ISSN 2249 –5460

INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES, TECHNOLOGY AND HUMANITIES

www.internationalejournals.com

Available online at www.internationaleJournals.com

International eJournals

International Journal of Mathematical Sciences, Technology and Humanities 16 (2011) 153 – 173

NUMERICAL INTEGRATION OF GENERAL SINGULAR PERTURBATION PROBLEMS- TECHNIQUE 2

Loka Pavani[a] **and Y.N Reddy**[b]

[a] *Department of mathematics, Chaitanya Bharathi Institute of Technology, Hyderabad, 500 075, India*

[b] *Department of mathematics, National Institute of Technology, Warangal, 506 004, India*

Abstract

In this paper, a numerical integration method is presented for solving general singularly perturbed two-point boundary value problems. This method does not depend on asymptotic expansions. The main feature of this method is that it does not require a very fine mesh size. The original second order differential equation is replaced by an approximate first-order differential equation called 'neutral type' equation with a small deviating argument. Simpsons 1/3 rule is used to obtain a three term recurrence relation. Thomas Algorithm is used to solve the resulting tridiagonal algebraic system of equations. The proposed method is iterative on the deviating argument. The method is to be repeated for different choices of the deviating argument. Two linear and one non-linear examples with left-end boundary layer, one example each for right end boundary layer, internal layer and two layers are solved and the computational results are presented. It is observed that the present method approximates the exact solution very well.

Keywords: Singular perturbation problems, boundary layer, neutral type equations, deviating argument, Thomas Algorithm.

1 Introduction

Singular perturbation problems are of common occurrence in many branches of applied mathematics, such as fluid dynamics, elasticity, chemical reactor theory, aerodynamics, magneto hydrodynamics and plasma dynamics. A few notable examples are boundary layer problems, WKB problems, the modeling of steady and unsteady viscous flow problems with large Reynolds number, convective heat transport problems with large peclet numbers etc.

It is well known fact that the solutions of these problems exhibit a multi-scale character. That is there is /are thin layer(s) where the solution varies rapidly (non-uniformly), while away from the layer, the solution behaves regularly and varies slowly. Therefore, the numerical treatment of singularly perturbed boundary value problems gives major computational difficulties.

A wide variety of papers and books have been published in the recent years, describing various methods for solving singular perturbation problems, among these we mention Bender and Orszag[1], Els golts and Norkin [2], Hemker and Miller [3], Kevorkian and Cole [4], Nayfeh [5], O'Malley[6], Y.N.Reddy [7].

In this paper, a numerical integration method is presented for solving general singularly perturbed two-point boundary value problems. This method does not depend on asymptotic expansions. The main feature of this method is that it does not require a very fine mesh size. The original second order differential equation is replaced by an approximate first-order differential equation called 'neutral type' equation with a small deviating argument. Simpsons 1/3 rule is used to obtain a three term recurrence relation. Thomas Algorithm is used to solve the resulting tridiagonal algebraic system of equations. The proposed method is iterative on the deviating argument. The method is to be repeated for different choices of the deviating argument. Two linear and one non-linear examples with left-end boundary layer, one example each for right end boundary layer, internal layer and two layers are solved and the computational results are presented. It is observed that the present method approximates the exact solution very well.

2 Description of the method

To describe the method consider a class of singular perturbation problems of the form

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \;\; x \in [0,1] \tag{1}$$

$$with\; y(0) = \alpha \; and \;\; y(1) = \beta \tag{2}$$

where ε is a small positive parameter$(0 < \varepsilon \ll 1)$; α, β are given constants, $a(x)$,$b(x)$, $f(x)$ are assumed to be sufficiently continuously differentiable functions in $[0, 1]$, $b(x) \leq 0$ and $a(x) \geq M > 0$ on $[0,1]$ where M is some positive constant. Under these assumptions (1), (2) has a unique solution, $y(x)$ which in general displays a boundary layer of width $o(\varepsilon)$ at $x = 0$ for small values of ε.

We will now present a numerical technique for replacing the original second order differential equation (1) by an approximate first order differential equation with a small deviating argument and then solving it efficiently by using finite differences.

Let δ be a small positive deviating argument$(0 < \delta \ll 1)$. By Taylor's series expansion in the neighborhood of a point , we have

$$y'(x-\delta) = y'(x) - \delta y''(x) \tag{3}$$

$$y''(x) = \frac{1}{\delta}\big(y'(x) - y'(x-\delta)\big)$$

Substituting in (1)

$$\frac{\varepsilon}{\delta}\big(y'(x) - y'(x-\delta)\big) + a(x)y'(x) + b(x)y(x) = f(x)$$

$$\varepsilon y'(x) + \delta a(x)y'(x) - \varepsilon y'(x-\delta) + \delta b(x)y(x) = \delta f(x) \tag{4}$$

This transition from equation (1) to (4) is admissible because of the condition that δ is small$(0 < \delta \ll 1)$. We rearrange terms in (4) to get following equation called a differential equation of "neutral type" with a small deviating argument.

$$y'(x) = p(x)y'(x-\delta) + q(x)y(x) + r(x) \tag{5}$$

For $\delta \le x \le 1$ where

$$p(x) = \frac{\varepsilon}{\varepsilon + \delta a(x)} \tag{6}$$

$$q(x) = \frac{-\delta b(x)}{\varepsilon + \delta a(x)} \tag{7}$$

$$r(x) = \frac{\delta f(x)}{\varepsilon + \delta a(x)} \tag{8}$$

Theory and discussion on differential equations with a small deviating argument can be found in Elsgolts [3] , Bellman etal [1] and Reddy Y.N. [9]

Now divide the interval $[0,1]$ into N equal parts with mesh size h. i.e., $h = \frac{1}{N}$ and $x_i = ih$, $i = 0,1,\ldots.,N$ integrates (5) in $[x_i, x_{i+1}]$ $(i = 1,2,\ldots\ldots,N-1)$ we get

$$y(x_{i+1}) - y(x_i) = \int_{x_i}^{x_{i+1}} \Big(p(x)y'(x-\delta) + q(x)y(x) + r(x)\Big)\,dx$$

$$= p(x_{i+1})y(x_{i+1} - \delta) - p(x_i)y(x_i - \delta) + \int_{x_i}^{x_{i+1}} \big(-p'(x)y(x-\delta) + q(x)y(x) + r(x)\big)\,dx$$

We use Simpson's 1/3 rule for evaluating the integral approximately.

$$\begin{aligned} y(x_{i+1}) - y(x_i) &= p(x_{i+1})y(x_{i+1}-\delta) - p(x_i)y(x_i-\delta) \\ &+ \frac{h}{6}\Big[-p'(x_i)y(x_i-\delta) + q(x_i)y(x_i) + r(x_i) \\ &- 4p'\left(\frac{x_i + x_{i+1}}{2}\right) y\left(\frac{x_i + x_{i+1}}{2} - \delta\right) + 4q\left(\frac{x_i + x_{i+1}}{2}\right) y\left(\frac{x_i + x_{i+1}}{2}\right) \\ &+ 4r\left(\frac{x_i + x_{i+1}}{2}\right) - p'(x_{i+1})y(x_{i+1}-\delta) + q(x_{i+1})y(x_{i+1}) \\ &+ r(x_{i+1})\Big] \end{aligned} \tag{9}$$

