INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES, TECHNOLOGY AND HUMANITIES

www.internationalejournals.com

ISSN 2249 –5460

Available online at www.internationaleJournals.com

International eJournals

International Journal of Mathematical Sciences, Technology and Humanities 112 (2014) 1209–1220

A Theorem on $\varphi-\left|\overline{N},p_n\right|_k$ Summability of Infinite Series

[1]B.P.Padhy, [2]Dattaram Bisoyi, [3]Mahendra Misra and [4]U.K.Misra

[1]Dept. of Mathematics, Roland Institute of Technology, Golanthara-761008, Odisha, India. e-mail: iraady@gmail.com

[2]Dept. of Mathematics, L.N.Maha Vidyalaya, Kodala, Ganjam, Odisha, India. e-mail: dbisoyi2@gmail.com

[3]Dept. of Mathematics, Binayak Acharya College, Berhampur, Odisha, India e-mail: mahendramisra@2007.gmail.com

[4]Dept. of Mathematics, National Institute of Science and Technology, Pallur Hills -761008, Odisha, India. e-mail: umakanta_misra@yahoo.com

ABSTRACT

In this paper we have proved a theorem on $\varphi-\left|\overline{N},p_n\right|_k, k\geq 1$ Summability.

Keywords: $\left|\overline{N},p_n\right|_k$ Summability, $\varphi-\left|C,1\right|_k$ Summability,, $\varphi-\left|\overline{N},p_n\right|_k, k\geq 1$ Summability.

2010-Mathematics subject classification: 40D25

1. Introduction:

Let $\{s_n\}$ denote the nth partial sum of an infinite series $\sum a_n$ and let $\{p_n\}$ be a sequence of positive real constants such that

(1.1) $$P_n=\sum_{\nu=0}^{n}p_\nu\to\infty,\ for\, n=0,1,2,...\ (P_i=p_i=0,\ i<0)$$

Then the sequence-to-sequence transformation given by

(1.2) $$T_n=\frac{1}{P_n}\sum_{\nu=0}^{n}p_\nu s_\nu\ ,$$

defines the $\left(\overline{N},p_n\right)$ mean of the sequence $\{s_n\}$.

The series $\sum a_n$ is said to be $\left|\overline{N},p_n\right|_k, k\geq 1$ [1] summable if

(1.3) $$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_n - T_{n-1}\right|^k < \infty.,$$

Taking p_n=1 for all n, $\left|\overline{N}, p_n\right|_k$ summability reduces to $\left|C,1\right|_k$ summability method.

Further suppose $\{\varphi_n\}$ be a sequence of positive real numbers, then the series $\sum a_n$ is said to be $\varphi - \left|\overline{N}, p_n\right|_k, k \ge 1$ summable if

(1.4) $$\sum_{n=1}^{\infty}\varphi_n^{k-1}\left|T_n - T_{n-1}\right|^k < \infty..$$

Taking p_n=1 for all n, $\varphi - \left|\overline{N}, p_n\right|_k$ Summability reduces to $\varphi - \left|C,1\right|_k$ Summability method.

2. Concerning the $\left|C,1\right|_k$ Summability of infinite series $\sum a_n$, in 1957 Flett [2] has established the following result. He proved

Theorem-A:

Let σ_n and τ_n denote the $(C,1)$ mean of the sequence $\{s_n\}$ and $\{na_n\}$ respectively that is

(i) $$\sigma_n = \frac{1}{n+1}\sum_{\nu=0}^{n} s_\nu$$

and

(ii) $$\tau_n = \frac{1}{n+1}\sum_{\nu=0}^{n} \nu\, a_\nu$$

Then the series $\sum a_n$ is summable $\left|C,1\right|_k, k \ge 1$,iff

(2.1) $$\sum_{n=1}^{\infty}\frac{1}{n}\left|\tau_n\right|^k < \infty.$$

Further in 1995, Seyhan [4] extended the result of Flett to $\varphi - \left|C,1\right|_k$ Summability. He established

Theorem-B:

Let σ_n and τ_n be as defined in theorem-A and let $\{\varphi_n\}$ be a sequence of positive real numbers. Then the series $\sum a_n$ is summable $\varphi - \left|C,1\right|_k, k \ge 1$,iff

(2.2) $$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} \left|\tau_n\right|^k < \infty.$$

