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STABILITY ANALYSIS OF A TYPICAL THREE SPECIES SYN-ECO-SYSTEM WITH UNLIMITED RESOURCES

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ABSTRACT: The present investigation is an analytical study of three species (S_1, S_2, S_3) syn-eco system with unlimited resources. The system comprises of a commensal (S_1), two hosts S_2 and S_3 ie, S_2 and S_3 both benefit S_1, without getting themselves affected either positively or adversely. The model equations constitute a set of three first order non-linear simultaneous coupled differential equations. All possible equilibrium points of the model are identified. The system would be stable, if all the characteristic roots are negative, in case they are real and have negative real parts, in case they are complex. Trajectories of the perturbations over the equilibrium states are illustrated and the numerical solutions for the growth rate equations are computed using Runge-Kutta fourth order scheme.

Keywords: Commensal, Equilibrium state, Host, Trajectories, Unstable.

AMS Classification: 92D25, 92D40

1. INTRODUCTION

Ecology is a branch of life and environment sciences which asserts the existence of diverse species in the same environment and habitat. It is natural that two or more species living in a common habitat interact in different ways. The Ecological interactions can be broadly classified as Ammensalism, Competition, Commensalism, Mutualism, Predation, Parasitism and so on. Lotka[8] and Volterra [21] pioneered theoretical ecology significantly and opened new eras in the field of life and biological sciences. The general concepts of modeling have been discussed by several authors, viz., [3, 4, 5, 7, 9]. Srinivas [20] studied competitive ecosystem of two species and three species with limited and unlimited resources. Later, Laxminarayan et al [10] studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Stability analysis of competitive species was carried out by [17, 19], while Ravindra Reddy [18] investigated mutualism between two species. Acharyulu et al [1, 2] derived some productive results on various mathematical models of ecological Ammensalism with multifarious resources in the manifold directions. Further, Kumar [6] studied some mathematical models of ecological commensalism. The present author Prasad et al [11-16] investigated on the stability of a three and four species syn-ecosystems.

2. NOTATION ADOPTED

$N_i(t)$	:	The population strength of S_i at time t, $i=1,2,3$
t	:	Time instant
a_i	:	Natural growth rate of S_i, $i=1,2,3$
a_{ii}	:	Self inhibition coefficients of S_i, $i=1,2,3$
a_{12}, a_{13}	:	Interaction coefficients of S_1 due to S_2 and S_1 due to S_3
a_{23}	:	Interaction coefficient of S_2 due to S_3
$k_i = \frac{a_i}{a_{ii}}$	:	Carrying capacities of S_i, $i=1,2,3$

Further the variables N_1, N_2, N_3 are non-negative and the model parameters $a_1, a_2, a_3, a_{11}, a_{12}$, $a_{22}, a_{33}, a_{13}, a_{23}, k_1, k_2, k_3$ are assumed to be non-negative constants.

3. THE FIRST SPECIES WITH UNLIMITED RESOURCES

The model equations for the growth rates of S_1, S_2, S_3 are

$$\frac{dN_1}{dt} = N_1\left(a_1 + a_{12}N_2 + a_{13}N_3\right) \tag{1}$$

$$\frac{dN_2}{dt} = N_2\left(a_2 - a_{22}N_2 + a_{23}N_3\right) \tag{2}$$

$$\frac{dN_3}{dt} = N_3\left(a_3 - a_{33}N_3\right) \tag{3}$$

3.1 Equilibrium States

The system under investigation has four equilibrium states given by $\frac{dN_i}{dt} = 0, i = 1, 2, 3$

$$E_1 : \bar{N}_1 = 0,\ \bar{N}_2 = 0,\ \bar{N}_3 = 0$$

$$E_2 : \bar{N}_1 = 0,\ \bar{N}_2 = 0,\ \bar{N}_3 = k_3$$

$$E_3 : \bar{N}_1 = 0,\ \bar{N}_2 = k_2,\ \bar{N}_3 = 0$$

$$E_4 : \bar{N}_1 = 0,\ \bar{N}_2 = k_2 + \frac{a_{23}k_3}{a_{22}},\ \bar{N}_3 = k_3$$

3.2 Stability Analysis

3.2.1 Stability of $E_1 : \bar{N}_1 = 0,\ \bar{N}_2 = 0,\ \bar{N}_3 = 0$

Let us consider small deviations from the steady state

i.e., $N_i(t) = \bar{N}_i + u_i(t), i = 1,2,3$ (4)

where $u_i(t)$ is a small perturbations in the species S_i.

