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Pre A*- Clones, "If-Then-Else" Structures over Pre A*- Algebra

[1]K. Suguna Rao and [2]P.Koteswara Rao

[1]Dept. of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar-522 510, Andhra Pradesh, INDIA. e-mail ID: sugunaraok@yahoo.com

[2]Dept. of Commerce, Acharya Nagarjuna University, Nagarjuna Nagar-522 510, Andhra Pradesh, INDIA

Abstract: In this paper we divide two sections. In the first section we introduce the Definition of Pre A*-Clone, Every Pre A*-Clone generates Pre A*-algebra, Pre A*-algebra forms Pre A*-Clone and some other theorems. Definition of Boolean Clone ,Every Boolean clone generates Boolean Algebra. In the Second Section ,we introduce the definition of if-then –else operation on pre A*-algebra and pointed set and some other theorems.

Key Words: Clone, A*-Clone, Boolean Algebra, Huntington's theorem.

1. Pre A*- Clone:

Definition: A Pre A*- Clone over a set 'S' is a set A of ternary operations on S satisfying the following conditions.

$\forall\, f, g \in A, x, y, z, x_1, y_1, z_1 \in S$

(i) $f(x, x, x) = x$

(ii) $f(f(x_1, y_1, z_1), f(x_2, y_2, z_2), f(x_3, y_3, z_3)) = f(x_1, y_2, z_3)$

(iii) $f(g(x_1, y_1, z_1), g(x_2, y_2, z_2), g(x_3, y_3, z_3)) = g(f(x_1, x_2, x_3), f(y_1, y_2, y_3), f(z_1, z_2, z_3))$

(iv) A contains the projections Π_1, Π_2 where $\Pi_1(x, y, z) = x$, $\Pi_2(x, y, z) = y$

Further 'A' is closed under the operations $f^{|}, f \wedge g, f \vee g$ defined by

(v) $f^{|}(x, y, z) = f(y, x, z)$

(vi) $(f \wedge g)(x, y, z) = f(g(x, y, z), g(y, y, z), z)$

(vii) $(f \vee g)(x, y, z) =? f(g(x, x, z), g(x, y, z), z)$

1.1 Example: Let X, Y be sets and 'S' be the set of all partial functions from X into Y. Then 3^x is a pre A*-Clone over S w. r. t to the operations defined by

$$a(x,y,z)(P) = \begin{cases} x(P) \text{ if } a(P) = 1 \\ y(P) \text{ if } a(P) = 0 \\ z(P) \text{ if } a(P) = 2 \end{cases}$$

For $a \in 3^x$, $x, y, z, \in S$ and P lies in the domains of x, y, z.

1.2Theorem: Every a pre A*-Clone on a set 'S' is a pre A*-Algebra.

Proof: Suppose A is a pre A*-Clone over a set 'S'. Clearly $f \wedge g = g \wedge f$