By Taylors series we have

$$y(x_i - \delta) \approx y(x_i) - \delta\, y'(x_i)$$

$$y(x_i-\delta) = y(x_i) - \delta\left(\frac{y(x_i)-y(x_{i-1})}{h}\right)$$

$$y(x_i-\delta) = \left(1-\frac{\delta}{h}\right)y(x_i) + \frac{\delta}{h}y(x_{i-1}) \tag{10}$$

Also

$$y(x_{i+1}-\delta) \approx \left(1-\frac{\delta}{h}\right)y(x_{i+1}) + \frac{\delta}{h}y(x_i) \tag{11}$$

$$y\left(\frac{x_i+x_{i+1}}{2}-\delta\right) = y\left(x_{i+\frac{1}{2}}-\delta\right)$$

$$\approx y\left(x_{i+\frac{1}{2}}\right) - \delta y'\left(x_{i+\frac{1}{2}}\right)$$

$$= y\left(x_{i+\frac{1}{2}}\right) - \delta\left(\frac{y(x_{i+1})- y(x_i)}{h}\right)$$

$$= y\left(x_{i+\frac{1}{2}}\right) - \frac{\delta}{h}y(x_{i+1}) + \frac{\delta}{h}y(x_i) \tag{12}$$

From Hermite interpolation

$$y\left(x_{i+\frac{1}{2}}\right) = \frac{y(x_i)+ y(x_{i+1})}{2} + \frac{h}{8}[y'(x_i) - y'(x_{i+1})] + o(h^4) \tag{13}$$

From (5)

$$y'(x_{i+1}) = p(x_{i+1})y'(x_{i+1}-\delta) + q(x_{i+1}) + y(x_{i+1}) + r(x_{i+1}) \tag{14.a}$$

$$y'(x_i) = p(x_i)y'(x_i-\delta) + q(x_i) + y(x_i) + r(x_i) \tag{14.b}$$

We shall write$p(x_i) = p_i; q(x_i) = q_i; r(x_i) = r_i; y(x_i) = y_i; y'(x_i) = y'_i$ and so on

Also

$$y'(x_{i+1}-\delta) = \frac{y(x_{i+1}-\delta) - y(x_i-\delta)}{h}$$

$$= \frac{1}{h}\left[\left(1-\frac{\delta}{h}\right)y_{i+1} + \frac{\delta}{h}y_i - \left(1-\frac{\delta}{h}\right)y_i - \frac{\delta}{h}y_{i-1}\right]$$

$$= \frac{1}{h}\left[\left(1-\frac{\delta}{h}\right)y_{i+1} + (\frac{2\delta}{h}-1)y_i - \frac{\delta}{h}y_{i-1}\right] \tag{15.a}$$

Similarly

$$y'(x_i-\delta) = \frac{y(x_{i+1}-\delta) - y(x_i-\delta)}{h}$$

$$= \frac{1}{h}\left[\left(1-\frac{\delta}{h}\right)y_{i+1} + (\frac{2\delta}{h}-1)y_i - \frac{\delta}{h}y_{i-1}\right] \qquad (15.b)$$

Substitute $(15\ a)$, $(15\ b)$ in $(14\ )$

$$y'_i - y'_{i+1=}\frac{p_i}{h}\left[\left(1-\frac{\delta}{h}\right)y_{i+1} + -\left(1-\frac{2\delta}{h}\right)y_i - \frac{\delta}{h}y_{i-1}\right] - \frac{p_{i+1}}{h}\left[\left(1-\frac{\delta}{h}\right)y_{i+1} - \left(1-\frac{2\delta}{h}\right)y_i - \frac{\delta}{h}y_{i-1}\right] + q_i y_i - q_{i+1}y_{i+1} + r_i - r_{i+1}$$

Substituting in (13)

$$y_{i+\frac{1}{2}} = y_{i-1}\left(\frac{-\delta}{8h}(p_i - p_{i+1})\right) + y_i\left(\frac{1}{2}+\frac{1}{8}\left(\frac{2\delta}{h}-1\right)(p_i-p_{i+1}) + \frac{h}{8}\ q_i\right) + y_{i+1}\ \left(\frac{1}{2}+\frac{1}{8}\left(1-\frac{\delta}{h}\right)(p_i-p_{i+1}) - \frac{h}{8}\ q_{i+1}\right) + \frac{h}{8}(r_i - r_{i+1}\ ) \qquad (16)$$

Substituting in (9) and rearranging terms we get

$$\begin{aligned} & y_{i-1}\left[\frac{p_i\delta}{h} + \frac{p_i{}'\delta}{6} + \frac{\delta}{12}\ (p_{i+1}-p_i)(p'_{i+\frac{1}{2}} - q_{i+\frac{1}{2}})\right] \\ & - y_i\left[1 + \frac{p_{i+1}\delta}{h} - p_i\left(1-\frac{\delta}{h}\right) - \frac{h}{6}p'_i\left(1-\frac{\delta}{h}\right) + \frac{hq_i}{6} - \frac{2}{3}\delta p'_{i+\frac{1}{2}} - \frac{\delta}{6}p'_{i+1} - \frac{2h}{3}\left(p'_{i+\frac{1}{2}} - q_{i+\frac{1}{2}}\right)(\frac{1}{2}+\frac{hq_i}{8}+\frac{1}{8}(\frac{2\delta}{h}-1)(p_i-p_{i+1})\right] \\ & + y_{i+1}\ \left[1 - p_{i+1}\left(1-\frac{\delta}{h}\right) - \frac{2\delta}{3}p'_{i+\frac{1}{2}} - \frac{h}{6}q_{i+1} + \frac{h}{6}p'_{i+1}\left(1-\frac{\delta}{h}\right) + \frac{2h}{3}\left(p'_{i+\frac{1}{2}} - q_{i+\frac{1}{2}}\right)(\frac{1}{2} - \frac{hq_{i+1}}{8} + \frac{1}{8}(p_i-p_{i+1})(1-\frac{\delta}{h})\ )\right] \\ & = \frac{h}{6}(r_i + r_{i+1}) + \frac{2h}{3}r_{i+\frac{1}{2}} - \frac{h^2}{12}\left(\ p'_{i+\frac{1}{2}} - q_{i+\frac{1}{2}}\right)(r_i - r_{i+1}\ ) \end{aligned} \qquad (17)$$

This is now in the following three term recurrence relation of the form

$$A_i y_{i-1} - \ B_i y_i + C_i y_{i+1} = D_i\ ; \quad i = 1,2,\ldots.,N-1 \qquad (18)$$