In 2010 Mishra U.K., Panda. S.P. and Panda S.P. [3] have proved theorem-A to $\left|\overline{N}, p_n\right|_k$ Summability methods. They proved:

Theorem-C:

Let $\{t_n\}$ denote the $(\overline{N}, p_n)$-mean of the sequence $\{na_n\}$ and $\{T_n\}$ be the sequence as defined in (1.2), where $\{p_n\}$ be a sequence of positive real constants satisfying the following conditions:

(a) $np_n = O(P_n)$

(b) $P_n = O(np_n)$

and

(c) $n\left|\Delta p_n\right| = O(p_n)$

Then $\sum a_n$ is summable $\left|\overline{N}, p_n\right|_k, k \geq 1$, iff

(2.3) $$\sum_{n=1}^{\infty} \frac{\left|t_n\right|^k}{n} < \infty.$$

3. In this chapter we establish a similar theorem for $\varphi - \left|\overline{N}, p_n\right|_k, k \geq 1$, Summability method. Hence we prove the following:

Theorem-3.1:

Let $\{T_n\}$ and $\{t_n\}$ denote the sequences $(\overline{N}, p_n)$-mean of the sequence $\{s_n\}$ and $\{na_n\}$ respectively where $\{\varphi_n\}, \{p_n\}$ b the sequences of positive real constants satisfying the following conditions:

(3.1) $np_n = O(P_n)$

(3.2) $P_n = O(np_n)$

(3.3) $n\left|\Delta p_n\right| = O(p_n)$

and

(3.4) $$\frac{\varphi_n}{n} = O(1)$$

Then the series $\sum a_n$ is summable $\varphi - \left|\overline{N}, p_n\right|_k, k \geq 1$, iff

(3.5) $$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} \left|t_n\right|^k < \infty.$$

This theorem generalizes theorem-A,B and C.

4. We require the following lemma to prove theorem-3.1.

Lemma-4.1[3]:

Let $\{p_n\}$ be a sequence of positive real constants satisfying (i) and (ii) of theorem-C then

(4.1) $$p_{n+1} = O(p_n)$$

and

(4.2) $$p_n = O(p_{n+1})$$

holds good.

5. Proof of Theorem-3.1:

Sufficient Part $(\Leftarrow)$:

Since $\{t_n\}$ is the $(\overline{N}, p_n)$-mean of the sequence $\{na_n\}$, we have

(5.1) $$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu \nu a_\nu = \frac{1}{P_n} \sum_{\nu=1}^{n} p_\nu \nu a_\nu$$

Then $$P_n t_n - P_{n-1} t_{n-1} = n p_n a_n$$

(5.2) $$\Rightarrow a_n = \frac{P_n t_n - P_{n-1} t_{n-1}}{n p_n}$$

Now we have

$$T_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu$$

$$= \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu \sum_{\lambda=0}^{\nu} a_\lambda$$

$$= \frac{1}{P_n}\sum_{\lambda=0}^{n} a_\lambda \sum_{\nu=\lambda}^{n} p_\nu$$

$$= \frac{1}{P_n}\sum_{\lambda=0}^{n} a_\lambda \left(P_n - P_{\lambda-1}\right)$$

$$= \sum_{\lambda=0}^{n} a_\lambda - \frac{1}{P_n}\sum_{\lambda=1}^{n} a_\lambda P_{\lambda-1}$$

Then

(5.3) $$\nabla T_n = T_n - T_{n-1}$$

$$= \sum_{\lambda=0}^{n} a_\lambda - \frac{1}{P_n}\sum_{\lambda=1}^{n} a_\lambda P_{\lambda-1} - \sum_{\lambda=0}^{n-1} a_\lambda + \frac{1}{P_{n-1}}\sum_{\lambda=1}^{n} a_\lambda P_{\lambda-1}$$

$$= a_n + \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right)\sum_{\lambda=1}^{n-1} a_\lambda P_{\lambda-1} - \frac{P_{n-1} a_n}{P_n}$$

$$= a_n\left(1 - \frac{P_{n-1}}{P_n}\right) + \frac{p_n}{P_n P_{n-1}}\sum_{\lambda=1}^{n-1} a_\lambda P_{\lambda-1}$$