$(f \vee g)(x,y,z) = f(g(x,x,z), g(x,y,z), z)$

(i) $f^{\sim\sim} = f$, $\forall f \in A$

$f^{\sim\sim}(x,y,z) = (f^{\sim})^{\sim}(x,y,z)$

$= f^{\sim}(y,x,z)$

$= f(x,y,z)$

$= f$

$\therefore f^{\sim\sim} = f$

(ii) $f \wedge f = f$

$(f \wedge f)(x,y,z) = f(f(x,y,z), f(y,y,z), z)$

$= f(x,y,z), f(y,y,z), z$

$= f(x,y,z).$

$\therefore f \wedge f = f$

(iii) $f \wedge g = g \wedge f$

$(f \wedge g)(x,y,z) = f(g(x,y,z), g(y,y,z), z)$

$= g(f(x,y,z), f(y,y,z), f(z,z,z))$

$(g \wedge f)(x,y,z) = g(f(x,y,z), f(y,y,z), z)$

$= g\big(f(x,y,z), f(y,y,z), f(z,z,z)\big)$

$\therefore f \wedge g = g \wedge f$

(iv) $(f \wedge g)^{\sim} = f^{\sim} \vee g^{\sim}$

$(f \wedge g)^{\sim}(x,y,z) = (f \wedge g)(y,x,z)$

$= f(g(y,x,z), g(x,x,z), z)$

$(f^{\sim} \vee g^{\sim})(x,y,z) = f^{\sim}(g(x,y,z), g(x,y,z), z)$

$$= f^{\sim}(g(x,y,z), g(x,y,z), z)$$

$$= f(g(y,x,z), g(x,x,z), z)$$

$\therefore (f \wedge g)^{\sim} = f^{\sim} \vee g^{\sim}$

(v) $\underline{f \wedge (g \wedge h) = (f \wedge g) \wedge h,}$

$$\big(f \wedge (g \wedge h)\big)(x,y,z) = f(g \wedge h\ (x,y,z))$$

$$= f(g(h(x,y,z), h(y,y,z), z))$$

$$= (f \wedge g)(h(x,y,z), h(y,y,z), z))$$

$$= (f \wedge g)(h(x,y,z))$$

$$= ((f \wedge g) \wedge h)\ (x,y,z)$$

$\therefore f \wedge (g \wedge h) = (f \wedge g) \wedge h$

(vi) $\underline{f \wedge (g \vee h) = (f \wedge g) \vee (f \wedge h)}$

Now $\big(f \wedge (g \vee h)\big)(x,y,z) = f((g \vee h)\ (x,y,z))$

$$= f(g(h(x,x,z), h(x,y,z), z))$$

$$= (f \wedge g)(f(h(x,x,z), h(x,y,z), z))$$

$$= (f \wedge g) \wedge f(h(x,y,z))$$

$$= (f \wedge g) \vee (f \wedge h)$$

Now $(f \wedge g)(x,y,z) = f(g(x,y,z), g(y,y,z), z)$

$$(f \wedge h)(x,y,z) = f(h(x,y,z), h(y,y,z), z)$$

$$\therefore \big((f \wedge g) \vee (f \wedge h)\big)(x,y,z) = f(g(x,y,z), g(y,y,z), z)\ \vee f(h(x,y,z), h(y,y,z), z)$$

$$= f((g(x,y,z), g(y,y,z), z) \vee f(h(x,y,z), h(y,y,z), z))$$

(vii) $x \wedge y = x \wedge (x^{\sim} \wedge y)$for all x

$f \wedge g = f \wedge (f^{\sim} \wedge g)$for all $f, g \in A$

$f \wedge g(x,y,z) = f(g(x,y,z), g(y,y,z), z)$

$f^{\sim} \wedge g(x,y,z) = f^{\sim}(g(x,y,z), g(y,y,z), z)$

$(f \wedge (f^{\sim} \wedge g))(x,y,z) = f(f(g(y,y,z), g(x,y,z), z)$

$\therefore$ A is a pre A*-Algebra.

1.3Theorem: Every pre A*-Algebra is a pre A*-Clone over a set ‘S’.

Proof: Suppose $(A, \wedge, \vee, (-)^{\sim})$ A is a pre A*-Algebra

Let $a \in A$, $x, y, z \in B(A)$ define $a(x, y, z) = (a_x) \vee (a_y) \vee (a_z)$ for all $x, y, z \in B(A)$

Then A is a set of ternary operations on B(A).

(i) $a(x, x, x) = (ax) \vee (ax) \vee (ax)$

$$= (a \vee a \vee a)(x)$$
$$= 1.x = x$$
$$\therefore a(x, x, x) = x$$

(ii) $a(a(x_1, y_1, z_1), a(x_2, y_2, z_2), a(x_3, y_3, z_3))$

$$= a((ax_1) \vee (ay_1) \vee (az_1)) \vee a((ax_2) \vee (ay_2) \vee (az_2)) \vee a((ax_3) \vee (ay_3) \vee (az_3))$$

$$= ax_1 \vee ay_2 \vee az_3$$

$$= a(x_1, y_2, z_3)$$

$$\therefore a(a(x_1, y_1, z_1), a(x_2, y_2, z_2), a(x_3, y_3, z_3)) = a(x_1, y_2, z_3)$$

(iii) $a(b(x_1, y_1, z_1), b(x_2, y_2, z_2), b(x_3, y_3, z_3))$

$$= a(bx_1 \vee by_1 \vee bz_1) \vee a(bx_2 \vee by_2 \vee bz_2) \vee a(bx_3 \vee by_3 \vee bz_3)$$

$$= b(ax_1 \vee ax_2 \vee ax_3) \vee b(ay_1 \vee ay_2 \vee ay_3) \vee b(az_1 \vee az_2 \vee az_3)$$

$$= b(a(x_1 \vee x_2 \vee x_3), a(y_1 \vee y_2 \vee y_3), a(z_1, z_2, z_3))$$

$$\therefore a(b(x_1, y_1, z_1), b(x_2, y_2, z_2), b(x_3, y_3, z_3)) = b(a(x_1 \vee x_2 \vee x_3), a(y_1 \vee y_2 \vee y_3), a(z_1, z_2, z_3))$$