The basic equations (1), (2) and (3) are quasi-linearized to obtain the equations for the perturbed state as,

$$\frac{du_i}{dt} = a_i u_i, i = 1,2,3 \tag{5}$$

The characteristic equation is $(\lambda - a_1)(\lambda - a_2)(\lambda - a_3) = 0$ (6)

The characteristic roots of (6) are $a_1, a_2, -d_3$. Since all the three are positive. Hence the state is **unstable** and the solutions of the equations (5) are

$$u_i = u_{i0}\, e^{a_i t},\ i = 1, 2, 3 \tag{7}$$

where u_{10}, u_{20}, u_{30} are the initial values of u_1, u_2, u_3 respectively.

Trajectories of perturbations

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are $\left(\frac{u_1}{u_{10}}\right)^{\frac{1}{a_1}} = \left(\frac{u_2}{u_{20}}\right)^{\frac{1}{a_2}} = \left(\frac{u_3}{u_{30}}\right)^{\frac{1}{a_3}}$ (8)

3.2.2 Stability of $E_2 : \bar{N}_1 = 0,\ \bar{N}_2 = 0,\ \bar{N}_3 = k_3$

In this state, the basic equations can be quasi-linearized,
We get

$$\frac{du_1}{dt}=(a_1+a_{13}k_3)u_1;\ \frac{du_2}{dt}=(a_2+a_{23}k_3)u_2;\ \frac{du_3}{dt}=-a_3u_3 \tag{9}$$

The characteristic roots are $a_1+a_{13}k_3$, $a_2+a_{23}k_3$ and $-a_3$. Since two of these three roots are positive, hence the state is **unstable** and the solutions are

$$u_1=u_{10}e^{(a_1+a_{13}k_3)\,t};u_2=u_{20}\ e^{(a_2+a_{23}e_3)\,t};u_3=u_{30}e^{-a_3t} \tag{10}$$

Trajectories of perturbations

The trajectories in the u_1-u_2 and u_2-u_3 planes are given by

$$\left(\frac{u_1}{u_{10}}\right)^{\frac{1}{a_1+a_{13}k_3}}=\left(\frac{u_2}{u_{20}}\right)^{\frac{1}{a_2+a_{23}k_3}}=\left(\frac{u_3}{u_{30}}\right)^{-\frac{1}{a_3}} \tag{11}$$

3.2.3 Stability of $E_3:\bar{N}_1=0,\bar{N}_2=k_2,\bar{N}_3=0$

The basic equations can be quasi-linearized,
We get

$$\frac{du_1}{dt}=(a_1+a_{12}k_2)u_1;\ \frac{du_2}{dt}=-a_2u_2+a_{23}k_2u_3;\ \frac{du_3}{dt}=a_3u_3 \tag{12}$$

The characteristic roots are $a_1+a_{12}k_2$, $-a_2$ and a_3. Since two of these three roots are positive, hence the state is **unstable**. The equations (12) yield the solutions,

$$u_1=u_{10}e^{(a_1+a_{12}k_2)\,t};u_2=\left(u_{20}-A\right)e^{-a_2t}+Ae^{a_3t};u_3=u_{30}e^{a_3t} \tag{13}$$

where $A=\dfrac{a_{23}k_2u_{30}}{a_3-a_2}$ (14)

Trajectories of perturbations

The trajectories in the u_1-u_2 and u_2-u_3 planes are

$$u_2=\left(u_{20}-A\right)\left(\frac{u_1}{u_{10}}\right)^{\frac{-a_2}{a_1+a_{12}k_2}}+A\left(\frac{u_1}{u_{10}}\right)^{\frac{a_3}{a_1+a_{12}k_2}};\ u_2=\left(u_{20}-A\right)\left(\frac{u_3}{u_{30}}\right)^{\frac{-a_2}{a_3}}+\frac{Au_3}{u_{30}} \tag{15}$$

