Fuzzy Translations of Fuzzy $p$-Ideals in BCI-Algebras

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ABSTRACT

In this paper, the concepts of fuzzy translation to fuzzy $p$-ideals in BCK/BCI algebras are introduced. The notion of fuzzy extensions and fuzzy multiplications of fuzzy $p$-ideals with several related properties are investigated. Also, the relationships between fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy ideals are investigated.

AMS Mathematics Subject Classifications: 06F35, 03G25, 08A72.

Keywords: Fuzzy ideal, fuzzy $p$-ideal, fuzzy translation, fuzzy extension, fuzzy multiplication.

1. Introduction

BCK/BCI-algebras are two important classes of logical algebras introduced by Iski in 1966 (see [9, 10, 20]). Since then, several works have been dedicated to the theory of BCI/BCK/MV/BL-algebras with a focus on ideals and filters of these classes of algebras. From the logical point of view, various ideals correspond to various sets of provable formulas [2, 11, 12]. In 1965, Zadeh [18] introduced the concept of fuzzy sets which has been successfully applied to many mathematical disciplines. In 1991, O. Xi [17] applied the concept of fuzzy sets to BCI-algebras and introduced the notion of fuzzy ideals in BCI-algebras. In 1994, Jun et al. [15] introduced fuzzy $p$-ideals in BCI-algebras and in 2010, Kordi et al. [16], extend it to notation of $(m,n) -$fold $p$-ideals, see also [17]. Lee et al. [18] and Jun [14] discussed fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy
sub algebras and ideals in BCK/BCI algebras. They investigated relations among fuzzy translations, fuzzy extensions and fuzzy multiplications. In this paper, fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy \( p \)-ideals in BAK/BCI-algebras are discussed. Relations among fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy \( p \)-ideals in BAK/BCI-algebras are also investigated.

2. Preliminaries

By a BCI-algebra we mean an algebra \((X; *, 0)\) of type \((2, 0)\) satisfying following axioms:

\[
\begin{align*}
(1) & \quad ((x * y) * (x * z)) * (z * y) = 0, \\
(2) & \quad (x * (x * y)) * y) = 0, \\
(3) & \quad x * x = 0, \\
(4) & \quad x * y = 0 and y * x = 0 imply x = y.
\end{align*}
\]

for all \( x, y, z \in X \). We can define a partial ordering "\( \leq \)" on \( X \) by \( x \leq y \) if and only if \( x * y = 0 \).

The following statements are true in any BCI-algebra \( X \):

\[
\begin{align*}
(1.1) & \quad (x * y) * z = (x * z) * y, \\
(1.2) & \quad x * 0 = x, \\
(1.3) & \quad (x * z) * (y * z) \leq x * y, \\
(1.4) & \quad x \leq y implies x * z \leq y * z and z * y \leq z * x, \\
(1.5) & \quad 0 * (x * y) = (0 * x) * (0 * y), \\
(1.6) & \quad x * (x * (x * y)) = x * y.
\end{align*}
\]

Definition 2.1. A nonempty subset \( I \) of \( X \) is called an ideal of \( X \) if it satisfies:

\[
\begin{align*}
(I_1) & \quad 0 \in I, \\
(I_2) & \quad x * y \in I and y \in I imply x \in I.
\end{align*}
\]

Definition 2.2. A nonempty subset \( I \) of \( X \) is called an ideal of \( X \) if it satisfies condition \((I_1)\) and

\[
\begin{align*}
(I_3) & \quad (x * z) * (y * z) \in I and y \in I imply x \in I.
\end{align*}
\]

Putting \( z = 0 \) in \((I_3)\), we can see that every \( p \)-ideal is an ideal.

Definition 2.3. A fuzzy set \( \mu \) of BCI-algebra \( X \) is called fuzzy ideal of \( X \) if it satisfies
(FI₁) \( \mu(0) \geq \mu(x) \)

(FL₁) \( \mu(x) \geq \min\{\mu(x \ast y), \mu(y)\} \).

**Definition 2.4.** A fuzzy set \( \mu \) of BCI-algebra \( X \) is called fuzzy \( p \)-ideal of \( X \) if it satisfies (FI₁) and

\[
(FI₃) \mu(x) \geq \min\{((x \ast z) \ast (y \ast z)), \mu(y)\}
\]

**Proposition 2.5.** ([17]) Let \( \mu \) be a fuzzy set in a BCI-algebra \( X \). Then \( \mu \) is a fuzzy \( p \)-ideal of \( X \) if and only if for all \( t \in [0,1] \),

\[
\mu_t \neq \emptyset \Rightarrow \mu_t \text{ is a } p \text{-ideal of } X,
\]

Where \( \mu_t = \{x \in X|\mu(x)t\} \).