(iv) $\Pi_1 = 1$, $\Pi_2 = 0$, since

$$\Pi_1(x, y, z) = 1\,(x, y, z)$$
$$= 1x \vee 1y \vee 1z$$
$$= x$$

$\|$ly, $\Pi_2(x, y, z) = y$

(v) $a(b(x, y, z), b(y, y, z), z)$

$$= (abx) \vee (aby) \vee (abz) \vee (aby) \vee (aby) \vee (abz) \vee (az)$$

$= (abx) \vee (aby) \vee)(aby) \vee (aby) \vee (a \vee a)bz) \vee az$

$= (abx) \vee (aby) \vee (aby) \vee (aby) \vee (ab \vee a)z$

$= (abx) \vee (aby) \vee (aby) \vee (aby) \vee (aby) \vee (a \wedge b)z$

$= (a \wedge b)x \vee (a \wedge b)y \vee (a \wedge b)z$

$= (a \wedge b)(x, y, z)$

$\therefore (a \wedge b)(x, y, z) = a(b(x, y, z), b(y, y, z), z)$

(vi) $a(b(x, x, z), b(x, y, z), z)$

$= (abx) \vee (abx) \vee (abz) \vee (abx) \vee (aby) \vee (abz) \vee (az)$

$= (abx) \vee (abx) \vee)(abx) \vee (aby) \vee (a \vee a)bz \vee az$

$= (abx) \vee (abx) \vee (abx) \vee (aby) \vee (ab \vee a)z$

$= (ab \vee ab \vee ab \vee a)$

$= (a \wedge b)\ (x,y,z)$

(vii)$a^{\sim}(x, y, z)$ $=(a^{\sim}x) \vee (a^{\sim}y) \vee (a^{\sim}z)$

$=a^{\sim}y \vee a^{\sim}x \vee a^{\sim}z$

$=ay \vee ax \vee az$

$=a(y, x, z)$

$\therefore a^{\sim}(x, y, z) = a(y, x, z)$

$\therefore$ A is a Pre A*-clone over B(A)

1. 4 Definition: A set B of binary operations on a set S satisfying the following conditions is called a Boolean Clone over ‘S’.

(i) $f(x, x) = x, \forall x \in S, \forall f \in B$

(ii) $f(f(x, y), f(u, v) = f(x, v)$

(iii) B contains projections π_1, π_2 where $\pi_1(x, y) = x$, $\pi_2(x, y) = g$

(iv) $f(g(x, y), g(u, v) = g(f(x, u), f(y, v)$ further B is closed under the operations $f^{|}$, $f \wedge g$ defined by

(v) $f^{|}(x, y) = f(y, x)$ (vi)$(f \wedge g)(x, y) = f(g(x, y), y)$

1.5 Example: For any let X, 2^x is a Boolean clone over the Set 'S' of all functions from X into a given let Y where we define for $f \in 2^x$, x, y ∈ S, P ∈ X.

$$f(x, y)(P) = x(P) \text{ if } f(P) = 1$$
$$= y(P) \text{ if } f(P) = 0$$

1.6 Theorem: Every Boolean Clone B over a Set 'S' becomes a Boolean algebra.

$$(f \wedge f^{|})(x, y) = f(f^{|}(x, y), y)$$
$$= f(f(y, x), y)$$
$$= f(f(y), x), f(y, y))$$
$$= f(y, y) = y = \Pi_2(x, y)$$

$\therefore (f \wedge f^{|}) = \Pi_2$.

$$[(f \wedge g) \vee (f \wedge g^{|})](x, y)$$
$$= (f \wedge g)[x, f \wedge g^{|}(x, y)]$$
$$= (f \wedge g)[x, f(g(y, x), y)]$$
$$= (f \wedge g)[x, g(y, f(x, f))]$$
$$= f[g(x, (g(y, f(x, y))), g(y, f(x, y))]$$
$$= f(g(x, f(x, y)), g(y, f(x, y))$$
$$= g(f(x, y), f(f(x, y), f(x, y))$$
$$= g(f(x, y), f(x, y))$$
$$= f(x, y)$$

$\therefore [(f \wedge g) \vee (f \wedge g^{|})] = f$

By Huntington's theorem, $(B, \wedge, \vee, (-)^1, \Pi_2)$ is a Boolean Algebra.

