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Solution of Second Order Initial Value Problem By Differential Transform Method

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ABSTRACT:

In this paper, we applied Differential Transform Method (DT Method) to solve a second order initial value problem $Y''= f(x, y)$ and comparing with other methods Multi-Step method and Implicit method. The methods are demonstrated and the superiority of the methods is discussed. A numerical experiment is given by applying DT Method, Numerov's method and the numerical differentiation methods of the corresponding orders.

Keywords: Differential Transform Method(DT Method), Multi-Step methods, Special Multi Step methods, Numerical Differentiation, Initial Value Problem.

INTRODUCTION: The differential transform is an analytic method for solving differential equations. The concept of the differential transform was first introduced by Zhou in 1986[10]. This method constructs an analytical solution in the form of a polynomial. It is different from the traditional higher order Taylor series method. The differential transform method is an alterative procedure for obtaining analytic Taylor series solution of the differential equations. A single step and multistep methods for the first order Initial value problem of ordinary differential equations has been made by several researchers and a detailed treatment of the subject has been provided by many authors ([1], [2], [4], [5], [6]).Special multistep methods based on numerical integration for the solution of the special second order differential equations have been derived in Henrici [3]. Methods of reduction of order for solving singularly perturbed two-point boundary value problems was considered in [5].We have derived the methods based on numerical differentiation for the special second order differential equation. Since the special methods that arise are Numerov's methods, Stormer's methods etc.we have studied the resulting implicit methods. The interesting feature of the implicit methods is that they contain only one derivative term, where as the Stormer's methods and Numerov's methods based on numerical integration contain more derivative terms. Obviously, application of the implicit methods is comparatively economical as there is only one function evaluation in each of them and more function evaluation in Stormer's and Numerov's methods. Further information can be had from ([7], [8], [9]).

DIFFERENTIAL TRANSFORM METHOD

Differential transformation of function $y(x)$ is defined as follows

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \quad (1)$$

In (1), $y(x)$ is the original function and $Y(k)$ is the transformed function. Differential inverse transform of $Y(k)$ is defined as follows

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k) \quad (2)$$

In fact, from (1) and (2), we obtain

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{d^k y(x)}{dx^k}\right] \quad (3)$$

Eq. (3) implies that the concept of differential transformation is derived from the Taylor series expansion. From the definitions (1) and (2), it is easy to obtain the following mathematical operations:

Original Function	Transform Function
$f(x) = g(x) \pm h(x)$	$F(k) = G(k) \pm H(k)$
$f(x) = cg(x)$	$F(k) = cG(k)$
$f(x) = e^x$	$F(k) = \frac{1}{k!}$
$f(x) = \frac{d^n g(x)}{dx^n}$	$F(k) = \frac{(k+n)!}{k!} G(k+n)$
$f(x) = g(x).h(x)$	$F(k) = \sum_{r=0}^{\infty} G(k) F(k-r)$
$f(x) = x^n$	$F(k) = \delta(k-n)$

GENERAL LINEAR MULTISTEP METHODS FOR SPECIAL SECOND ORDER DIFFERENTIAL EQUATIONS

The special second order differential equation

$$Y'' = f(x, y)\,,\; Y(0) = Y_0\,,\; Y'(0) = Y_0' \qquad (4)$$

Often arises in a number of applications such as mechanical problems without dissipation. A general linear multistep method of step number k for the numerical solution of (4) is given by

$$Y_{n+1} = \sum_{j=1}^{k} a_j y_{n+1-j} + h^2 \sum_{j=0}^{k} b_j y_{n+1-j} \qquad (5)$$

Where a_j, b_j are constants and h is the step size.

Introducing the polynomials

$$\rho(\xi) = \xi^k - \sum_{j=1}^{k} a_j \xi^{k-1} \text{ and } \sigma(\xi) = \sum_{j=1}^{k} b_j \xi^{k-1} \qquad (6)$$

(5) can be written as $\rho(E) y_{n-k+1} - h^2 \sigma(E) y''_{n-k+1} = 0$ (7)

Where E is the shift operator defined by $E(y_n) = y_{n+1}$.

Applying (7) to $y'' = \lambda y$, we get the characteristic equation

$$\rho(\xi) - \bar{h}\sigma(\xi) = 0\text{, where } \bar{h} = \lambda h^2 \qquad (8)$$

The roots ξ_1 of the characteristic equation (8) and $\bar{h}$ are in general, complex and the region of absolute stability is defined to be the region of the complex $\bar{h}$-plane such that the roots of the characteristic equation (8) lie within the unit circle whenever $\bar{h}$ lies in the interior of the region of absolute stability of R and its boundary by ∂R, then the locus of ∂R is given by

$$\bar{h}(Q) = \rho(e^{i\theta}) / \sigma(e^{i\theta}),\; 0 \le \theta \le 2\pi \qquad (9)$$

DERIVATIVE OF THE METHODS

Let p(x) be the backward difference interpolating polynomial of Y(x) at the (k+1) abscissas $x_{n+1}, x_n, \ldots\ldots\ldots\ldots, x_{n-k+1}$. Then p(x) is given by

$$p(x) = \sum_{m=0}^{k} (-1)^m \binom{-s}{m} \nabla^m y_{n+1} s = (x - x_{n+1}) / h \qquad (10)$$

Differentiating (10) twice with respect to x, we get

$$p''(x) = \left(\frac{1}{h^2}\right)\sum_{m=0}^{k}\frac{d^2}{ds^2}\left[(-1)^m\left({}^{-s}{}_{m}\right)\right]\nabla^m y_{n+1}$$