Where

$$A_i = \frac{p_i\delta}{h} + \ \frac{p'_i\delta}{6} + \frac{\delta}{12}(\ p_{i+1} - p_i)\ \left(p'_{i+\frac{1}{2}} - q_{i+\frac{1}{2}}\right) \qquad (19)$$

$$B_i = 1 + \frac{p_{i+1}\delta}{h} - p_i\left(1-\frac{\delta}{h}\right) - \frac{h}{6}p'_i\left(1-\frac{\delta}{h}\right) + \frac{hq_i}{6} - \frac{2}{3}\delta p'_{i+\frac{1}{2}} - \frac{\delta}{6}p'_{i+1} - \frac{2h}{3}\left(p'_{i+\frac{1}{2}} - q_{i+\frac{1}{2}}\right)(\frac{1}{2}+\frac{hq_i}{8}+\frac{1}{8}(\frac{2\delta}{h}-1)(p_i-p_{i+1}) \qquad (20)$$

$$C_i = 1 - \left(1-\frac{\delta}{h}\right)p_{i+1} + \frac{h}{6}\left(1-\frac{\delta}{h}\right)p'_{i+1} - \frac{2\delta}{3}p'_{i+\frac{1}{2}} - \frac{h}{6}q_{i+1} + \frac{2h}{3}\left(p'_{i+\frac{1}{2}} - q_{i+\frac{1}{2}}\right)\left(\frac{1}{2} - \frac{h}{8}q_{i+1} + \frac{(p_i - p_{i+1})}{8}\left(1-\frac{\delta}{h}\right)\right) \quad (21)$$

$$D_i = \frac{h}{6}(r_i + r_{i+1}) + \frac{2h}{3}r_{i+\frac{1}{2}} + \frac{h^2}{12}\left(p'_{i+\frac{1}{2}} - q_{i+\frac{1}{2}}\right)(r_i - r_{i+1}) \quad (22)$$

(18) gives a system of $(N-1)$ equations with $(N+1)$ unknowns y_0 to y_N . The two given boundary conditions (2) together with these $(N-1)$ equations are sufficient to solve for y_0 to y_N. The solution is obtained using Thomas Algorithm. We repeat the numerical scheme for different choices of δ satisfying the condition $0 < \delta \ll 1$ until the solution profiles do not differ materially from iteration to iteration. From computational point of view, we use an absolute error criterion namely:

$$|y(x)^{m+1} - y(x)^m| \leq \sigma \qquad 0 \leq x \leq 1$$

where $y(x)^m$ is the solution for the m^{th} iterate of δ and σ is the prescribed tolerance bound.

3 Numerical examples

To demonstrate the applicability of the method , we apply it to four singular perturbation problems with left end boundary layer. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison.

Example 3.1: Consider the following singular perturbation problem from Bender and Orszag [2, pp.480; problem 9.17 with α=0]

$$\varepsilon y''(x) + y'(x) - y(x) = 0; \qquad x\epsilon[0,1]$$

with y(0)=1 and y(1)=1

The exact solution is given by $y(x) = \frac{[(e^{m_2}-1)e^{m_1 x} + (1-e^{m_1})e^{m_2 x}]}{[e^{m_2} - e^{m_1}]}$,

where $m_1 = \frac{-1+\sqrt{1+4\varepsilon}}{2\varepsilon}$, $m_2 = \frac{-1-\sqrt{1+4\varepsilon}}{2\varepsilon}$

The numerical results are given in **Table 1(a)** and **1(b)** for ε= 10^{-3} and $\varepsilon = 10^{-4}$ respectively.

Example 3.2: Now consider the following non homogeneous singular perturbation problem from fluid dynamics for fluid of small viscosity (Reinhardt [11],example 2)

$$\varepsilon y''(x) + y'(x) = 1 + 2x ; \qquad x\epsilon[0,1]$$

with $y(0) = 1$ and $y(1) = 1$.

The exact solution is given by $y(x) = x(x + 1 - 2\varepsilon) + \frac{(2\varepsilon-1)(1-e^{-x/\varepsilon})}{(1-e^{-1/\varepsilon})}$

The numerical results are given in **Table 2(a)** and **2(b)** for $\varepsilon=10^{-3}$ and $\varepsilon=10^{-4}$ respectively.

4 Non- linear problems

We use quasi-linearization process to linearise the non-linear singular perturbation problem. We apply our method to three problems.

Example 4.1: Consider the following example from Kevorkian and Cole[5,p. 56,Eq (2.5.1)];

$\varepsilon y''(x) + y(x)y'(x) - y(x) = 0 \qquad x\epsilon[0,1]$

with $y(0) = -1$ and $y(1) = 3.9995$.

We have chosen to use Kevorkian and Coles uniformly valid approximation [5, pp 57 and 58, Eqs.(2.5.5),(2.5.11),and (2.5.14)] for comparison.

$$y(x) = x + c_1 tanh \frac{c_1}{2}\left(\frac{x}{\varepsilon} + c_2\right)$$

where $c_1 = 2.9995$ and $c_2 = \frac{1}{c_{1_1}} \log(\frac{c_1-1}{c_2+1})$

For this example, also we have boundary layer of width $O(\varepsilon)$ at x=0 (cf Kevorkian and Cole [5]) .The numerical results are given in **Table 3(a)** and **3(b)** for $\varepsilon= 10^{-3}$ and $\varepsilon=10^{-4}$ respectively.

5 Right end boundary layer problems

We now describe the numerical integration method for solving problems with the boundary layer at the right end of the underlying internal. To be specific we consider the following singular perturbation problem

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x); \;\; x \in [0,1] \tag{23}$$
$$with\, y(0) = \alpha \; and \; y(1) = \beta \tag{24}$$

Where ε is a small positive parameter$(0 < \varepsilon \ll 1)$; α, β are given constants, (x) , $b(x)$ and $f(x)$ are assumed to be sufficiently continuously differentiable functions in $[0,1]$. We now assume that $a(x) \leq M < 0$ through out the interval $[0,1]$ where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 1$.

The evaluation of right end boundary layer for (23) and (24) is similar to that of left end boundary layer but these are some differences worth noting. Let δ be a small positive deviating argument$(0 < \delta \ll 1)$. By Taylor's series expansion in the neighborhood of point , we have

$$y'(x + \delta) = y'(x) + \delta y''(x) \tag{25}$$

Substituting in (23)

$$\frac{\varepsilon}{\delta}\big(y'(x + \delta) - y'(x)\big) + a(x)y'(x) + b(x)y(x) = f(x)$$

$$\varepsilon y'(x+\delta) - \varepsilon y'(x) + \delta a(x) y'(x) + \delta b(x) y(x) = \delta f(x) \tag{26}$$

This transition from equation (23) to (26) is admissible because of the condition that δ is small$(0 < \delta \ll 1)$. The following equation is called a differential equation of "neutral type" with a small deviating argument.

$$y'(x) = p(x) y'(x+\delta) + q(x) y(x) + r(x) \tag{27}$$

For $\delta \le x \le 1$ where

$$p(x) = \frac{-\varepsilon}{\delta a(x) - \varepsilon} \tag{28}$$

$$q(x) = \frac{-\delta b(x)}{\delta a(x) - \varepsilon} \tag{29}$$

$$r(x) = \frac{\delta f(x)}{\delta a(x) - \varepsilon} \tag{30}$$

We will now discuss the numerical scheme for solving (27). As usual now divide the interval [0,1] in to N equals parts with mesh size h; i.e. $h = \frac{1}{N}$ and $x_i = ih, i = 0,1, \dots, N$. Integrates (27) in $[x_i, x_{i+1}]$ $(i = 1,2, \dots, N-1)$. We get

$$y_i(x_i) - y(x_{i-1}) = \int_{x_{i-1}}^{x_i} \left(p(x) y'(x+\delta) + q(x) y(x) + r(x)\right) dx$$

$$= p(x_i) y(x_i + \delta) - p(x_{i-1}) y(x_{i-1} + \delta) + \int_{x_{i-1}}^{x_i} \left(-p'(x) y(x+\delta) + q(x) y(x) + r(x)\right) dx$$