1.7 Corollary: If A is Pre A*-clone over 'S', then every Sun pre A*-Algebra of A is a Pre A*-clone over S.

1.8 Theorem: If A is Pre A*-clone over S then there exists a set X such that A is isomorphic to a pre sub clone of 3^x.

Proof: Since A is a Pre A*-Algebra

⇒ A is isomorphic to a sub Pre A*-Algebra P F_n (*x, 2*) for some X and since P F_n (*x, 2*) can be identified with 3^x, A is isomorphic to sub Pre A*-Algebra of 3^x. Let Y be any set and S be the set of all partial functions from X to Y. Then 3^x is a Pre A*-clone over S. Since every sub Pre A*-Algebra of 3^x is an A*-clone over S, A is isomorphic to a sub Pre A*-clone of 3^x.

2 If-Then-Else Operation:

2.1 Definition: Let 'A' be a pre $A*$ −algebra with 1. If $xa_1b \in A$; $define\ the\ If - then -$ *else* operation on A as $\text{if}_x(a, b) = (x \wedge a) \vee (x^\sim \wedge b)$.

($\text{if}_x(a, b) =$ should be viewed as conditional "$\text{ifx}_1\text{then a}_1\text{else b}$")

2.2 Lemma: Every pre A*-algebra which indicates Constants satisfies the following laws.

(i) $\text{if}_2(a_1b) = 2$

(ii) $\text{if}_x(2,2) = 2$

(iii) $\text{if}_1(a_1b) = a(b \neq 2)$

(iv) $\text{if}_0(a_1b) = b(p \neq 2)$

(v) $\text{if}_x(1,0) = x$

Proof:

(i) $\text{if}_2(a, b) = (2 \wedge a) \vee (2^\sim \wedge b) = 2 \vee (2 \wedge b) = 2 \vee 2 = 2$

(ii) $\text{if}_x(2,2) = (x \wedge 2) \vee (x^\sim \wedge 2) = 2 \vee 2 = 2$

(iii) $\text{if}_1(a, b) = (1 \wedge a) \vee (1^\sim \wedge b) = a \vee (0 \wedge b) = a \vee 0\ (b \neq 2) = a$

(iv) $\text{if}_0(a, b) = (0 \wedge a) \vee (0^\sim \wedge b) = o \vee (1 \wedge b)(p \neq 2) = 0 \vee b = b$

(v) If $x = 2$, then $\text{if}_x(1,0) = (2 \wedge 1) \vee (2^\sim \wedge 0) = 2 = x$

If $x \neq 2$, then $\text{if}_x(1,0) = (x \wedge 1) \vee (x^\sim \wedge 0) = x \vee 0 = x$

One can observe that either x or a or b is 2 then $\text{if}_x(a, b) = 2$

2.3 Lemma: Every pre A*-algebra satisfies

(i) $\text{if}_{x^\sim}(a, b) = \text{if}_x(b, a)$

(ii) $\text{if}_x(a, b) \wedge c = \text{if}_x(a \wedge c), (b \wedge c)$

(iii) $\text{if}_x(a, b) \vee c = \text{if}_x(a \vee c), (b \vee c)$

(iv) $\text{if}_x\left(\text{if}_y(a, b), \text{if}_y(c, d)\right) = \text{if}_y(\text{if}_x(a, c), \text{if}_x(b, d))$

(v) $\text{if}_{x \vee y}(a, b) = \text{if}_x(a, \text{if}_y(a, b))$

(vi) $\text{if}_{x \wedge y}(a, b) = \text{if}_x(\text{if}_y(a, b), b)$

(vii) $\text{if}_p(a, a) = a$

Proof:

(i) $\text{if}_{x^{\sim}}(a, b) = (x^{\sim} \wedge a) \vee (x^{\sim\sim} \wedge b)$

$= (x^{\sim} \wedge a) \vee (x \wedge b)$

$= (x \wedge b) \vee (x^{\sim} \wedge a)$

$= \text{if}_x(b, a)$

Therefore $\text{if}_{x^{\sim}}(a, b) = \text{if}_x(b, a)$

(ii) $\text{if}_x(a, b) \wedge c = \{(x \wedge a) \vee (x^{\sim} \wedge b)\} \wedge c$