$$= \frac{p_n}{P_n P_{n-1}}\sum_{\lambda=1}^{n-1} a_\lambda P_{\lambda-1}$$

Using (5.2) we get

$$\nabla T_n$$

$$= \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n} P_{\nu-1}\frac{P_\nu t_\nu - P_{\nu-1} t_{\nu-1}}{\nu p_\nu}$$

$$= \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n}\frac{P_{\nu-1} P_\nu t_\nu}{\nu p_\nu} - \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n}\frac{P_{\nu-1}^2 t_{\nu-1}}{\nu p_\nu}$$

$$= \frac{t_n}{n} + \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n-1}\frac{P_{\nu-1} P_\nu t_\nu}{\nu p_\nu} - \frac{p_n}{P_n P_{n-1}}\sum_{\nu=0}^{n-1}\frac{P_\nu^2 t_\nu}{(\nu+1) p_{\nu+1}}$$

$$= \frac{t_n}{n} + \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu t_\nu \left(\frac{P_{\nu-1}}{\nu p_\nu} - \frac{p_\nu}{(\nu+1) p_{\nu+1}} \right)$$

$$= \frac{t_n}{n} - \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} \frac{P_\nu t_\nu}{\nu} + \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu^2 t_\nu \left(\frac{1}{\nu p_\nu} - \frac{1}{(\nu+1) p_{\nu+1}} \right)$$

$$= \frac{t_n}{n} - \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} \frac{P_\nu t_\nu}{\nu} + \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} \frac{P_\nu^2 t_\nu}{\nu(\nu+1) p_{\nu+1}}$$

$$+ \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} \frac{P_\nu^2 t_\nu}{\nu} \left(\frac{1}{p_\nu} - \frac{1}{p_{\nu+1}} \right)$$

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{(Say)}$$

To complete the proof of the sufficient part, by using Minokowski's inequality it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} \left| T_{n,i} \right|^k < \infty \ for\ i = 1,2,3,4.$$

Now we have

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} \left| T_{n,1} \right|^k = \sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} \left| t_n \right|^k < \infty. \text{ By (3.5).}$$

Next we have

$$\sum_{n=2}^{m+1} \varphi_n^{k-1} \left| T_{n,2} \right|^k$$

$$\leq \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k \left(\sum_{\nu=1}^{n-1} \frac{P_\nu |t_\nu|}{\nu} \right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{\nu=1}^{n-1} \frac{\varphi_\nu p_\nu}{\nu} |t_\nu| \right)^k$$

Using (3.1) and (3.4)

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{\nu=1}^{n-1} p_\nu \right)^k \left(\sum_{\nu=1}^{n-1} \frac{\varphi_\nu^k p_\nu |t_\nu|}{\nu^k} \right)$$

Using Holder's inequality

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left(\sum_{\nu=1}^{n-1}\frac{\varphi_\nu^k}{\nu^k}|t_\nu|^k p_\nu\right)$$

$$= O(1)\sum_{\nu=1}^{m} p_\nu \frac{\varphi_\nu^k}{\nu^k}|t_\nu|^k \sum_{n=\nu+1}^{m+1}\frac{p_n}{P_nP_{n-1}}$$

$$= O(1)\sum_{\nu=1}^{m} p_\nu \frac{\varphi_\nu^k}{\nu^k}|t_\nu|^k \sum_{n=\nu+1}^{m+1}\left(\frac{1}{P_{n-1}}-\frac{1}{P_n}\right)$$

$$= O(1)\sum_{\nu=1}^{m} p_\nu \frac{\varphi_\nu^k}{\nu^k}|t_\nu|^k \frac{1}{P_\nu}$$

$$= O(1)\sum_{\nu=1}^{m} \frac{p_\nu\varphi_\nu}{P_\nu}\frac{\varphi_\nu^{k-1}}{\nu^k}|t_\nu|^k$$

$$= O(1)\sum_{\nu=1}^{m} \frac{np_\nu}{P_\nu}\varphi_\nu^{k-1}\frac{|t_\nu|^k}{\nu^k}, \text{ Using (3.4)}$$

$$= O(1)\sum_{\nu=1}^{m} \varphi_\nu^{k-1}\frac{|t_\nu|^k}{\nu^k}, \text{ Using (3.1)}$$