3. **Main Results**

Throughout this paper, we take \( \dagger = 1 - \sup\{\mu(x) | x \in X\} \) for any fuzzy set \( \mu \) of \( X \).

**Definition 3.1.** ([18]) Let \( \mu \) be a fuzzy subset of \( X \) and let \( \alpha \in [0, \dagger] \). A mapping \( \mu_\alpha \dagger \rightarrow [0,1] \) is called a fuzzy \( \alpha \)-translation of \( \mu \) if it satisfies \( \mu_\alpha(x) = \mu(x) + \alpha \) for all \( x \in X \).

**Theorem 3.2.** if \( \mu \) is a fuzzy \( p \)-ideal of \( X \), then the fuzzy \( \alpha \)-translation \( \mu_\alpha \dagger \) of \( \mu \) is a fuzzy \( p \)-ideal of \( X \), for all \( \alpha \in [0, \dagger] \).

**Proof.** Assume that \( \mu \) is a fuzzy \( p \)-ideal \( X \) and let \( \alpha \in [0, \dagger] \). Then we have

\[
\mu_\alpha = \mu(0) + \alpha \geq \mu(x) + \alpha = \mu_\alpha(x),
\]

and for all \( x, y, z \in X \) we have

\[
\mu_\alpha(x) = \mu(x) + \alpha \\ \geq \min\{\mu((x \ast z) \ast (y \ast z)), \mu(y)\} + \alpha \\ = \min\{\mu((x \ast z) \ast (y \ast z)) + \alpha, \mu(y)\} + \alpha \\ \min\{\mu_\alpha((x \ast z) \ast (y \ast z)), \mu_\alpha(y)\}.
\]

Hence, the fuzzy \( \alpha \)-translation \( \mu_\alpha \dagger \) of \( \mu \) is a fuzzy \( p \)-ideal of \( X \).

**Theorem 3.3.** let \( \mu \) be a fuzzy subset of \( X \) such that the fuzzy \( \alpha \)-translation \( \mu_\alpha \dagger \) of \( \mu \) is a fuzzy \( p \)-ideal of \( X \), for some \( \alpha \in [0, \dagger] \). Then, \( \mu \) is a fuzzy of \( X \).
Proof. Assume that $\mu^\alpha_\alpha$ is a fuzzy $p$-ideal of $X$, for some $\alpha \in [0, \dag]$. Let $x \in X$, then

$$\mu(0) + \alpha = \mu^\alpha_\alpha(0) \geq \mu^\alpha_\alpha(x) = \mu(x) + \alpha,$$

So $\mu(0) \geq \mu(x)$. Also, for all $x, y, z \in X$ we have

$$\mu(x) + \alpha = \mu^\alpha_\alpha(x) \geq \min\{\mu^\alpha_\alpha((x * z) * (y * z)), \mu^\alpha_\alpha(y)\} = \min\{\mu((x * z) * (y * z)) + \alpha, \mu(y)\} + \alpha = \min\{\mu((x * z) * (y * z)), \mu(y)\} + \alpha.$$

So, $\mu(x) \geq \min\{\mu((x * z) * (y * z)), \mu(y)\}$. Therefore $\mu$ is a fuzzy $p$-ideal of $X$.

**Theorem 3.4.** If the fuzzy $\alpha$-translation $\mu^\alpha_\alpha$ of $\mu$ is a fuzzy $p$-ideal of $X$, for some $\alpha \in [0, \dag]$. Then, $\mu$ is a fuzzy of $X$.

**Proof.** Let the fuzzy $\alpha$-translation $\mu^\alpha_\alpha$ of $\mu$ is a fuzzy $p$-ideal of $X$. Then, we have $\mu^\alpha_\alpha(x) \geq \min\{\mu^\alpha_\alpha((x * z) * (y * z)), \mu^\alpha_\alpha(y)\}$. Since by ([15]), $\mu$ is a subalgebra, we have

$$\mu^\alpha_\alpha(x) = \mu(x) + \alpha \geq \min\{\mu(x), \mu(y)\} + \alpha = \min\{\mu(x) + \alpha, \mu(y) + \alpha\} = \min\{\mu^\alpha_\alpha(x), \mu^\alpha_\alpha(y)\}.$$

Therefore, $\mu^\alpha_\alpha$ is a fuzzy sub algebra of $X$.