$= \{(x \wedge a) \wedge c\} \vee \{(x^{\sim} \wedge b) \wedge c\}$

$= \{x(a \wedge c\} \vee \{x^{\sim}(b \wedge c)\}$

$= \text{if}_x(a \wedge c), (b \wedge c)$

(iii) If $x \neq 2$, then

$\text{if}_x(a \vee c), (b \vee c) = \{x \wedge (a \vee c)\} \vee \{x^{\sim} \wedge (b \vee c)\}$

$= \{(x \wedge a) \vee (x \wedge c)\} \vee \{(x^{\sim} \wedge b) \vee (x^{\sim} \wedge c)\}$

$= \{(x \wedge a) \vee (x^{\sim} \wedge b)\} \vee \{(x \wedge c) \vee (x^{\sim} \wedge c)\}$

$= \{(x \wedge a) \vee (x^{\sim} \wedge b)\} \vee \{(x \vee x^{\sim}) \wedge c\}$

$= \text{if}_x(a, b) \vee (1 \wedge c) \quad (x \neq 2 \Rightarrow x \vee x^{\sim} = 1)$

$= \text{if}_x(a, b) \vee c$

$\text{If}_x = 2$, the proof is clear.

(iv) $\text{if}_x\left(\text{if}_y(a, b), \text{if}_y(c, d)\right) = \left\{x \wedge \text{if}_y(a, b)\right\} \vee \left\{x^{\sim} \wedge \text{if}_y(c, d)\right\}$

$= \{x \wedge [(y \wedge a) \vee (y^{\sim} \wedge b)]\} \vee \{x^{\sim} \wedge [(y \wedge c) \vee (y^{\sim} \wedge d)]\}$

$= \{x \wedge (y \wedge a)\} \vee \{x \wedge (y^{\sim} \wedge b)\} \vee \{x^{\sim} \wedge (y \wedge c)\} \vee \{x^{\sim} \wedge (y^{\sim} \wedge d)\}$

$= \{y \wedge (x \wedge a)\} \vee \{y \wedge (x^{\sim} \wedge c)\} \vee \{y^{\sim} \wedge (x \wedge b)\} \vee \{y^{\sim} \wedge (x^{\sim} \wedge d)\}$

$= [y \wedge \{(x \wedge a) \wedge (x^{\sim} \wedge c)\}] \vee [y^{\sim} \wedge \{(x \wedge b) \wedge (x^{\sim} \wedge d)\}]$

$= [y \wedge \text{if}_x(a, c)] \vee [y^{\sim} \wedge \text{if}_x(b, d)]$

$= \text{if}_y(\text{if}_x(a, c), \text{if}_x(b, d))$

(v) $\text{if}_x\left(a, \text{if}_y(a, b)\right) = (x \wedge a) \vee (x^{\sim} \wedge \text{if}_y(a, b))$

$= (x \wedge a) \vee [x^{\sim} \wedge \{(y \wedge a) \vee (y^{\sim} \wedge b)\}]$

$= (x \wedge a) \vee [\{x^{\sim} \wedge (y \wedge a)\} \vee \{x^{\sim} \wedge (y^{\sim} \wedge b)\}]$

$= [(x \wedge a) \vee \{x^{\sim} \wedge (y \wedge a)\}] \vee \{x^{\sim} \wedge (y^{\sim} \wedge b)\}$

$= [(x \wedge a) \vee \{(x^{\sim} \wedge y) \wedge a\}] \vee \{x^{\sim} \wedge (y^{\sim} \wedge b)\}$

$= [\{x \vee \{(x^{\sim} \wedge y)\} \wedge a] \vee \{x^{\sim} \wedge (y^{\sim} \wedge b)\}$

$= \{(x \vee y) \wedge a\} \vee \{(x \vee y)^{\sim} \wedge b\}$

$= if_{x \vee y}(a, b)$

(vi) $if_x\big(if_y(a, b), b\big) = \big(x \wedge if_y(a, b)\big) \vee (x^{\sim} \wedge b)$

$= [x \wedge \{(y \wedge a) \vee (y^{\sim} \wedge b)\}] \vee (x^{\sim} \wedge b)$

$= [(x \wedge (y \wedge a)) \vee (x \wedge (y^{\sim} \wedge b))] \vee (x^{\sim} \wedge b)$

$= \big(x \wedge (y \wedge a)\big) \vee [\big(x \wedge (y^{\sim} \wedge b)\big) \vee (x^{\sim} \wedge b)]$

$= \big(x \wedge (y \wedge a)\big) \vee [((x \wedge y^{\sim}) \wedge b) \vee (x^{\sim} \wedge b)]$