We use Simpson's 1/3 rule for evaluating the integral approximately.

$$\begin{aligned} y(x_i) - y(x_{i-1}) &= p(x_i) y(x_i+\delta) - p(x_{i-1}) y(x_{i-1}+\delta) \\ &+ \frac{h}{6}\Big[-p'(x_{i-1}) y(x_{i-1}+\delta) + q(x_{i-1}) y(x_{i-1}) + r(x_{i-1}) \\ &- 4p'\left(\frac{x_i + x_{i-1}}{2}\right) y\left(\frac{x_i + x_{i-1}}{2} + \delta\right) \\ &+ 4q\left(\frac{x_{i-1} + x_i}{2}\right) y\left(\frac{x_{i-1} + x_i}{2}\right) + 4r\left(\frac{x_{i-1} + x_i}{2}\right) \\ &- p'(x_i) y(x_i+\delta) + q(x_i) y(x_i) + r(x_i)\Big] \end{aligned} \tag{31}$$

By Taylors series we have

$$y(x_{i-1}+\delta) \approx y(x_{i-1}) + \delta\, y'(x_{i-1})$$

$$= y(x_{i-1}) + \delta \left(\frac{y(x_i) - y(x_{i-1})}{h}\right)$$

$$= \left(1-\frac{\delta}{h}\right)y(x_{i-1})+\frac{\delta}{h}y(x_i) \tag{32}$$

Similarly

$$y(x_i+\delta) = \left(1-\frac{\delta}{h}\right)y(x_i)+\frac{\delta}{h}y(x_{i+1}) \tag{33}$$

Also

$$y\left(\frac{x_{i-1}+x_i}{2}+\delta\right) = y\left(x_{i-\frac{1}{2}}+\delta\right)$$

$$\approx y\left(x_{i-\frac{1}{2}}\right)+\delta y'\left(x_{i-\frac{1}{2}}\right)$$

$$= y\left(x_{i-\frac{1}{2}}\right)+\delta\left(\frac{y(x_i)-y(x_{i-1})}{h}\right)$$

$$= y\left(x_{i-\frac{1}{2}}\right)+\frac{\delta}{h}y(x_i)-\frac{\delta}{h}y(x_{i-1}) \tag{34}$$

From Hermite interpolation

$$y\left(x_{i-\frac{1}{2}}\right) = \frac{y(x_{i-1})+y(x_i)}{2}+\frac{h}{8}[y'(x_{i-1})-y'(x_i)]+o(h^4) \tag{35}$$

From (27)

$$y'(x_{i-1}) = p(x_{i-1})y'(x_{i-1}+\delta)+q(x_{i-1})y(x_{i-1})+r(x_{i-1}) \tag{36.a}$$

$$y'(x_i) = p(x_i)y'(x_i+\delta)+q(x_i)y(x_i)+r(x_i) \tag{36.b}$$

We shall write $p(x_i) = p_i; q(x_i) = q_i; r(x_i) = r_i; y(x_i) = y_i; y'(x_i) = y'_i$ and so on

Also

$$y'(x_{i-1}+\delta) = \frac{y(x_i+\delta)-y(x_{i-1}+\delta)}{h}$$

$$= \frac{1}{h}\left[\left(1-\frac{\delta}{h}\right)y_i+\frac{\delta}{h}y_{i+1}-\left(1-\frac{\delta}{h}\right)y_{i-1}-\frac{\delta}{h}y_i\right]$$

$$= \frac{1}{h}\left[\frac{\delta}{h}y_{i+1}+\left(1-\frac{2\delta}{h}\right)y_i-\left(1-\frac{\delta}{h}\right)\ y_{i-1}\right] \tag{37.a}$$

Similarly

$$y'(x_i+\delta) = \frac{y(x_i+\delta)-y(x_{i-1}+\delta)}{h}$$

$$= \frac{1}{h}\left[\frac{\delta}{h}y_{i+1}+\left(1-\frac{2\delta}{h}\right)y_i-\left(1-\frac{\delta}{h}\right)y_{i-1}\right] \tag{37.b}$$

Substitute $(37\ a)$, $(37\ b)$ in (36)

$$y'_{i-1} - y'_{i} = \frac{p_{i-1} - p_i}{h}\left[\frac{\delta}{h} y_{i+1} + \left(1 - \frac{2\delta}{h}\right) y_i - \left(1 - \frac{\delta}{h}\right) y_{i-1}\right] + q_{i-1} y_{i-1} - q_i y_i + r_{i-1} - r_i$$

Substituting in (35)

$$y_{i-\frac{1}{2}} = y_{i-1}\left(\frac{1}{2} - \frac{1}{8}(p_{i-1} - p_i)\left(1 - \frac{\delta}{h}\right) + \frac{h}{8} q_{i-1}\right) + y_i\left(\frac{1}{2} + \frac{1}{8}\left(1 - \frac{2\delta}{h}\right)(p_{i-1} - p_i) - \frac{h}{8} q_i\right) + y_{i+1}\left(\frac{\delta}{8h}(p_{i-1} - p_i)\right) + \frac{h}{8}(r_{i-1} - r_i) \quad (38)$$

Substituting in (31) and rearranging terms we get

$$y_{i-1}\left(-1 + p_{i-1}\left(1 - \frac{\delta}{h}\right) + \frac{h}{6} p'_{i-1}\left(1 - \frac{\delta}{h}\right) - \frac{h}{6} q_{i-1} - \frac{2\delta}{3} p'_{i-\frac{1}{2}} + \frac{2h}{3}\left(\frac{1}{2} - \frac{1}{8}(p_{i-1} - p_i)\left(1 - \frac{\delta}{h}\right) + \frac{h}{8} q_{i-1}\right)\left(p'_{i-\frac{1}{2}} - q_{i-\frac{1}{2}}\right)\right)$$
$$- y_i\left(-1 + p_i\left(1 - \frac{\delta}{h}\right) - \frac{p_{i-1}\delta}{h} - \frac{\delta}{6} p'_{i-1} - \frac{2\delta}{3} p'_{i-\frac{1}{2}} - \frac{2h}{3}\left(p'_{i-\frac{1}{2}} - q_{i-\frac{1}{2}}\right)\left(\frac{1}{2} + \frac{1}{8}(p_{i-1} - p_i)\left(1 - \frac{2\delta}{h}\right) - \frac{h q_i}{8}\right) - \frac{h}{6} p'_i\left(1 - \frac{\delta}{h}\right) + \frac{h}{6} q_i\right)$$
$$+ y_{i+1}\left(-\frac{p_i \delta}{h} + \frac{\delta}{12}(p_{i-1} - p_i)\left(p'_{i-\frac{1}{2}} - q_{i-\frac{1}{2}}\right) + \frac{p'_i \delta}{6}\right)$$
$$= \frac{h}{6}(r_{i-1} + r_i) + \frac{2h}{3} r_{i-\frac{1}{2}} - \frac{h^2}{12}\left(p'_{i-\frac{1}{2}} - q_{i-\frac{1}{2}}\right)(r_{i-1} - r_i) \quad (39)$$