**Theorem 3.5** let $\mu$ be a fuzzy subset of $X$ such that the fuzzy $\alpha$-translation $\mu^\alpha_\alpha$ of $\mu$ is a fuzzy $p$-ideal of $X$, for some $\alpha \in [0, \dag]$. Then, $\mu$ is a fuzzy of $X$.

**Proof.** Clearly, $0 \in I_\mu$. Assume that $x, y, z \in X$ such that $(x * z) * (y * z), y \in I_\mu$, then

$$\mu^\alpha_\alpha((x * z) * (y * z)) = \mu^\alpha_\alpha(x) = \mu^\alpha_\alpha(y)$$

Thus, we have

$$\mu^\alpha_\alpha(x) \geq \min\{\mu^\alpha_\alpha((x * z) * (y * z)), \mu^\alpha_\alpha(y)\} = \mu^\alpha_\alpha(0)$$
Since, $\mu_\alpha^+$ of $\mu$ is a fuzzy $p$-ideal of $X$, we conclude that $\mu_\alpha^+(x) = \mu_\alpha^+(0)$. Therefore $\mu(x) + \alpha = \mu(0) + \alpha$, i.e., $\mu(x) = \mu(0)$, so that $x \in I_\mu$. Therefore, $I_\mu$ is an $p$-ideal of $X$.

**Proposition 3.6.** ([25]), If the fuzzy $\alpha$-translation $\mu_\alpha^+$ of $\mu$ is an $p$-ideal of $X$, then it is order reversing.

**Theorem 3.7.** let $\mu$ be a fuzzy subset of $X$ such that the fuzzy $\alpha$-translation $\mu_\alpha^+$ of $\mu$ is a fuzzy ideal of $X$, then the following statements are equivalent:

(i) $\mu_\alpha^+$ is a fuzzy $p$-ideal of $X$,

(ii) $\mu_\alpha^+ (0 * (0 * x)) \leq \mu_\alpha^+ (x)$

**Proof.** (i) $\Rightarrow$ (ii): It is enough to put $x=z=0$ and $y=x$ in definition of fuzzy $p$-ideal. 

(ii) $\Rightarrow$ (i): for all $x, y, z \in X$ we have

$$(0 * (0 * x) * y) * ((x * z) * (y * z)) = (0 * (x * z) * (y * z)) * (0 * x) * y =$$

$$\left(\left(0 * (0 * z) * (0 * y) * (0 * z)\right) * (0 * x)\right) * y$$

$$\leq \left((0 * x) * (0 * y) * (0 * x)\right) * y$$

$$= (0 * (0 * y)) * y = 0$$

Now, by Corollary 3.6 of ([25]), we have:

$$\mu_\alpha^+(0 * (0 * x)) \geq \min \{\mu_\alpha^+(x * z) * (y * z) ; \mu_\alpha^+(y)\},$$

So $\mu_\alpha^+(x) \geq \min \{\mu_\alpha^+(x * z) * (y * z) ; \mu_\alpha^+(y)\}$. Hence $\mu_\alpha^+$ is a fuzzy $p$-ideal of $X$.

**Definition 3.8.** ([18]) Let $\mu_1$ and $\mu_2$ be fuzzy subsets of $X$. If $\mu_1 \leq \mu_2$, for all $x \in X$, then we say that $\mu_2$ is a fuzzy extension of $\mu_1$.

**Definition 3.9.** Let $\mu_1$ and $\mu_2$ be fuzzy subsets of $X$. Then $\mu_2$ is called a fuzzy $p$-ideal extension of $\mu_1$ if following statements are valid:

(i) $\mu_2$ is a fuzzy extension of $X$, then $\mu_1$,

(ii) If $\mu_1$ is a fuzzy $p$-ideal of $X$, then $\mu_2$ us a fuzzy $p$-ideal of $X$.

**Theorem 3.10.** let $\mu$ be a fuzzy $p$-ideal of $X$ subset of $X$ and $\alpha \in [0, \dagger]$. then the fuzzy $\alpha$-translation $\mu_\alpha^+$ of $\mu$ is a fuzzy $p$-ideal extension of $\mu$. 


Proof. It's clear from the definition of fuzzy $\alpha$-translation.