$= \big(x \wedge (y \wedge a)\big) \vee [\{(x \wedge y^{\sim}) \vee x^{\sim}\} \wedge b]$

$= \{(x \wedge y) \wedge a\} \vee \{(x^{\sim} \vee y^{\sim}) \wedge b\}.$

$= \{(x \wedge y) \wedge a\} \vee \{(x \wedge y)^{\sim} \wedge b\}.$

$= if_{x \wedge y}(a, b)$

(vii) $if_p(a, a) = (a \wedge a) \vee (a^{\sim} \wedge a)$

$= a \vee (a^{\sim} \wedge a)$

$= a$

Observe that $if_x(a, a) = a(x \wedge a) \vee (x^{\sim} \wedge a)$

$= (x \vee x^{\sim}) \wedge a, \text{if } x \neq 2,$

then $x \vee x^{\sim} = 1$ and $if_x(a, a) = a$ Otherwise the value is 2.

2.4 Definition:A pointed set $(S, \perp)$ it is said to be an "if-then-else" structure over pre A*-Algebra A; if the ternary operation

$(x, a, y) \to xay, \forall s, y \in S, \forall a \in A$

$SXAXS \to S$ satisfied

(i) $xax = xa^{\sim}\perp$

(ii) $x^{|}y = x$

(iii)$xa^{\sim}y = yax$

(iv) $(xay)bz = xab[(\perp bz)a(ybz)]$ where $ab = a \wedge b$

2.5 Note: $a^{\sim} = (a \vee 1)$ is called a hault.

2.6 Lemma: If $(S, \perp)$ is an "if-then-else" structure over pre A*-Algebra A, then

(i) $xay = xa(\perp ay)$

(ii) $xab[(\perp bz)a(ybz)] = (\perp bz)a(ybz)]$

(iii) $(xby)a(\mu b\vartheta) = (xa\mu)b(ya\vartheta)$

(iv) $x(a \wedge b)y = (xay)b(yay)$

Proof:

(i) $xay = x(a \vee a)y$

$= xa(\perp ay)$

$\therefore xay = xa(\perp ay)$

(ii) $xab[(\perp bz)a(ybz)]$

$= (\perp bz)a[xab(yba(\perp bz))]$

$[\because xa(ybz) = yb(xaz)]$ where $ab = a \wedge b$

$= (\perp bz)a[\perp b(xab(ybz))]$

$= (\perp bz)a[\perp b(xay)b(zaz)]$

$= (\perp bz)a[\perp b\{(xa(yay))b(za(zaz))\}]$

$= (\perp bz)a[\perp b\{(xbz)a(ybz)a(ybz)\}]$

$= (\perp bz)a[\perp b\{(xbz)a(yay)b(zaz)\}]$

$= (\perp bz)a[(\perp a \perp)b\{(xbz)a[(ybz)a(ybz)]\}]$

$= (\perp bz)a[(xbz)a(ybz)]$

$\therefore xab[(\perp bz)a(ybz)] = (\perp bz)a[(xbz)a(ybz)]$

(iii) $(xby)a(\mu b\vartheta)$

$= [xb(\perp by)]a[\mu b(\perp b\vartheta)]$

$= [xb(\perp by)]a[\perp a[\mu b(\perp b\vartheta)]]$

$= [xb(\perp by)]a\{[\perp b(\perp b \perp)]a[\mu b(\perp b\vartheta)]\}$

$= [xb(\perp by)]a\{(\perp a\mu)b[(\perp b \perp)a(\perp b\vartheta)]\}$

$= [xb(\perp by)]a\{(\perp a\mu)b[(\perp a \perp)b(\perp a\vartheta)]\}$

$= [xa(\perp a\mu)]b\{(\perp by)a[(\perp a \perp)b(\perp a\vartheta)]\}$

$= [xa(\perp a\mu)]b\{[\perp a[(\perp a \perp)]b[ya(\perp a\vartheta)]\}$

$= (xa\mu)b[\perp b[ya(\perp a\vartheta)]]$

$= (xa\mu)b[\perp b(ya\vartheta)]$

$= (xa\mu)b(ya\vartheta)$

$\therefore (xby)a(\mu b\vartheta) = (xa\mu)b(ya\vartheta)$

(iv) $(xay)b(yay)$

$= [\perp a(xay)]b[\perp a(yay)]$

$= [\perp a(xay)]b[\perp b(\perp ay)]$

$=\perp b\{[\perp a(xay)]b(\perp ay)\}$

$=\perp b\{[xa(\perp ay)]b(\perp ay)\}$

$=\perp b\{xab(\perp ay)\}$

$= xab(\perp b(\perp ay))$

$= xab(\perp a \vee by)$

$= x(a \wedge b)y$

$\therefore x(a \wedge b)y = (xay)b(yay)$

2.7 Theorem: pre A*-Algebra A is a pre A*-clone iff$(S, \perp)$ is an "if-then-else" structure over A for some distinguished element$\perp$.