This is now in the following three term recurrence relation of the form

$$A_i y_{i-1} - B_i y_i + C_i y_{i+1} = D_i; \quad i = 1,2,\ldots,N-1 \quad (40)$$

Where

$$A_i = -1 + p_{i-1}\left(1 - \frac{\delta}{h}\right) + \frac{h}{6} p'_{i-1}\left(1 - \frac{\delta}{h}\right) - \frac{h}{6} q_{i-1} - \frac{2\delta}{3} p'_{i-\frac{1}{2}} + \frac{2h}{3}\left(\frac{1}{2} - \frac{1}{8}(p_{i-1} - p_i)\left(1 - \frac{\delta}{h}\right) + \frac{h}{8} q_{i-1}\right)\left(p'_{i-\frac{1}{2}} - q_{i-\frac{1}{2}}\right) \quad (41)$$

$$B_i = -1 + p_i\left(1-\frac{\delta}{h}\right) - \frac{p_{i-1}\delta}{h} - \frac{\delta}{6}p'_{i-1} - \frac{2\delta}{3}p'_{i-\frac{1}{2}} - \frac{2h}{3}\left(p'_{i-\frac{1}{2}} - q_{i-\frac{1}{2}}\right)\left(\frac{1}{2}+\frac{1}{8}(p_{i-1}-p_i)\left(1-\frac{2\delta}{h}\right)-\frac{hq_i}{8}\right) - \frac{h}{6}p'_i\left(1-\frac{\delta}{h}\right)+\frac{h}{6}q_i \quad (42)$$

$$C_i = -\frac{p_i\delta}{h} + \frac{\delta}{12}(p_{i-1}-p_i)\left(p'_{i-\frac{1}{2}} - q_{i-\frac{1}{2}}\right) + \frac{p'_i\delta}{6} \quad (43)$$

$$D_i = \frac{h}{6}(r_{i-1}+r_i) + \frac{2h}{3}r_{i-\frac{1}{2}} - \frac{h^2}{12}\left(p'_{i-\frac{1}{2}} - q_{i-\frac{1}{2}}\right)(r_{i-1}-r_i) \quad (44)$$

(40) gives a system of $(N-1)$ equations with $(N+1)$ unknowns y_0 to y_N . The two given boundary conditions (24) together with these $(N-1)$ equations are sufficient to solve for y_0 to y_N. The solution is obtained using Thomas Algorithm. We repeat the scheme for various choices of δ until the solution profile does not differ materially from iteration to iteration.

6 Numerical example

Example 6.1: Consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) = 0; \quad x\epsilon[0,1]$$

with $y(0) = 1$ and $y(1) = 0$ .For this example, we have boundary layer at $x = 1$.

The exact solution is given by $y(x) = \frac{\left(e^{\frac{x-1}{\varepsilon}}-1\right)}{\left(e^{\frac{-1}{\varepsilon}}-1\right)}$.

The numerical results are given in **Table 4(a)** and **4(b)** for $\varepsilon=10^{-3}$ and $\varepsilon=10^{-4}$ respectively.

7 Internal layer problems

We will now discuss the singular perturbation problem with an internal layer of the underlying interval. In this case $a(x)$ changes sign in the domain of interest. Without loss of generality, we can take $a(0) = 0$ and the interval to be $[-1,1]$. With the help of one model example we demonstrate the applicability of the numerical integration method for solving singular perturbation problem with internal layer.

Example 7.1: Consider the following SPP

$$\varepsilon y'' + xy'(x) - y(x) = 0; \ x \in [-1,1] \quad (45)$$

$$with\ y(-1) = 1\ and\ y(1) = 2 \quad (46)$$

For this example we have $a(x) = x$, $b(x) = -1$ and $f(x) = 0$. Further we have an internal layer of width $o(\sqrt{\varepsilon})$ at $x = 0$ (for details, see O'Malley [7,pp168-172,eq 8.1 case (i)] and Kevorkian and Cole [5,pp 41-43,eqs (2.3.76) and (2.3.77)] we see that the function

$a(x) = x < 0$	$-1 \leq x < 0$
$a(x) = x = 0$	$x > 0$
$a(x) = x > 0$	$0 < x \leq 1$

By making use of transitions suggested for left end and right end boundary layer we replace (45) by following first-order differential equations with a small deviating argument.

$$y'(x) = p(x)y'(x+\delta) + q(x)y(x) + r(x) \tag{47}$$

for $-1 \leq x \leq -\delta$

where $p(x), q(x)$ and $r(x)$ are given by

$$p(x) = \frac{-\varepsilon}{\delta a(x) - \varepsilon} \tag{48}$$

$$q(x) = \frac{-\delta b(x)}{\delta a(x) - \varepsilon} \tag{49}$$

$$r(x) = \frac{\delta f(x)}{\delta a(x) - \varepsilon} \tag{50}$$

and

$$y'(x) = p(x)y'(x-\delta) + q(x)y(x) + r(x) \tag{51}$$

For $\delta \leq x \leq 1$ where

$$p(x) = \frac{\varepsilon}{\varepsilon + \delta a(x)} \tag{52}$$

$$q(x) = \frac{-\delta b(x)}{\varepsilon + \delta a(x)} \tag{53}$$

$$r(x) = \frac{\delta f(x)}{\varepsilon + \delta a(x)} \tag{54}$$

We now divide the interval $[-1,1]$ into N equal parts with mesh size h; i.e., $h = 2/N$ and $x_i = -1 + ih$ for $i = 0,1, \ldots N$. Let us denote $\frac{N}{2} = L$. Then integrating, using the Simpsons formula, equation (47) in $[x_{i-1}, x_i]$ for $i = 1,2, \ldots \ldots . . L-1$ and equation (51) in $[x_i, x_{i-1}]$ for $i = L+1, L+2, \ldots . N-1$ we get a system of $(N-2)$ equations with $(N+1)$ unknowns. From given boundary conditions we get two more equations.

$$y_0 = y(-1) = 1$$

$$y_N = y(1) = 2$$

We need one more equation to solve for the unknowns $(y_0, y_1, \ldots \ldots \ldots y_N)$. For this, we consider the original equation at $x = x_L = 0$. Since $a(x) = 0$ at $x = x_L = 0$ we get

$$\varepsilon y''(x_L) + b(x_L)y(x_L) = f(x_L) \tag{55}$$

By making use of the second order central finite difference approximation for the second order derivative in (55) we get

$$\frac{\varepsilon[y_{L-1} - 2y_L + y_{L+1}]}{h^2} + b(x_L)y_L(x_L) = f(x_L)$$

$$[\varepsilon]y_{L-1} - [2\varepsilon - h^2 b_L]y_L + [\varepsilon]y_{L+1} = h^2 f(x_L) \tag{56}$$

With this we have $(N+1)$ equations for $(N+1)$ unknowns. We can now solve by 'Thomas Algorithm'. We repeat the scheme for various choices of δ until the solution profile does not differ materially from iteration to iteration. The numerical results are presented in **Table 5(a)** and **5(b)** for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$ respectively.