The following example show that a fuzzy $p$-ideal extension of a fuzzy $p$-ideal $\mu$ may not be represented as a fuzzy $\alpha$-translation of $\mu$:

Example 3.11. Consider a BIC-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>A</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>b</td>
</tr>
<tr>
<td>B</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>C</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Let $\mu$ be a fuzzy subset of $X$ defined by:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>0.6</td>
<td>0.6</td>
<td>0.4</td>
<td>0.4</td>
</tr>
</tbody>
</table>

Then, $\mu$ is a fuzzy $p$-ideal of $X$. Let $\theta$ be a fuzzy subset of $X$ given by:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>0.63</td>
<td>0.63</td>
<td>0.41</td>
<td>0.41</td>
</tr>
</tbody>
</table>

Then, $\theta$ is a fuzzy $p$-ideal extension of $\mu$. But it is not the fuzzy $\alpha$-translation $\mu_\alpha^+$ of $\mu$, for all $\alpha \in [0, \dagger]$.

Theorem 3.12. Let $\mu$ be a fuzzy subset of $X$ and $\alpha \in [0, \dagger]$. Then, the fuzzy $\alpha$-translation $\mu_\alpha^+$ of $\mu$ is a fuzzy $p$-ideal of $X$ if and only if $U_\alpha(\mu; t)$ is a $p$-ideal of $X$, for all $t \in \text{Im}(\mu)$ with $t > \alpha$.

Proof. Suppose that $\mu_\alpha^+$ is a fuzzy $p$-ideal of $X$ and $t \in \text{Im}(\mu)$ with $t > \alpha$. Since $\mu_\alpha^+(0) \geq \mu_\alpha^+(x)$, for all $x \in X$, we have $\mu(0) + \alpha = \mu_\alpha^+(0) \geq \mu_\alpha^+(x) = \mu(x) + \alpha \geq t$, for $x \in U_\alpha(\mu; t)$, so $0 \in U_\alpha(\mu; t)$. Let, $x, y, z \in X$ such that $(x * z) * (y * z), y \in U_\alpha(\mu; t)$, then

$$\mu((x * z) * (y * z)) \succeq t - \alpha, \quad \mu(y) \succeq t - \alpha$$

i.e.,
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\[ \mu^+(x * z * (y * z)) \geq t, \quad \mu^+(y) \geq t \]

Since \( \mu^+ \) is a fuzzy \( p \)-ideal. So, we have

\[ \mu(x) + \alpha = \mu^+(x) \geq \min\{\mu^+(x * z * (y * z)), \mu^+(y)\} \geq t, \]

that is \( \mu(x) \geq t - \alpha \) so that \( x \in U_{\alpha}(\mu; t) \). Therefore, \( U_{\alpha}(\mu; t) \) is a \( p \)-ideal of \( X \).

Conversely, suppose that for all \( t \in \text{Im} (\mu) \) with \( t > \alpha \), \( U_{\alpha}(\mu; t) \) is a \( p \)-ideal of \( X \). If there exists \( x \in X \) such that \( \mu^+(0) < \beta \leq \mu^+(\alpha) \), then \( \mu(\alpha) \geq \beta - \alpha \) but \( \mu(0) \geq \beta - \alpha \).

Therefore \( \alpha \in U_{\alpha}(\mu; t) \) and \( 0 \notin U_{\alpha}(\mu; t) \). Hence it's contradiction and so for all \( x \in X, \mu^+(0) \geq \mu^+(x) \). Now assume that there exist \( a, b, c \in X \) such that,

\[ \mu^+(\alpha) < \gamma \leq \min\{\mu^+(a * c * (b * c)), \mu^+(b)\}. \]

Then \( \mu(a * c * (b * c)) \geq \gamma - \alpha \) and \( \mu(b) \geq \gamma - \alpha \). Therefore \( (a * c) * (b * c), b \in U_{\alpha}(\mu; t) \) but \( b \notin U_{\alpha}(\mu; t) \), which is a contradiction. Hence, \( \mu^+ \) is a fuzzy \( p \)-ideal of \( X \).

**Theorem 3.13.** Let \( \mu \) be a fuzzy \( p \)-ideal of \( X \) and let \( \alpha, \beta \in [0, \dagger] \). If \( \alpha \geq \beta \) Then, the fuzzy \( \alpha \)-translation \( \mu^+_\alpha \) of \( \mu \) is a fuzzy \( p \)-ideal extension of the fuzzy \( \beta \)-translation \( \mu^+_\beta \) of \( \mu \).

**Proof.** It's straightforward.

**Theorem 3.14.** Let \( \mu \) be a fuzzy \( p \)-ideal of \( X \) and \( \beta \in [0, \dagger] \). For every fuzzy \( p \)-ideal extension \( \nu \) of the fuzzy \( \beta \)-translation \( \mu^+_\beta \) of \( \mu \), there exists \( \alpha \in [0, \dagger] \). Such that \( \alpha \geq \beta \) and \( \nu \) is a fuzzy \( p \)-ideal extension of the fuzzy \( \alpha \)-translation \( \mu^+_\alpha \) of \( \mu \).

