ-ideal of S is prime.

COROLLARY 4.21: In a Γ-semigroup S, the following conditions are equivalent.
1) Every Γ-ideal of S is prime.
2) S is a semisimple Γ-semigroup.
3) Γ-ideals of S form a chain.

***Proof* :** (1) ⇒ (2) : Suppose that every Γ-ideal of S is prime. By theorem 4.20, S is a semisimple primary Γ-semigroup.
(2) ⇒ (3) : Suppose that S is a semisimple Γ-semigroup. By theorem 4.19, Γ-ideals of S form a chain.
(3) ⇒ (1) : Suppose that S is a semisimple Γ-semigroup and (prime) Γ-ideals of S form a chain. Then by theorem 4.19, every Γ-ideal of S is prime.

THEOREM 4.22: In a duo Γ-semigroup S, the following conditions are equivalent.

1) Every Γ-ideal of S is prime.
2) S is a regular primary Γ-semigroup.
3) S is a regular semiprimary Γ-semiprimary.
4) S is a regular Γ-semigroup and Γ-idempotents in S form a chain under natural ordering.

Proof **:** (1) ⇒ (2): Suppose that every Γ-ideal of S is prime. By theorem 4.20, S is semisimple primary Γ-semigroup and S is semisimple semiprimary Γ-semigroup. By theorem 4.17, S is regular. Therefore S is regular primary Γ-semigroup.
(2) ⇒ (3): Suppose that S is a regular primary Γ-semigroup. Since every primary Γ-semigroup is a semiprimary Γ-semigroup. Therefore S is a regular semiprimary Γ-semigroup.
(3) ⇒ (4): Suppose that S is regular semiprimary Γ-semigroup. By theorem 4.18, Γ-idempotents in S form a chain under natural ordering.
(4) ⇒ (1): Suppose that S is a regular Γ-semigroup and Γ-idempotents in S form a chain under natural ordering. Then by theorem 4.18, every Γ-ideal of S is prime.

DEFINITION 4.23: An element a of a Γ-semigroup S is said to be ***regular*** provided $a = a\alpha x\beta a$, for some $x \in S$, $\alpha, \beta \in \Gamma$. i.e, $a \in a\Gamma S\Gamma a$ [34].
DEFINITION 4.24: A Γ- semigroup S is said to be ***a regular Γ- semigroup*** provided every element is regular.

THEOREM 4.25: A Γ-semigroup S is regular Γ-semigroup if and only if every principal Γ-ideal is generated by an idempotent.

***Proof*:** Suppose that S is a regular Γ-semigroup. Let $< a >$ be a principal Γ-ideal of S. Since S is a regular Γ-semigroup, there exists $x \in S$, $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Let $a\alpha x = e$, since $a = a\alpha x\beta a = e\beta a \in < e >$ and $e = a\alpha x \in < a >$. Therefore $< a > = < e >$ and hence every principal Γ-ideal is generated by an idempotent.

Conversely, suppose that every principal Γ-ideal os S generated by an idempotent. Therefore $< a > = < e >$. Suppose $a \in < a > \subseteq a\Gamma S\Gamma a \Rightarrow a = a\alpha x\beta a$, for some $x \in S$, $\alpha, \beta \in \Gamma$.

DEFINITION 4.26: An element a of a Γ-semigroup S is said to be ***left regular*** provided $a = a\alpha a\beta x$, for some $x \in S$, $\alpha, \beta \in \Gamma$. i.e, $a \in a\Gamma a\Gamma S$.

DEFINITION 4.27: An element a of a Γ-semigroup S is said to be ***right regular*** provided $a = x\alpha a\beta a$, for some $x \in S$, $\alpha, \beta \in \Gamma$. i.e, $a \in S\Gamma a\Gamma a$.

DEFINITION 4.28: An element a of a Γ-semigroup S is said to be ***completely regular*** provided, there exists an element $x \in S$ such that $a = a\alpha x\beta a$ for $\alpha, \beta \in \Gamma$ and $a\alpha x = x\beta a$, for all $\gamma \in \Gamma$. i.e, $a \in a\Gamma x\Gamma a$ and $a\Gamma x = x\Gamma a$.

DEFINITION 4.29: A Γ- Semigroup S is said to be ***completely regular Γ- Semigroup*** provided every element is completely regular.

DEFINITION 4.30: Let S be a Γ-semigroup, $a \in S$ and $\alpha, \beta \in \Gamma$. An element $b \in S$ is said to be an **(α, β)*-inverse*** of a if $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$.