8 Problems with two boundary layers

The suggestions given for internal layer problems apply mutatis mutandis to problems with two boundary layers. To illustrate this we will again consider the case where $a(x)$ changes sign in the domain of interest. Without loss of generality, we can take $a(0) = 0$and the interval to be $[-1,1]$. We demonstrate with an example.

Example 8.1: Consider the following SPP

$$\varepsilon y'' - xy'(x) - y(x) = 0; \;\; x \in [-1,1] \tag{57}$$

$$with\; y(-1) = 1 \; and \;\; y(1) = 2 \tag{58}$$

For this example we have $a(x) = -x$, $b(x) = 1$ and $f(x) = 0$. Further we have two boundary layers one at $x = -1$ and one at $x = 1$ (for details, see O'Malley [7, pp168-173,eq 8.1 case (i)] we see that the function

$$a(x) = -x > 0 \qquad -1 \le x < 0$$

$$a(x) = -x = 0 \qquad x = 0$$

$$a(x) = -x > 0 \qquad 0 < x \le 1$$

By making use of transitions suggested for left end and right end boundary layer we replace (57) by following first-order differential equations with a small deviating argument.

$$y'(x) = p(x)y'(x-\delta) + q(x)y(x) + r(x) \tag{59}$$

For $-1 + \delta \le x \le 0$

Where $p(x), q(x)$ and $r(x)$ are given by

$$p(x) = \frac{\varepsilon}{\delta a(x) + \varepsilon} \tag{60}$$

$$q(x) = \frac{-\delta b(x)}{\delta a(x) + \varepsilon} \tag{61}$$

$$r(x) = \frac{\delta f(x)}{\delta a(x) + \varepsilon} \tag{62}$$

and

$$y'(x) = p(x)y'(x+\delta) + q(x)y(x) + r(x) \tag{63}$$

for $0 \le x \le 1-\delta$ where

$$p(x) = \frac{-\varepsilon}{\delta a(x) - \varepsilon} \tag{64}$$

$$q(x) = \frac{-\delta b(x)}{\delta a(x) - \varepsilon} \tag{65}$$

$$r(x) = \frac{\delta f(x)}{\delta a(x) - \varepsilon} \tag{66}$$

We now divide the interval $[-1,1]$ in Nequal parts with mesh size h, i.e., $h = \frac{2}{N}$ and $x_i = -1 + ih$ for $i = 0,1,\dots.N$. Let us denote $\frac{N}{2} = L$ then integrating using the Simpsons one third formula equation (59) in $[x_i, x_{i+1}]$ for $i = 1,2,\dots.L-1$ and equation (63) in $[x_{i-1}, x_i]$ for $i = L+1, L+2, \dots.N-1$ we get a system of $(N-2)$ equations with $(N+1)$ unknowns. From given boundary conditions we get two more equations.

$$y_0 = y(-1) = 1$$

$$y_N = y(1) = 2$$

We need one more equation to solve for the unknowns $(y_0, y_1, \dots\dots\dots y_N)$. For this, we consider the original equation at $x = x_L = 0$. Since $a(x) = 0$ at $x = x_L = 0$ we get

$$\varepsilon y''(x_L) + b(x_L)y(x_L) = f(x_L) \tag{67}$$

By making use of the second order central finite difference approximation for the second order derivative in (67) we get

$$\frac{\varepsilon[y_{L-1} - 2y_L + y_{L+1}]}{h^2} + b(x_L)y_L(x_L) = f(x_L)$$

$$[\varepsilon]y_{L-1} - [2\varepsilon - h^2 b_L]y_L + [\varepsilon]y_{L+1} = h^2 f(x_L) \tag{68}$$

With this we have $(N+1)$ equations for $(N+1)$ unknowns. We can now solve by 'Thomas Algorithm' for different choices of δ $(0 < \delta \ll 1)$.The numerical results are presented in **Table 6(a)** and **6(b)** for $\varepsilon = 10^{-3}$and $\varepsilon = 10^{-4}$ respectively.

9 Discussion and conclusions

We have presented the numerical integration method (technique 2) for general singularly perturbed two point boundary value problems. We have implemented this method on three linear problems; three non-linear problems, one with right layer, and one each with internal and two layers by taking different values of ε. The process is repeated for different choices of δ until the solution profile stabilizes. The choice of δ is not unique but can assume any value satisfying the condition $0 < \delta \ll 1$. To reduce computational time we fix the mesh size h and vary δ. The numerical results are presented in tables. They are compared with the exact solution. It is observed that the present method approximates the exact solution very well. It appears to be one of the best choices for solving singular perturbation problem with less computational time.

International Journal of Mathematical Sciences, Technology and Humanities 10 (2011) 153 – 173
Loka Pavani and Y.N Reddy

Table 1 (a): computational results for **Example 3.1** with $\varepsilon = 10^{-3}$ and $h = 0.01$

x	$y(x)$			Exact Solution
	$\delta = 0.001$	$\delta = 0.005$	$\delta = 0.01$	
0.000	1.0000000	1.0000000	1.0000000	1.0000000
0.020	0.3808455	0.3808392	0.3808392	0.3756778
0.040	0.3833047	0.3832984	0.3832984	0.3832590
0.060	0.3909974	0.3909911	0.3909911	0.3909939
0.080	0.3988877	0.3988814	0.3988814	0.3988845
0.100	0.4069376	0.4069313	0.4069313	0.4069344
0.200	0.4496904	0.4496841	0.4496841	0.4496873
0.300	0.4969348	0.4969288	0.4969288	0.4969318
0.400	0.5491427	0.5491370	0.5491370	0.5491398
0.500	0.6068356	0.6068303	0.6068303	0.6068329
0.600	0.6705897	0.6705850	0.6705850	0.6705873
0.700	0.7410418	0.7410378	0.7410378	0.7410397
0.800	0.8188955	0.8188925	0.8188925	0.8188939
0.900	0.9049284	0.9049268	0.9049268	0.9049276
1.000	1.0000000	1.0000000	1.0000000	1.0000000

Table 1(b): computational results for **Example 3.1** for $\varepsilon = 10^{-4}$ and $h = 0.01$

x	$y(x)$			Exact Solution
	$\delta = 0.0001$	$\delta = 0.0005$	$\delta = 0.001$	
0.000	1.0000000	1.0000000	1.0000000	1.0000000
0.020	0.3754103	0.3754079	0.3754079	0.3753323
0.040	0.3829307	0.3829283	0.3829283	0.3829141
0.060	0.3906656	0.3906632	0.3906632	0.3906490
0.080	0.3985568	0.3985544	0.3985544	0.3985401
0.100	0.4066073	0.4066049	0.4066049	0.4065907
0.200	0.4493660	0.4493636	0.4493636	0.4493497
0.300	0.4966211	0.4966188	0.4966188	0.4966053
0.400	0.5488455	0.5488433	0.5488433	0.5488306
0.500	0.6065618	0.6065599	0.6065599	0.6065481
0.600	0.6703476	0.6703458	0.6703458	0.6703355
0.700	0.7408410	0.7408397	0.7408397	0.7408310
0.800	0.8187476	0.8187466	0.8187466	0.8187402
0.900	0.9048468	0.9048462	0.9048462	0.9048426
1.000	1.0000000	1.0000000	1.0000000	1.0000000