3.2.4 Stability of $E_4:\bar{N}_1=0,\bar{N}_2=k_2+\dfrac{a_{23}k_3}{a_{22}},\bar{N}_3=k_3$

In this state, the basic equations can be quasi-linearized,
We have

$$\frac{du_1}{dt} = \alpha_1 u_1; \ \ \frac{du_2}{dt} = -\alpha_2 u_2 + \frac{a_{23}\alpha_2}{a_{22}} u_3; \ \ \frac{du_3}{dt} = -a_3 u_3 \tag{16}$$

where $\alpha_1 = (a_1 + a_{13}k_3 + \frac{a_{12}\alpha_2}{a_{22}}) > 0; \ \alpha_2 = a_2 + a_{23}k_3 > 0$ (17)

The characteristic roots are α_1 , $-\alpha_2$ and $-a_3$. Since one of these three roots is positive, hence the state is **unstable**. The equations (17) yield the solutions.

$$u_1 = u_{10}e^{\alpha_1 t}; u_2 = \left(u_{20} - B\right) e^{-\alpha_2 t} + Be^{-a_3 t}; u_3 = u_{30}e^{-a_3 t} \tag{18}$$

where $B = \frac{a_{23}\alpha_2 u_{30}}{a_{22}(\alpha_2 - a_3)}$ (19)

Trajectories of perturbations

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$u_2 = \left(u_{20} - B\right)\left(\frac{u_1}{u_{10}}\right)^{\frac{-\alpha_2}{\alpha_1}} + B\left(\frac{u_1}{u_{10}}\right)^{\frac{-a_3}{\alpha_1}}; \ u_2 = \left(u_{20} - B\right)\left(\frac{u_3}{u_{30}}\right)^{\frac{\alpha_2}{a_3}} + \frac{Bu_3}{u_{30}} \tag{20}$$

4. THE SECOND SPECIES WITH UNLIMITED RESOURCES

The model equations for the growth rates of S_1, S_2, S_3 are

$$\frac{dN_1}{dt} = N_1\left(a_1 - a_{11}N_1 + a_{12}N_2 + a_{13}N_3\right) \tag{21}$$

$$\frac{dN_2}{dt} = N_2\left(a_2 + a_{23}N_3\right) \tag{22}$$

$$\frac{dN_3}{dt} = N_3\left(a_3 - a_{33}N_3\right) \tag{23}$$

4.1 Equilibrium States

The system under investigation has four equilibrium states given by $\frac{dN_i}{dt} = 0, i = 1, 2, 3$

$$E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0$$

$E_2 : \bar{N}_1 = 0,\ \bar{N}_2 = 0,\ \bar{N}_3 = k_3$

$E_3 : \bar{N}_1 = k_1,\ \bar{N}_2 = 0,\ \bar{N}_3 = 0$

$E_4 : \bar{N}_1 = k_1 + \frac{a_{13}k_3}{a_{11}},\ \bar{N}_2 = 0,\ \bar{N}_3 = k_3$

4.2 Stability Analysis

The equilibrium states E_1 and E_2 are **unstable** as established in **3.2.1** and **3.2.2**. Now we will discuss the stability of other equilibrium states.

4.2.1 Stability of $E_3 : \bar{N}_1 = k_1,\ \bar{N}_2 = 0,\ \bar{N}_3 = 0$

The basic equations can be quasi-linearized,
We get

$$\frac{du_1}{dt} = -a_1u_1 + a_{12}k_1u_2 + a_{13}k_1u_3;\ \ \frac{du_2}{dt} = a_2u_2;\ \ \frac{du_3}{dt} = a_3u_3 \qquad (24)$$

The characteristic roots are $-a_1$, a_2 and a_3. Since two of these three roots are positive, hence the state is **unstable** and the solutions of (24) are

$$u_1 = (u_{10} - C - D)e^{-a_1t} + Ce^{a_2t} + De^{a_3t};\ u_2 = u_{20}e^{a_2t};\ u_3 = u_{30}e^{a_3t} \qquad (25)$$

where $C = \frac{a_{12}k_1u_{20}}{a_1 + a_2}; D = \frac{a_{13}k_1u_{30}}{a_1 + a_3}$ (26)