$$= O(1) \ as \ m \to \infty .$$

Further we have

$$\sum_{n=2}^{m+1}\varphi_n^{k-1}\left|T_{n,3}\right|^k$$

$$\leq \sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left(\sum_{\nu=1}^{n-1}\frac{P_\nu^2|t_\nu|}{\nu(\nu+1)p_{\nu+1}}\right)^k$$

$$\leq \sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}\frac{P_\nu P_{\nu+1}|t_\nu|}{\nu(\nu+1)p_{\nu+1}}\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}\frac{P_\nu}{\nu}\frac{\varphi_{\nu+1}}{\nu+1}|t_\nu|\right)^k \text{,Using (3.2)}$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}\frac{P_\nu}{\nu}|t_\nu|\right)^k \text{, Using (3.4)}$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}\frac{\varphi_\nu p_\nu}{\nu}|t_\nu|\right)^k \text{, Using (3.2)}$$

$$= O(1) \; as \; m \to \infty \text{, Proceeding as above}$$

Next we have

$$\sum_{n=2}^{m+1}\varphi_n^{k-1}\left|T_{n,4}\right|^k$$

$$\le \sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left(\sum_{\nu=1}^{n-1}\frac{P_\nu^2|t_\nu|}{\nu}\left|\frac{1}{p_\nu}-\frac{1}{p_{\nu+1}}\right|\right)^k$$

$$\le \sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}^k}{}^k\left(\sum_{\nu=1}^{n-1}\frac{P_\nu^2|t_\nu|}{\nu}\left|\frac{p_{\nu+1}-p_\nu}{p_{\nu+1}p_\nu}\right|\right)^k,$$

Using (3.4) & (3.1)

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}\frac{\varphi_\nu p_\nu}{\nu p_\nu}\frac{P_\nu}{p_{\nu+1}}|\Delta p_\nu||t_\nu|\right)^k \text{,Using(3.2)}$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}\frac{\varphi_\nu}{\nu}\frac{\varphi_\nu P_\nu}{p_{\nu+1}}|\Delta p_\nu||t_\nu|\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}\frac{\varphi_\nu}{\nu}\varphi_\nu|\Delta p_\nu||t_\nu|\right)^k \text{, Using Lemma-1}$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}\frac{\varphi_\nu}{\nu}\nu|\Delta p_\nu||t_\nu|\right)^k \text{, Using (3.4)}$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}\frac{\varphi_\nu}{\nu}p_\nu|t_\nu|\right)^k \text{, Using (3.3)}$$

$$= O(1) \; as \; m \to \infty \text{, proceeding as above}$$

This proves the sufficient part of the theorem.

Necessary Part $(\Rightarrow)$:

From (5.3) we have

$$\frac{P_{n-1}P_n}{p_n}\nabla T_n = \sum_{\nu=1}^{n} P_{\nu-1}a_\nu$$

$$a_n = \frac{1}{P_{n-1}}\left[\frac{P_{n-1}P_n}{p_n}\nabla T_n - \frac{P_{n-2}P_{n-1}}{p_{n-1}}\nabla T_{n-1}\right]$$

$$= \frac{P_n}{p_n}\nabla T_n - \frac{P_{n-2}}{p_{n-1}}\nabla T_{n-1}$$

Now

$$t_n = \frac{1}{P_n}\sum_{\nu=1}^{n} p_\nu \nu\, a_\nu$$

$$= \frac{1}{P_n}\sum_{\nu=1}^{n}\left(\nu\, P_\nu \nabla T_\nu - \frac{\nu\, p_\nu P_{\nu-2}}{p_{\nu-1}}\nabla T_{\nu-1}\right)$$

$$= \frac{1}{P_n}\sum_{\nu=1}^{n}\nu\, P_\nu \nabla T_\nu - \frac{1}{P_n}\sum_{\nu=0}^{n-1}\frac{(\nu+1)p_{\nu+1}P_{\nu-1}}{p_\nu}\nabla T_\nu$$

$$= n\nabla T_n + \frac{1}{P_n}\sum_{\nu=1}^{n}\nu\, P_\nu \nabla T_\nu - \frac{1}{P_n}\sum_{\nu=1}^{n-1}\frac{(\nu+1)p_{\nu+1}P_{\nu-1}}{p_\nu}\nabla T_\nu$$

$$= n\nabla T_n + \frac{1}{P_n}\sum_{\nu=1}^{n}\nu\,\nabla T_\nu\left(P_\nu - P_{\nu-1}\right)$$

$$+\frac{1}{P_n}\sum_{\nu=1}^{n-1}\nu\,\nabla T_\nu P_{\nu-1}\left(1-\frac{p_{\nu+1}}{p_\nu}\right)+\frac{1}{P_n}\sum_{\nu=1}^{n-1}P_{\nu-1}\,\frac{p_{\nu+1}}{p_\nu}\nabla T_\nu$$

$$= t_{n,1} + t_{n,2} + t_{n,3} + t_{n,4}\ \text{,say.}$$

To complete the necessary part by using Minokowski's inequality we need to show only