International Journal of Mathematical Sciences, Technology and Humanities 10 (2011) 153 – 173
Loka Pavani and Y.N Reddy

Table 2 (a): computational results for **Example 3.2** with $\varepsilon = 10^{-3}$ and $h = 0.01$

x	$y(x)$			Exact Solution
	$\delta = 0.001$	$\delta = 0.005$	$\delta = 0.01$	
0.000	0.0000000	0.0000000	0.0000000	0.0000000
0.020	-0.9693931	-0.9693914	-0.9693914	-0.9776400
0.040	-0.9564127	-0.9564111	-0.9564111	-0.9564111
0.060	-0.9345200	-0.9345187	-0.9345187	-0.9345187
0.080	-0.9117604	-0.9117592	-0.9117592	-0.9117592
0.100	-0.8882002	-0.8881992	-0.8881992	-0.8881992
0.200	-0.7583990	-0.7583992	-0.7583992	-0.7583992
0.300	-0.6085985	-0.6085993	-0.6085993	-0.6085993
0.400	-0.4387979	-0.4387993	-0.4387993	-0.4387993
0.500	-0.2489976	-0.2489994	-0.2489994	-0.2489994
0.600	-0.0391975	-0.0391995	-0.0391995	-0.0391995
0.700	0.1906024	0.1906004	0.1906004	0.1906004
0.800	0.4404019	0.4404003	0.4404003	0.4404003
0.900	0.7102012	0.7102002	0.7102002	0.7102002
1.000	1.0000000	1.0000000	1.0000000	1.0000000

Table 2(b): computational results for **Example 3.2** with $\varepsilon = 10^{-4}$ and $h = 0.01$

x	$y(x)$			Exact Solution
	$\delta = 0.0001$	$\delta = 0.0005$	$\delta = 0.001$	
0.000	0.0000000	0.0000000	0.0000000	0.0000000
0.020	-0.9798795	-0.9798795	-0.9798795	-0.9798795
0.040	0.9582072	0.9582072	0.9582072	0.9582072
0.060	-0.9362112	-0.9362112	-0.9362112	-0.9362112
0.080	-0.9134152	-0.9134152	-0.9134152	-0.9134152
0.100	-0.8898192	-0.8898192	-0.8898192	-0.8898192
0.200	-0.7598393	-0.7598393	-0.7598393	-0.7598393
0.300	-0.6098593	-0.6098593	-0.6098593	-0.6098593
0.400	-0.4398794	-0.4398794	-0.4398794	-0.4398794
0.500	-0.2498994	-0.2498994	-0.2498994	-0.2498994
0.600	-0.0399195	-0.0399195	-0.0399195	-0.0399195
0.700	0.1900604	0.1900604	0.1900604	0.1900604
0.800	0.4400403	0.4400403	0.4400403	0.4400403
0.900	0.7100202	0.7100202	0.7100202	0.7100202
1.000	1.0000000	1.0000000	1.0000000	1.0000000

International Journal of Mathematical Sciences, Technology and Humanities 10 (2011) 153 – 173
Loka Pavani and Y.N Reddy

Table 3 (a): computational results for **Example 4.1** for $\varepsilon = 10^{-3}$ and $h = 0.01$

x	$y(x)$			Exact Solution
	$\delta = 0.002$	$\delta = 0.005$	$\delta = 0.01$	
0.00	-1.0000000	-1.0000000	-1.0000000	-1.0000000
0.02	3.0195081	3.0182605	3.0174482	3.0195000
0.04	3.0457075	3.0427003	3.0412421	3.0395000
0.06	3.0654664	3.0625739	3.0611746	3.0595000
0.08	3.0852230	3.0824459	3.0811040	3.0795000
0.10	3.1049836	3.1023202	3.1010354	3.0994999
0.20	3.2038591	3.2017312	3.2007146	3.0995001
0.40	3.4019752	3.4007554	3.4001825	3.3994999
0.60	3.6006048	3.6000557	3.5998023	3.5994999
0.80	3.7997720	3.7996356	3.7995741	3.7995000
1.00	3.9995000	3.9995000	3.9995000	3.9995000

Table 3(b): computational results for **Example 4.1** with $\varepsilon = 10^{-4}$ and $h = 0.01$

x	$y(x)$			Exact Solution
	$\delta = 0.0002$	$\delta = 0.0005$	$\delta = 0.001$	
0.00	-1.0000000	-1.0000000	-1.0000000	-1.0000000
0.02	3.0255888	3.0226586	3.0212131	3.0195000
0.04	3.0457203	3.0427110	3.0412514	3.0395000
0.06	3.0654716	3.0625796	3.0611794	3.0595000
0.08	3.0852277	3.0824506	3.0811088	3.0795000
0.10	3.1049883	3.1023247	3.1010394	3.0994999
0.20	3.2038627	3.2017341	3.2007163	3.0995001
0.40	3.4019766	3.4007571	3.4001822	3.3994999
0.60	3.6006055	3.6000564	3.5995015	3.5994999
0.80	3.7997735	3.7996359	3.7995739	3.7995000
1.00	3.9995000	3.9995000	3.9995000	3.9995000

Table 4 (a): computational results for **Example 6.1** with $\varepsilon = 10^{-3}$ and $h = 0.01$

x	$y(x)$			Exact Solution
	$\delta = 0.002$	$\delta = 0.005$	$\delta = 0.01$	
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.10	1.0000000	1.0000002	0.9999997	1.0000000
0.20	1.0000000	1.0000002	0.9999997	1.0000000
0.30	1.0000000	1.0000002	0.9999997	1.0000000
0.40	1.0000000	1.0000002	0.9999997	1.0000000
0.50	1.0000000	1.0000002	0.9999997	1.0000000
0.60	1.0000000	1.0000002	0.9999997	1.0000000
0.70	1.0000000	1.0000002	0.9999997	1.0000000
0.80	1.0000000	1.0000002	0.9999997	1.0000000
0.90	1.0000000	1.0000002	0.9999997	1.0000000
0.92	1.0000000	1.0000002	0.9999997	1.0000000
0.94	0.9999995	0.9999996	0.9999992	1.0000000
1.00	0.0000000	0.0000000	0.0000000	0.0000000

Table 4(b): computational results for **Example 6.1** with $\varepsilon = 10^{-4}$ and $h = 0.01$

x	$y(x)$			Exact Solution
	$\delta = 0.0002$	$\delta = 0.0005$	$\delta = 0.001$	
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.10	1.0000001	1.0000001	1.0000001	1.0000000
0.20	1.0000001	1.0000001	1.0000001	1.0000000
0.30	1.0000001	1.0000001	1.0000001	1.0000000
0.40	1.0000001	1.0000001	1.0000001	1.0000000
0.50	1.0000001	1.0000001	1.0000001	1.0000000
0.60	1.0000001	1.0000001	1.0000001	1.0000000
0.70	1.0000001	1.0000001	1.0000001	1.0000000
0.80	1.0000001	1.0000001	1.0000001	1.0000000
0.90	1.0000001	1.0000001	1.0000001	1.0000000
0.92	1.0000001	1.0000001	1.0000001	1.0000000
0.94	1.0000001	1.0000001	1.0000001	1.0000000
1.00	0.0000000	0.0000000	0.0000000	0.0000000