Proof: Case-I: Suppose $(S, \perp)$ is an "if-then-else" structure over A.

For $x, y, z \in S, a \in A$, define

$a(x, y, z) = za(xay)$

For $a(x, x, x) = x$ is clear

$a(a(x_1, y_1, z_1), a(x_2, y_2, z_2), a(x_3, y_3, z_3))$

$= [z_3\, a(x_3\, a\, y_3)]\, a\{[z_1\, a(x_1\, a\, y_1)]a[z_2\, a(x_2\, a\, y_2)]\}$

$= [z_3\, a\, \{[z_1\, a(x_1\, a\, y_1)]a[z_2\, a(x_2\, a\, y_2)]\}$

$= [z_3\, a\, \{(x_1\, a\, y_1)\, a[z_2\, a(x_2\, a\, y_2)]\}$

$= z_3\, a\, \{x_1 a[z_2\, a(x_2\, a\, y_2)]\}$

$= z_3\, a\, \{\, z_2\, a[x_1 a\, (x_2\, a\, y_2)]\}$

$= z_3\, a\, \{\, z_2\, a[x_1 a\, y_2]\}$

$= z_3\, a\, (x_1 a\, y_2) = a(x_1, y_2, z_3)$

$\therefore a(a(x_1, y_1, z_1), a(x_2, y_2, z_2), a(x_3, y_3, z_3)) = a(x_1, y_2, z_3)$

$a(b(x_1, y_1, z_1), b(x_2, y_2, z_2), b(x_3, y_3, z_3))$

$= a[z_1 b\, (x_1 b y_1), z_2 b\, (x_2 b y_2), z_3 b\, (x_3 b y_3)]$

$= [z_3 b\, (x_3 b y_3)]a\{[z_1 b\, (x_1 b y_1)]\, a\, [z_2 b\, (x_2 b y_2)]\}$

$= [z_3 b\, (x_3 b y_3)]a\{(z_1 a z_2)\, b[(x_1 b y_1)\, a\, (x_2 b y_2)]\}$

$= [z_3 b\, (x_3 b y_3)]a\{(z_1 a z_2) b[(x_1 a\, x_2)\, b\, (y_1 a\, y_2)]\}$

$= [z_3 a\, (z_1 a z_2)]\, b\{(x_3 b y_3)]a[(x_1 a\, x_2)\, b\, (y_1 a\, y_2)]\}$

$= a\, (z_1, z_2, z_3)\, b[a(x_1, x_2, x_3)\, ba(y_1, y_2, y_3)]\}$

$= b[a(x_1, x_2, x_3), a(y_1, y_2, y_3), a\ (z_1, z_2, z_3)]$

$\therefore a(b(x_1, y_1, z_1), b(x_2, y_2, z_2), b(x_3, y_3, z_3)) = b[a(x_1, x_2, x_3), a(y_1, y_2, y_3), a\ (z_1, z_2, z_3)]$

and clearly$1 = \Pi_1, 0 = \Pi_2$

$\therefore$ A is a Pre A*-clone over S

Case II:

Conversely Suppose

Since A is pre A*-clone
$\Rightarrow$ A is a pre A*-Algebra
Define for $a \in A, x, y, z \in S$
$xay = a(x, y, \perp)$
$a(x, y, z) = za_1(xa_2y)$
Now $za_1(xa_2y) \quad = a_1\ (z, za_2, y, \perp)$
$= a_1\ (z, a_2(x, y, \perp), \perp)$
$= a\ (a_2(x, y, \perp), a_1(x, y, \perp), z)$
$= a\ (a(x, y, y), a(x, y, y), z)$
$= a(x, y, z)$
$\therefore a(x, y, z) \ = za_1(xa_2y)$
$xa_{\tilde{1}} \perp \ = a_{\tilde{1}}(x, \perp, \perp)$

$= a_1\ (\perp, x, \perp)$

$= a(x, x, \perp)$

$= xax$.