THEOREM 4.31: If a is a regular element if and only if a has an (α, β)-inverse.

Proof : Let a be a regular element then $a = a\alpha b\beta a$, for some $b \in S$, $\alpha, \beta \in \Gamma$. Therefore $a\alpha(b\beta a\alpha b)\beta a = (a\alpha b\beta a)\alpha b\beta a = a\alpha b\beta a = a$ and $(b\beta a\alpha b)\beta a\alpha\ (b\beta a\alpha b) = b\beta(a\alpha b\beta a)\alpha\ (b\beta a\alpha b) = b\beta a\alpha(b\beta a\alpha b) = b\beta(a\alpha b\beta a)\alpha b = b\beta a\alpha b$. Therefore $b\beta a\alpha b$ is the (α, β)-inverse of a.
Conversely suppose that b is an (α, β)-inverse of a. Then $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$. Therefore $a = a\alpha b\beta a$ and hence a is regular.

DEFINITION 4.32: An element a of Γ- semigroup S is said to be ***semisimple*** provided $a \in < a > \Gamma < a >$, that is, $< a > \Gamma < a > = < a >$.

DEFINITION 4.33: A Γ- semigroup S is said to be ***semisimple Γ- semigroup*** provided every element is a semisimple.

DEFINITION 4.34: An element a of a Γ- semigroup S is said to be ***intra regular*** provided $a = x\alpha a\beta a\gamma y$ for some $x,y \in S$ and $\alpha,\beta,\gamma \in \Gamma$.

EXAMPLE 4.35: Let S = {0, a, b}. then define the caylay table as in the example 1.4.32, S is intra regular Γ-semigroup.

THEOREM 4.36: If '*a*' is a completely regular element of a Γ- Semigroup S, then *a* is regular and semisimple.

Proof: Let a is completely regular of a Γ- Semigroup S. i.e. $a \in a\Gamma x\Gamma a$, $a\Gamma x = x\Gamma a$ for some $x \in S$, implies a is regular. Suppose a is regular, implies $a \in a\Gamma x\Gamma a \subseteq < a > \Gamma < a >$. Therefore a is semisimple.

THEOREM 4.37: Let S is a Γ-semigroup and *a* ∈ S. Then *a* is completely regular if and only if *a* is both left regular and right regular.

Proof **:** Suppose a is completely regular. Then $a \in a\Gamma S\Gamma a$ and $a\Gamma S = S\Gamma a$.
Now $a \in a\Gamma S\Gamma a = a\Gamma a\Gamma S$. Therefore a is left regular.
Also $a \in a\Gamma S\Gamma a = S\Gamma a\Gamma a$. Therefore a is right regular.
Conversely suppose that a is both left regular and right regular.
Then $a \in a\Gamma a\Gamma S$, $a \in S\Gamma a\Gamma a$. If $a \in a\Gamma a\Gamma S$, then $a = a\alpha a\beta x$ for some $x \in S$, $\alpha, \beta \in \Gamma$.
If $a \in S\Gamma a\Gamma a$, then $a = y\gamma a\delta a$ for some $y \in S$, $\gamma, \delta \in \Gamma$.
Now $a = a\alpha a\beta x = a\alpha(a)\beta x = a\alpha(y\gamma a\delta a)\beta x = a\alpha y\gamma(a\delta a\beta x) \Rightarrow a \in a\Gamma S\Gamma S = a\Gamma S$.
And $a = y\gamma a\delta a = y\gamma(a)\delta a = y\gamma(a\alpha a\beta x)\delta a = (y\gamma a\alpha a)\beta x\delta a \Rightarrow a \in S\Gamma S\Gamma a = S\Gamma a$.
Therefore $S = a\Gamma S = S\Gamma a$.
Now $a\alpha(y\beta a\beta x)\alpha a = (a\alpha y\beta a)\beta x\alpha a = a\beta x\alpha a = a$.
And $(y\beta a\beta x)\alpha a\alpha(y\beta a\beta x) = y\beta(a\beta x\alpha a)\alpha y\beta a\beta x = y\beta a\alpha y\beta a\beta x = y\beta(a\alpha y\beta a)\beta x = y\beta a\beta x$.
Also $a\alpha(y\beta a\beta x) = (a\alpha y\beta a)\beta x = a\beta x = y\beta a\alpha a\beta x = y\beta a = y\beta(a\beta x\alpha a) = (y\beta a\beta x)\alpha a$.
Therefore a is completely regular.