Trajectories of perturbations

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$u_1 = \left(u_{20} - C - D\right)\left(\frac{u_2}{u_{20}}\right)^{\frac{-a_1}{a_2}} + \frac{Cu_2}{u_{20}} + D\left(\frac{u_2}{u_{20}}\right)^{\frac{a_3}{a_2}};\ \left(\frac{u_2}{u_{20}}\right)^{a_3} = \left(\frac{u_3}{u_{30}}\right)^{a_2} \qquad (27)$$

4.2.2 Stability of $E_4 : \bar{N}_1 = k_1 + \frac{a_{13}k_3}{a_{11}},\ \bar{N}_2 = 0,\ \bar{N}_3 = k_3$

In this state, the basic equations can be quasi-linearized,
We have

$$\frac{du_1}{dt} = -(a_1 + a_{13}k_3)u_1 + \frac{a_{12}}{a_{11}}(a_1 + a_{13}k_3)u_2 + \frac{a_{13}}{a_{11}}(a_1 + a_{13}k_3)u_3; \frac{du_2}{dt} = (a_2 + a_{23}k_3)u_2; \frac{du_3}{dt} = -a_3u_3 \qquad (28)$$

The characteristic roots are $-(a_1+a_{13}k_3)$, $a_2+a_{23}k_3$ and $-a_3$. Since two of these three roots are positive, hence the state is **unstable**. The equations (28) yield the solutions.

$$u_1=(u_{10}-E-F)e^{-(a_1+a_{13}k_3)t}+Ee^{(a_2+a_{23}k_3)t}+Fe^{-a_3t}; u_2=u_{20}e^{(a_2+a_{23}k_3)t}; u_3=u_{30}e^{-a_3t} \quad (29)$$

where $$E=\frac{a_{12}(a_1+a_{13}k_3)u_{20}}{a_{11}(a_1+a_2+a_{13}k_3+a_{23}k_3)}; F=\frac{a_{13}(a_1+a_{13}k_3)u_{30}}{a_{11}(a_1+a_{13}k_3-a_3)} \quad (30)$$

Trajectories of perturbations

The trajectories in the u_1-u_2 and u_2-u_3 planes are given by

$$u_1=\left(u_{20}-E-F\right)\left(\frac{u_2}{u_{20}}\right)^{-\frac{a_1+a_{13}k_3}{a_2+a_{23}k_3}}+\frac{Eu_2}{u_{20}}+F\left(\frac{u_2}{u_{20}}\right)^{\frac{-a_3}{a_2+a_{23}k_3}}; \left(\frac{u_2}{u_{20}}\right)^{-a_3}=\left(\frac{u_3}{u_{30}}\right)^{a_2+a_{23}k_3} \quad (31)$$

5. THE THIRD SPECIES WITH UNLIMITED RESOURCES

The model equations for the growth rates of S_1, S_2, S_3 are

$$\frac{dN_1}{dt}=N_1\left(a_1-a_{11}N_1+a_{12}N_2+a_{13}N_3\right) \quad (32)$$

$$\frac{dN_2}{dt}=N_2\left(a_2-a_{22}N_2+a_{23}N_3\right) \quad (33)$$

$$\frac{dN_3}{dt}=a_3N_3 \quad (34)$$

5.1 Equilibrium States

The system under investigation has four equilibrium states given by $\frac{dN_i}{dt}=0, i=1, 2, 3$

$$E_1:\bar{N}_1=0, \bar{N}_2=0, \bar{N}_3=0$$

$$E_2:\bar{N}_1=k_1, \bar{N}_2=0, \bar{N}_3=0$$

$$E_3:\bar{N}_1=0, \bar{N}_2=k_2, \bar{N}_3=0$$

$$E_4:\bar{N}_1=k_1+\frac{a_{12}k_2}{a_{11}}, \bar{N}_2=k_2, \bar{N}_3=0$$

5.2 Stability Analysis

The equilibrium states E_1, E_2 and E_3 are **unstable** as established in **3.1.1, 4.2.1** and **3.2.3**. Now we will discuss the stability of E_4.