$x1x = \Pi_1(x, x, \perp) = 2$

$xa^{\sim} y = a^{\sim}\ (x, x, \perp) = a(y, x, \perp) = yax$

$(xay)bz \qquad = b[xay, z, \perp]$

$= b[a(x, y, \perp), z, \perp]$

$= b[b(x, z, \perp), b(y, z, \perp), b(\perp, z, \perp)]$

$= a[xbz, ybz, \perp bz]$

$= (\perp bz)a_1\ [(xbz)a_2\ (ybz)]$

$= xa_2\ b_2\ [(\perp bz), a_1\ [(ybz)]$

$\therefore (S, \perp)$is an "if-Then-else" structure over $'A'$.

2.8 Corollary: Let *x, y* be two sets and S=Rel (*x, y*) the set of all relations from X to Y, $\perp$ be the empty relation. Then $(S, \perp)$is an "if-Then-else" structure over Every sub A*-algebra. T of T_x.

Proof: Since S=Rel $(x, y) = PF_n(X, 2^Y)$, T_x can be identified with 3^x; and $(S, \perp)$is an "if-Then-else" structure of 3^x

2.9 Theorem: If S is an "if-then-else" structure over a Boolean algebra B and ⊥ is a distinguished element then (S,⊥) is an "if-then-else" structure over A(B) generated by B where $xay = xa_1 (\perp a_2 y)$, where $a = (a_1, a_2)$.

Proof:

$$xax = xa_1 \left(\perp a_2 x\right) = \perp a_2 (xa_1 x)$$

$$= \perp a_2 x = xa_2^{\sim} \perp.$$

$$xly = xl_1 \left(\perp l_2 y\right) = xl_1 y = x$$

$$xa^{\sim} y = xa_1^{\sim}(\perp a_2 y)$$

$$= x(a_1 \vee a_2)^{\sim}(\perp a_2 y)$$

$$= (\perp a_2 y)(a_2 \vee a_1)x$$

$$= \perp a_2(ya_1 x)$$

$$= yax$$

$$(xay)bz = \perp b_2[(xay)b_1 z]$$

$$= [\perp a_2(\perp a_1 \perp)]b_2\{[\perp a_2(xa_1 y)]b_1[za_2(za_1 z)]\}$$

$$= [\perp a_2(\perp a_1 \perp)]b_2\{(\perp b_1 z)a_2[(xb_1 y)b_1(za_1 z)]\}$$

$$= [\perp a_2(\perp a_1 \perp)]b_2\{(\perp b_1 z)a_2[(xb_1 z)a_1(yb_1 z)]\}$$

$$= [\perp b_2(\perp b_2 z)]a_2\{(\perp a_1 \perp)b_2[(xb_1 z)a_1(yb_1 z)]\}$$

$$= (\perp bz)a_2\{[\perp b_2(xb_1 z)]a_1[\perp b_2(yb_1 z)]\}$$

$$= (\perp bz)a_2[(xbz)]a_1(ybz)$$

$$= (\perp bz)a_2[(xbz)]a_1(ybz)$$

$$= xa_1 b_1[(\perp bz)a_2(ybz)\}$$

$$\therefore (xay)bz = xa_1 b_1[(\perp bz)a_2(ybz)\}$$

$$x(a * b)y = x(a * b)_1[\perp [x(a * b)_2 y]$$

$$= xa_1[\perp a_1^{\sim}(b_1^{\sim})^{\sim} y]$$

$$= xa_1\{[\perp (b_1^{\sim})^{\sim} y]a_1^{\sim} y\}$$

$$= \{[\perp (b_1^{\sim})^{\sim} y](a_1)^{\sim} y]a_1^{\sim} x\}$$

$$= [\perp (b_1^{\sim})^{\sim} y]a_1^{\sim} x$$

$$= xa_1[\perp (b_1 \vee b_2)y]$$

$$= xa_1[\perp b_2(\perp b_1 y)]$$

$$= xa_1(\perp b_1 y)$$

$$\therefore x(a * b)y = xa_1(\perp b_1 y)$$

2.10 Lemma: Let A is a pre A*-algebra, $x \in B(A)$, then $if_x(a, a) = a, \forall\, a \in A$.

Proof: If $x \in B(A)$, then

$$if_x(a,a) = (x \wedge a) \vee (x^{\sim} \wedge a)$$
$$= (x \vee x^{\sim}) \wedge a \qquad (\because x \in B(A) \Rightarrow x \vee x^{\sim} = 1)$$
$$= 1 \wedge a = a\,.$$

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