Table 5 (a): computational results for **Example 7.1** with $\varepsilon = 10^{-3}$ and $h = 0.01$

x	$y(x)$			
	$\delta = 0.007$	$\delta = 0.008$	$\delta = 0.009$	$\delta = 0.01$
-1.00	1.0000000	1.0000000	1.0000000	1.0000000
-0.50	0.0025122	0.0016190	0.0010595	0.0070530
-0.10	0.0003587	0.0003146	0.0002784	0.0002474
-0.06	0.0086968	0.0081033	0.0075458	0.0070277
-0.04	0.0329158	0.0312147	0.0295829	0.0280422
-0.02	0.1014188	0.0975045	0.0936900	0.0900513
0.00	0.2508561	0.2434001	0.2356640	0.2284435
0.02	0.4578396	0.4452446	0.4328584	0.4210136
0.04	0.5846501	0.5702885	0.5563590	0.5431778
0.06	0.6430109	0.6290327	0.6157150	0.6032785
0.10	0.6831617	0.6718040	0.6613125	0.6517375
0.50	1.1570197	1.1553547	1.1538976	1.1526208
1.00	2.0000000	2.0000000	2.0000000	2.0000000

Table 5 (b): computational results for **Example 7.1** with $\varepsilon = 10^{-4}$ and $h = 0.01$

x	$y(x)$			
	$\delta = 0.0007$	$\delta = 0.0008$	$\delta = 0.0009$	$\delta = 0.001$
-1.00	1.0000000	1.0000000	1.0000000	1.0000000
-0.50	0.0002377	0.0001487	0.0000929	0.0000581
-0.10	0.0000008	0.0000003	0.0000001	0.0000001
-0.06	0.0000104	0.0000096	0.0000090	0.0000084
-0.04	0.0005205	0.0004973	0.0004742	0.0004521
-0.02	0.0153355	0.0148465	0.0143454	0.0138531
0.00	0.2124776	0.2074935	0.2022241	0.1969613
0.02	0.6816692	0.6655294	0.6490229	0.6328771
0.04	0.6783371	0.6627832	0.6476966	0.6334481
0.06	0.6671236	0.6533850	0.6404150	0.6283963
0.10	0.6823759	0.6715323	0.6615660	0.6525509
0.50	1.1567472	1.1551044	1.1536657	1.1524048
1.00	2.0000000	2.0000000	2.0000000	2.0000000

International Journal of Mathematical Sciences, Technology and Humanities 10 (2011) 153 – 173
Loka Pavani and Y.N Reddy

Table 6 (a): computational results for **Example 8.1** with $\varepsilon = 10^{-3}$ and $h = 0.01$

x	$y(x)$			
	$\delta = 0.002$	$\delta = 0.005$	$\delta = 0.008$	$\delta = 0.01$
-1.000	1.0000000	1.0000000	1.0000000	1.0000000
-0.980	0.0085510	0.0087472	0.0090601	0.0093280
-0.960	0.0000759	0.0000794	0.0000852	0.0000903
-0.940	0.0000007	0.0000007	0.0000001	0.0000009
-0.900	0.0000000	0.0000000	0.0000000	0.0000000
-0.500	0.0000000	0.0000000	0.0000000	0.0000000
0.000	0.0000000	0.0000000	0.0000000	0.0000000
0.500	0.0000000	0.0000000	0.0000000	0.0000000
0.900	0.0000000	0.0000000	0.0000000	0.0000000
0.940	0.0000014	0.0000014	0.0000014	0.0000013
0.960	0.0001494	0.0001486	0.0001482	0.0001481
0.980	0.0169659	0.0169206	0.0169019	0.0168947
1.000	2.0000000	2.0000000	2.0000000	2.0000000

Table 6 (b): computational results for **Example 8.1** with $\varepsilon = 10^{-4}$ and $h = 0.01$

x	$y(x)$			
	$\delta = 0.0002$	$\delta = 0.0005$	$\delta = 0.0008$	$\delta = 0.001$
-1.000	1.0000000	1.0000000	1.0000000	1.0000000
-0.980	0.0001015	0.0010212	0.0001027	0.0001030
-0.960	0.0000000	0.0000000	0.0000000	0.0000000
-0.940	0.0000000	0.0000000	0.0000000	0.0000000
-0.900	0.0000000	0.0000000	0.0000000	0.0000000
-0.500	0.0000000	0.0000000	0.0000000	0.0000000
0.000	0.0000000	0.0000000	0.0000000	0.0000000
0.500	0.0000000	0.0000000	0.0000000	0.0000000
0.900	0.0000000	0.0000000	0.0000000	0.0000000
0.940	0.0000000	0.0000000	0.0000000	0.0000000
0.960	0.0000000	0.0000000	0.0000000	0.0000000
0.980	0.0002084	0.0002053	0.0002039	0.0002033
1.000	2.0000000	2.0000000	2.0000000	2.0000000

REFERENCE

[1] **Bellman R.** Perturbation Techniques in Mathematics,Physics and Engineering [Book]. - New York : Holt,Rinehart,Winston, 1964.

[2] **Bender C. M. and Orszag S.A.** Advanced Mathematical Methods for Scientists and Engineers [Book]. - New York : McGraw-Hill, 1978.

[3] **El'sgol'ts L.E. and Norkin S.B.** Introduction to the Theory and Application of Differential Equations with Deviating Arguments [Book]. - New York : Academic Press, 1973.

[4] **Hemker P.W and Miller J.J.H** Numerical Analysis of Singular Perturbation Problems [Book]. - New York : Academic Press, 1979.

[5] **Kevorkian J. and ColeJ.D. and** Perturbation Methods in Applied Mathematics [Book]. - New York : Sringer Verlag, 1981.

[6] **Nayfeh A.H** Perturbation Methods [Book]. - New York : Wiley, 1979.

[7] **O'Malley R.E.** Introduction to Singular Perturbations [Book]. - New York : Academic Press, 1974.

[8] **Reddy Y.N.** A Note on the Numerical Integration Method for solving Singular Perturbation Problems [Journal] // Applied Mathematics and Computation. - 1991. - 2 : Vol. 43.

[9] **Reddy Y.N.** A Numerical Integration Method for solving Singular Perturbation Problems [Journal] // Applied Mathematics and Computation. - 1990. - Vol. 37. - pp. 83-95.

[10] **Reddy Y.N.** Numerical Treatment of Singularly Perturbed two-point boundary value problems [Report] : Ph.D Thesis / IIT. - Kanpur : [s.n.], 1986.

[11] **Reinhardt H.J.** Singular Perturbations of difference method for linear ordinary differential equations [Journal] // Applicable Anal.. - 1980. - Vol. 10. - pp. 53-70.