5.2.1 Stability of $E_4 : \bar{N}_1 = k_1 + \frac{a_{12}k_2}{a_{11}}, \bar{N}_2 = k_2, \bar{N}_3 = 0$

The basic equations can be quasi-linearized,
We have

$$\frac{du_1}{dt} = -(a_1 + a_{12}k_2)u_1 + \frac{a_{12}}{a_{11}}(a_1 + a_{12}k_2)u_2 + \frac{a_{13}}{a_{11}}(a_1 + a_{12}k_2)u_3; \frac{du_2}{dt} = -a_2u_2 + a_{23}k_2u_3; \frac{du_3}{dt} = a_3u_3 \quad (35)$$

The characteristic roots are $-(a_1 + a_{12}k_2)$, $-a_2$ and a_3. Since one of these three roots is positive, hence the state is **unstable**. The equations (35) yield the solutions.

$$u_1 = (u_{10} - G - H)e^{-(a_1 + a_{12}k_2)t} + Ge^{-a_2t} + He^{a_3t}; u_2 = (u_{20} - I)e^{-a_2t} + Ie^{a_3t}; u_3 = u_{30}e^{a_3t} \quad (36)$$

where $G = \frac{a_{12}(a_1 + a_{12}k_2)(u_{20} - I)}{a_{11}(a_1 - a_2 + a_{12}k_2)}; H = \frac{(a_1 + a_{12}k_2)(a_{12}I + a_{13}u_{30})}{a_{11}(a_1 + a_3 + a_{12}k_2)}; I = \frac{a_{23}k_2u_{30}}{a_2 + a_3}$ (37)

Trajectories of perturbations

The trajectories in the $u_1 - u_3$ and $u_2 - u_3$ planes are given by

$$u_1 = \left(u_{20} - G - H\right)\left(\frac{u_3}{u_{30}}\right)^{-\frac{a_1 + a_{12}k_2}{a_3}} + G\left(\frac{u_3}{u_{30}}\right)^{\frac{-a_2}{a_3}} + \frac{Hu_3}{u_{30}}; u_2 = (u_{20} - I)\left(\frac{u_3}{u_{30}}\right)^{-\frac{a_2}{a_3}} + \frac{Iu_3}{u_{30}} \quad (38)$$

6. NUMERICAL APPROACH

The numerical solution of the growth rate equations computed employing the fourth order Runge-Kutta method for specific values of the various parameters that characterize the model and the initial conditions. For this MAT LAB has been used and the results are illustrated in Figures 1 to 3.

6.1 Numerical example for the first species with unlimited resources

Consider the parameters,
$a_1 = 0.08, a_{12} = 1.2, a_{13} = 0.1, a_2 = 0.42, a_{22} = 10, a_{23} = 1.4, a_3 = 0.8, a_{33} = 2$

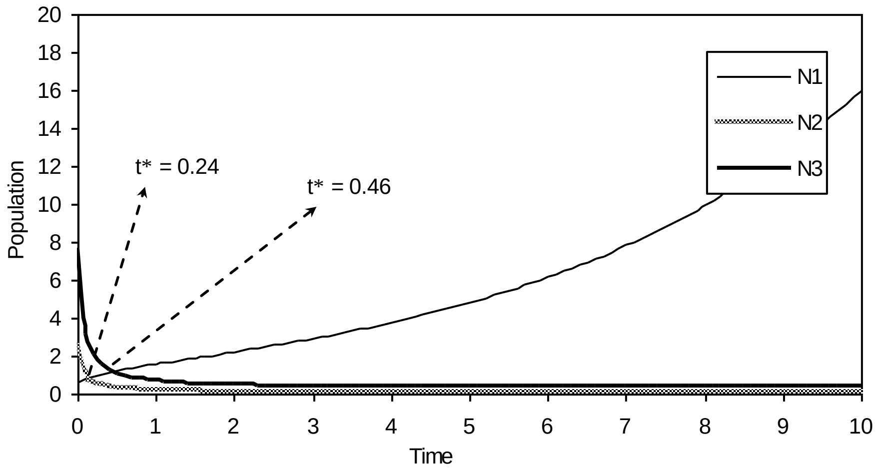

Figure 1: Variation of N_1, N_2, N_3 against time (t) for $N_{10} = 0.68, N_{20} = 2.52, N_{30} = 7.64$