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^k}\left|t_{n,i}\right|^k < \infty.\ for\ i = 1,2,3,4.$$

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^k}\left|t_{n,1}\right|^k=\sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}n^k}{n^k}\left|\nabla T_n\right|^k$$

$$=\sum_{n=1}^{\infty}\varphi_n^{k-1}\left|\nabla T_n\right|^k$$

$$=O(1).$$

Further,

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^k}\left|t_{n,2}\right|^k\le\sum_{n=2}^{m+1}\frac{\varphi_n^{k-1}}{n^k}\frac{1}{P_n^k}\Big(\sum_{\nu=1}^{n-1}\nu\left|\nabla T_\nu\right|p_\nu\Big)^k$$

$$\le\sum_{n=2}^{m+1}\frac{\varphi_n^{k-1}}{n^k}\frac{P_n}{P_n^kP_{n-1}}\Big(\sum_{\nu=1}^{n-1}\nu\left|\nabla T_\nu\right|p_\nu\Big)^k$$

$$=O(1)\sum_{n=2}^{m+1}\frac{\varphi_n^kp_n}{n^kP_n^kP_{n-1}}\Big(\sum_{\nu=1}^{n-1}\nu\left|\nabla T_\nu\right|p_\nu\Big)^k\text{, using (3.2)}$$

$$=O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_n^kP_{n-1}}\Big(\sum_{\nu=1}^{n-1}\nu\left|\nabla T_\nu\right|p_\nu\Big)^k\text{, using (3.4)}$$

$$=O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_n^kP_{n-1}}\Big(\sum_{\nu=1}^{n-1}p_\nu\Big)^{k-1}\Big(\sum_{\nu=1}^{n-1}\nu^k\left|\nabla T_\nu\right|^kp_\nu\Big),$$

Using Holder's inequality

$$=O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\Big(\sum_{\nu=1}^{n-1}\nu^k\left|\nabla T_\nu\right|^kp_\nu\Big)$$

$$=O(1)\sum_{\nu=1}^{m}p_\nu\nu^k\left|\nabla T_\nu\right|^k\sum_{n=\nu+1}^{m+1}\left(\frac{1}{P_{n-1}}-\frac{1}{P_n}\right)$$

$$=O(1)\sum_{\nu=1}^{m}p_\nu\nu^k\left|\nabla T_\nu\right|^k\left(\frac{1}{P_\nu}-\frac{1}{P_{m+1}}\right)$$

$$=O(1)\sum_{\nu=1}^{m}\frac{p_\nu}{P_\nu}\nu^k\left|\nabla T_\nu\right|^k$$

$$=O(1)\sum_{\nu=1}^{m}\nu^{k-1}\left|\nabla T_\nu\right|^k$$

$$= O(1)\sum_{\nu=1}^{m}\frac{P_n^{k-1}}{p_n^{k-1}}\left|\nabla T_\nu\right|^k \text{, using (3.1)}$$

$$= O(1)\sum_{\nu=1}^{m}\varphi_n^{k-1}\left|\nabla T_\nu\right|^k$$

$$= O(1) \text{ as } m \to \infty .$$

Again

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^k}\left|t_{n,3}\right|^k$$

$$\leq \sum_{n=2}^{m+1}\frac{\varphi_n^{k-1}}{n^k}\frac{1}{P_n^k}\left(\sum_{\nu=1}^{n-1}\nu P_{\nu-1}\left|\nabla T_\nu\right|\frac{\left|p_\nu - p_{\nu-1}\right|}{p_\nu}\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{1}{nP_n^k}\left(\sum_{\nu=1}^{n-1}\nu P_{\nu-1}\left|\nabla T_\nu\right|\frac{\left|\Delta p_\nu\right|}{p_\nu}\right)^k \text{, using (3.4)}$$