THEOREM 4.38: If '*a*' is a left regular element of a Γ- Semigroup S, then *a* is semisimple.

Proof: Suppose that a is left regular. Then $a \in a\Gamma a\Gamma x$ and hence $a \in < a > \Gamma < a >$. Therefore a is semisimple.

THEOREM 4.39: If '*a*' is a right regular element of a Γ- Semigroup S, then *a* is semisimple.

Proof**:** Suppose that a is right regular. Then $a \in a\Gamma a\Gamma x$ and hence $a \in < a > \Gamma < a >$. Therefore a is semisimple.

THEOREM 4.40: If '*a*' is a regular element of a Γ- Semigroup S, then *a* is semisimple.

Proof: Suppose that a is regular element of Γ-semigroup S. Then $a = a\alpha x\beta a$, for some $x \in S$, $\alpha, \beta \in \Gamma$ and hence $a \in \langle a \rangle \Gamma \langle a \rangle$. Therefore a is semisimple.

THEOREM 4.41: If '*a*' is a intra regular element of a Γ- Semigroup S, then *a* is semisimple.

***Proof*:** Suppose that a is intra regular. Then $a \in x\Gamma a\Gamma a\Gamma y$ for $x,y \in S$ and hence $a \in \langle a \rangle^2$ Therefore a is semisimple.

THEOREM 4.42: If S is a duo Γ-semigroup, then the following are equivalent for any element *a* ∈ S.

1) ***a* is completely regular.**
2) ***a* is regular.**
3) ***a* is left regular.**
4) ***a* is right regular.**
5) ***a* is intra regular.**
6) ***a* is semisimple.**

***Proof* :** Since S is duo Γ-senugroup, $a\Gamma S^1 = S^1\Gamma a$.
We have $a\Gamma S^1\Gamma a = a\Gamma a\Gamma S^1 = S^1\Gamma a\Gamma a = \langle a\Gamma a \rangle = \langle a \rangle \Gamma \langle a \rangle$.
(1) ⇒ (2): Suppose that a is completely regular. Then for some $x \in S$, $\alpha, \beta \in \Gamma$; $a = a\alpha x\beta a$ and $a\alpha x = x\alpha a$. Therefore a is regular.
(2) ⇒ (3): Suppose that a is regular. Then for some $x \in S$, $\alpha, \beta \in \Gamma$; $a = a\alpha x\beta a$.
Therefore $a \in a\Gamma S^1\Gamma a = a\Gamma a\Gamma S^1 \Rightarrow a = a\gamma a\delta y$ for some $y \in S^1$, $\gamma, \delta \in \Gamma$.
Therefore a is left regular.
(3) ⇒ (4): Suppose that a is left regular. Then for some $x \in S$, $\alpha, \beta \in \Gamma$; $a = a\alpha a\beta x$.
Therefore $a \in a\Gamma a\Gamma S^1 = S^1\Gamma a\Gamma a \Rightarrow a = y\gamma a\delta a$ for some $y \in S^1$, $\gamma, \delta \in \Gamma$.
Therefore a is right regular.
(4) ⇒ (5): Suppose that a is right regular. Then for some $x \in S$, $\alpha, \beta \in \Gamma$; $a = x\alpha a\beta a$.
Therefore $a \in S^1\Gamma a\Gamma a = \langle a \rangle \Gamma \langle a \rangle \Rightarrow a = x\alpha a\beta a\gamma y$ for some $x, y \in S^1$ and $\alpha, \beta, \gamma \in \Gamma$.
Therefore a is intra regular.
(5) ⇒ (6): Suppose that a is intra regular.
Then for some $x, y \in S^1$ and $\alpha, \beta, \gamma \in \Gamma$; $a = x\alpha a\beta a\gamma y$. Therefore $a \in \langle a \rangle \Gamma \langle a \rangle$. Therefore a is semisimple.
(6) ⇒ (1): Suppose that a is semisimple. Then $a \in \langle a \rangle \Gamma \langle a \rangle = a\Gamma a\Gamma S^1 = S^1\Gamma a\Gamma a$
$\Rightarrow a \in a\alpha a\beta x = y\gamma a\delta a$ for some $x, y \in S^1$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Therefore a is both left regular and right regular. By theorem 1.4.46, a is completely regular.

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