6.2 Numerical example for the second species with unlimited resources

Consider the parameters,
$a_1 = 1.82, a_{11} = 0.46, a_{12} = 0.72, a_{13} = 1.48, a_2 = 0.24, a_{23} = 0.34, a_3 = 0.76, a_{33} = 3.85$

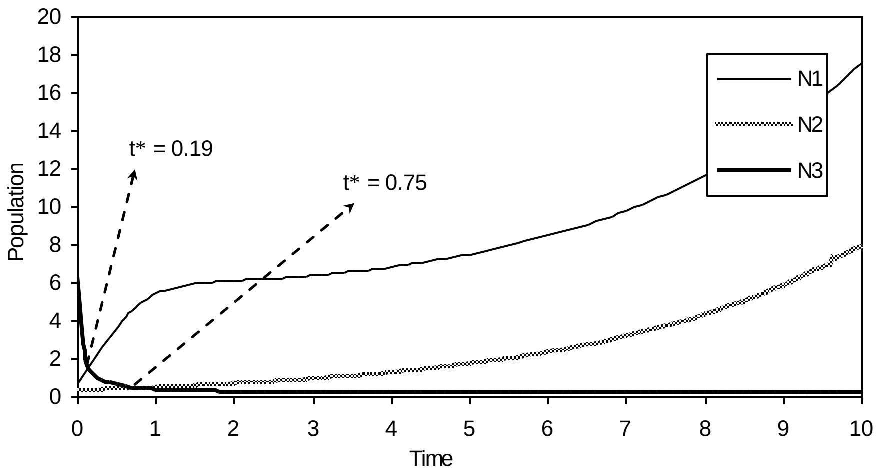

Figure 2: Variation of N_1, N_2, N_3 against time (t) for $N_{10} = 0.74, N_{20} = 0.27, N_{30} = 6.32$

6.3 Numerical example for the third species with unlimited resources

Consider the parameters,

$a_1 = 0.432, a_{11} = 2.664, a_{12} = 0.431, a_{13} = 1.224, a_2 = 2.07, a_{22} = 7.326, a_{23} = 5.292, a_3 = 0.288$

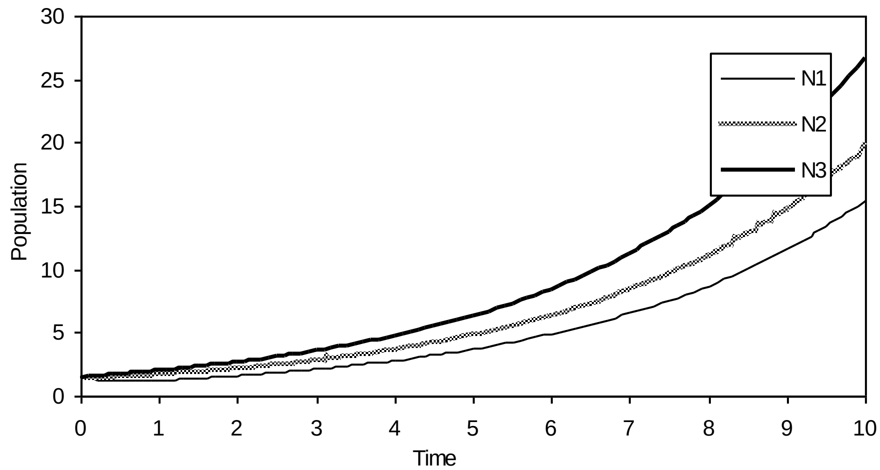

Figure 3: Variation of N_1, N_2, N_3 against time (t) for $N_{10} = 1.5, N_{20} = 1.5, N_{30} = 1.5$

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