**Proof.** For every fuzzy \( p \)-ideal \( \mu \) of \( X \) and \( \beta \in [0, \dagger] \), the fuzzy \( \beta \)-translation \( \mu^+_\beta \) of \( \mu \) is a fuzzy \( p \)-ideal extension of \( \mu^+_\beta \) then there exists \( \alpha \in [0, \dagger] \) such that \( \alpha \geq \beta \) and for all \( x \in X, \nu(x) \geq \mu^+_\alpha \).

**Definition 3.15.** Let \( \mu \) be a fuzzy subset of \( X \) and \( \gamma \in [0,1] \). A fuzzy \( \gamma \)-multiplication of \( \mu \) denoted by \( \mu^m_\gamma \), is defined to be a mapping \( \mu^m_\gamma: X \rightarrow [0,1] \) by \( \mu^m_\gamma(x) = \mu(x), \gamma \).

**Theorem 3.16.** If \( \mu \) is a fuzzy \( p \)-ideal of \( X \), then the \( \gamma \)-multiplication of \( \mu \) is a fuzzy \( p \)-ideal of \( X \) for all \( \gamma \in [0,1] \).

**Proof.** It's clear.
Theorem 3.17. Let $\mu$ be a fuzzy subset of $X$. Then $\mu$ is a fuzzy $p$-ideal of $X$ if and only if the fuzzy $\gamma$-multiplication $\mu^{m}_{\gamma}$ of $\mu$ is a fuzzy $p$-ideal of $X$, for all $\gamma \in [0,1]$.

Proof. ($\Rightarrow$) By Theorem 3.15, is clear.

($\Leftarrow$) Assume that $\mu^{m}_{\gamma}$ of $\mu$ is a fuzzy $p$-ideal of $X$, for all $\gamma \in [0,1]$. Thus,

$$\mu(0), \gamma = \mu^{m}_{\gamma}(0) \geq \mu^{m}_{\gamma}(x) = \mu(x), \gamma$$

i.e., for all $x \in X, \mu(0) \geq \mu(x)$ Also, for $x, \gamma, z \in X$, we have

$$\mu(x), \gamma = \mu^{m}_{\gamma}(x) \geq \min\{\mu^{m}_{\gamma}(x \ast z) \ast (\gamma \ast z), \mu^{m}_{\gamma}(\gamma)\}$$

$$= \min\{\mu((x \ast z) \ast (\gamma \ast z)), \mu(\gamma), \gamma\}$$

$$= \min\{\mu((x \ast z) \ast (\gamma \ast z)), \mu(\gamma)\}, \gamma$$

Which implies that $\mu(x) \geq \min\{\mu((x \ast z) \ast (\gamma \ast z)), \mu(\gamma)\}$. Therefore $\mu$ is a fuzzy $p$-ideal of $X$.

Theorem 3.18. Let $\mu$ be a fuzzy subset of $X$. $\alpha \in [0,\dagger]$ and $\gamma \in [0,1]$. Then, every fuzzy $\alpha$-translation $\mu^{\dagger}_{\alpha}$ of $\mu$ is a fuzzy $p$-ideal extension of the fuzzy $\gamma$-multiplication $\mu^{m}_{\gamma}$ of $\mu$.

Proof. For all $x \in X$, we have

$$\mu^{\dagger}_{\alpha}(x) = \mu(x) + \alpha \geq \mu(x) > \mu(x), \gamma = \mu^{m}_{\gamma}(x)$$

and so $\mu^{\dagger}_{\alpha}$ is s fuzzy extension of $\mu^{m}_{\gamma}$. Assume that $\mu^{m}_{\gamma}$ is a fuzzy $p$-ideal of $X$. Then by Theorem 3.16, $\mu$ is a fuzzy $p$-ideal of $X$. It follows from Theorem 3.2 that the fuzzy $\alpha$-translation $\mu^{\dagger}_{\alpha}$ of $\mu$ is a fuzzy $p$-ideal of $X$, for all $\alpha \in [0,\dagger]$.

Therefore, every fuzzy $\alpha$-translation $\mu^{\dagger}_{\alpha}$ of $\mu$ is a fuzzy $p$-ideal extension of the fuzzy $\gamma$-multiplication $\mu^{m}_{\gamma}$ of $\mu$.

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