$$= O(1)\sum_{n=2}^{m+1}\frac{1}{nP_n^k}\left(\sum_{\nu=1}^{n}P_{\nu-1}\left|\nabla T_\nu\right|\right)^k \text{, using (3.3)}$$

$$= O(1)\sum_{n=2}^{m+1}\frac{1}{nP_n^k}\left(\sum_{\nu=1}^{n}P_\nu\left|\nabla T_\nu\right|\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{1}{\varphi_n P_n^k}\left(\sum_{\nu=1}^{n}\varphi_\nu P_\nu\left|\nabla T_\nu\right|\right)^k \text{, using (3.2) and (3.4)}$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_n P_n^k}\left(\sum_{\nu=1}^{n}\varphi_\nu P_\nu\left|\nabla T_\nu\right|\right)^k \text{, using (3.2)}$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_{n-1} P_n^k}\left(\sum_{\nu=1}^{n}\varphi_\nu P_\nu\left|\nabla T_\nu\right|\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_{n-1} P_n^k}\left(\sum_{\nu=1}^{n}p_\nu\right)^{k-1}\left(\sum_{\nu=1}^{n}\varphi_\nu^k p_\nu\left|\nabla T_\nu\right|^k\right)$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_{n-1} P_n^k}\left(\sum_{\nu=1}^{n}p_\nu\right)^{k-1}\left(\sum_{\nu=1}^{n}\varphi_\nu^k p_\nu\left|\nabla T_\nu\right|^k\right)$$

$$= O(1)\sum_{\nu=1}^{m}p_\nu\varphi_\nu^k\left|\nabla T_\nu\right|^k\sum_{n=\nu+1}^{m+1}\left(\frac{1}{P_{n-1}}-\frac{1}{P_n}\right)$$

$$= O(1)\sum_{\nu=1}^{m} p_\nu \varphi_\nu^k \left|\nabla T_\nu\right|^k \left(\frac{1}{P_\nu} - \frac{1}{P_{m+1}}\right)$$

$$= O(1)\sum_{\nu=1}^{m} \frac{p_\nu}{P_\nu} \varphi_\nu^k \left|\nabla T_\nu\right|^k$$

$$= O(1)\sum_{\nu=1}^{m} \frac{\varphi_\nu^k}{\nu} \left|\nabla T_\nu\right|^k \text{, using (3.1)}$$

$$= O(1)\sum_{\nu=1}^{m} \varphi_\nu^{k-1} \left|\nabla T_\nu\right|^k \text{, using (3.4)}$$

$$= O(1) \; as \; m \to \infty .$$

Finally,

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} \left|t_{n,4}\right|^k$$

$$\leq \sum_{n=2}^{m+1} \frac{\varphi_n^{k-1}}{n^k} \frac{1}{P_n^k} \left(\sum_{\nu=1}^{n-1} P_{\nu-1} \frac{p_{\nu+1}}{p_\nu} \left|\nabla T_\nu\right|\right)^k$$

$$= O(1)\sum_{n=2}^{m+1} \frac{1}{nP_n^k} \left(\sum_{\nu=1}^{n-1} P_{\nu-1} \left|\nabla T_\nu\right|\right)^k \text{, using (3.3) and (3.4)}$$

$$= O(1)\sum_{n=2}^{m+1} \frac{1}{nP_n^k} \left(\sum_{\nu=1}^{n-1} P_\nu \left|\nabla T_\nu\right|\right)^k$$

$$= O(1) \; as \; m \to \infty \text{ , proceeding as above}$$

This completes the proof of the theorem.

REFERENCES

1. **Bor.H**: A Note on the Summability methods, Math.Proc. of Cambridge Phil. Doc.,Vol 97 (1975),pp:147-149.

2. Flett,T.M: On an extension of absolute Summability and some theorem of Littlewood and Pabey , Proc. Lond. Math. Soc. , 7(1957), pp: 113-141.

3. Misra U.K., Panda S.P., and Panda S.P.: A theorem on $\left|\overline{N}, p_n\right|_k$ summabilty of infinite series, Journal. Of Comp. and Math. Soc., Vol. 1(2) (2010), pp: 103-110.

4. Seyhan, H: The absolute Summability methods, Ph.D Thesis,Kayseri(1995), pp:01-57.