Replacing $y''(x)$ by $p''(x)$ in the equation (4) and putting $x = x_{n+1-r}$ i.e. s = -r, we get

$$\left(\frac{1}{h^2}\right)\sum_{m=0}^{k}\frac{d^2}{ds^2}\left[(-1)^m\left({}^{-s}{}_{m}\right)\right]_{s=-r}\nabla^m y_{n+1} = f_{n+1-r}$$ which takes the form

$$\sum_{m=0}^{k}\delta_{r,m}\nabla^m y_{n+1} = h^2 f_{n+1-r} \qquad (11)$$

Where $\delta_{r,m} = \frac{d^2}{ds^2}\left[(-1)^m\left({}^{-s}{}_{m}\right)\right]$ (12)

GENERATING FUNCTION FOR THE CCOEFFICIENTS $\delta_{r,m}$

Let $D_{r,t} = \sum_{m=0}^{\infty}\delta_{r,m}t^m$ (13)

Substituting (11) in (12), we get

$$D_{r,t} = \sum_{m=0}^{\infty}(-t)^m\frac{d^2}{ds^2}({}^{-s}c_m) \text{ at s = -r}$$

$$= \frac{d^2}{ds^2}\sum_{m=0}^{\infty}(-t)^m({}^{-s}c_m) \text{ at s = -r}$$

$$= \frac{d^2}{ds^2}e^{-s\log(1+t)} \text{ at s = -r}$$

$$\therefore \sum_{m=0}^{\infty}\delta_{t,m}t^m = (1-t)^{-s}\left[\log(1-t)\right]^2 \text{ at s = -r} \qquad (14)$$

IMPLICIT METHODS

Putting r = 0 is (14), we get the implicit method.

$$\sum_{m=0}^{k}\delta_{0,m}\nabla^m y_{n+1} = h^2 f_{n+1} \qquad (15)$$

Where from (15) $\delta_{0,m}$ is the coefficient of t^m in the expansion of $\left[\log(1-t)\right]^2$ in powers of 't'. The coefficient $\delta_{0,m}$ are tabulated below.

Table 1: Coefficients $\delta_{0,m}$; m = 0 (1) 8

m	0	1	2	3	4	5	6	7	8
$\delta_{0,m}$	0	0	1	1	11/12	5/6	137/180	7/10	1089/1680

Differences in (15) are expressed in terms of the function values,(15) takes of the form

$$\sum_{j=0}^{k} a_j y_{n+1-j} = h^2 f_{n+1} \qquad (16)$$

The coefficients a_j are tabulated below.

Table 2: Coefficients a_j ; j = 0 (1) k, k = 2 (1) 4

k	**0**	**1**	**2**	**3**	**4**
2	1	-2	1		
3	2	-5	4	-1	
4	35/12	-104/12	114/12	-56/12	11/12

The coefficients a_j in a slightly different from are given in[1] for 2(1)6. It can be seen that the local truncation error of the formula (16) is given by

$$\text{L.T.E} = \delta_{0,k+1} h^{k+1} y^{(k+1)}(\eta) \qquad (17)$$

It follows that the k-step method (16) has order (k-1). For the method (16) we have

$$\rho(\xi) = \sum_{j=0}^{k} a_j \xi^{k-j} \text{ and } \sigma(\xi) = \xi^k \qquad (18)$$

The roots of the characteristic equation $\rho(\xi) - \bar{h}\sigma(\xi) = 0$ tend to those of $\sigma(\xi) = 0$ which lie at the origin as $\bar{h} \to \infty$.

NUMERICAL EXPERIMENT

In this section, we have applied the DT Method, the Numerov's method given in Henrici[3] and the Implicit method to solve the differential equation

$$y'' = 2e^x + y \text{, with initial conditions } y(0) = 0 \text{, } y'(0) = -1 \qquad (19)$$

Taking the differential transform both sides of (19), we obtain

$$(k+2)(k+1)Y(k+2) = 2\frac{1}{k!} + Y(k) \qquad (20)$$

$$\text{and } Y(0) = 0, \; Y(1) = -1 \qquad (21)$$

where $Y(k)$ is the differential transform of y(x) .

Taking Eq.(21) in Eq.(20) and by recursive method, we have

$$Y(2) = 1 \qquad Y(3) = \frac{1}{6} \qquad Y(4) = \frac{1}{6} \qquad Y(5) = \frac{1}{40} \qquad Y(6) = \frac{1}{120}$$

$$Y(7) = \frac{1}{1007} \quad Y(8) = \frac{1}{5040} \quad Y(9) = \frac{1}{51840} \quad Y(10) = \frac{1}{362880} \ldots\ldots$$

The solution of equation(19) is

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k) = -x + x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{40}x^5 + \frac{1}{120}x^6 + \frac{1}{1008}x^7 + \frac{1}{5040}x^8 + \frac{1}{51840}x^9 + \ldots\ldots$$

In the interval [0, 2] with h = 0.01 the results are given in tables 3 & 4 and the corresponding graphs are shown in figure 1,2.

Now applying the implicit 4th order Numerov's method (Henrici[3])

$$y_{n+1} - 2y_n + y_{n-1} = (h^2/12)(f_{n+1} + 10y_n + y_{n-1}) \qquad (22)$$

The implicit 3rd order N.D. method (16) with k = 4 we get

$$y_{n+1} = (104/35)y_n - (114/35)y_{n-1} + (56/35)y_{n-2} - (11/35)y_{n-3} + (12/35)h^2 f_{n+1} \qquad (23)$$

In the interval [0, 2] with h = 0.01 and h = 0.02 and the results are given in tables 3 & 4 and the corresponding graphs are shown in figure 1,2.

Table 3

Solution by DT Method, 4th order Numerov and the Implicit 3rd order (k=4) N.D with h=0.01

x	Exact Solution	Solution by the D.T method	Absolute error	Numerical Solution by the Numerov method	Absolute error	Numerical Solution by Implicit 3rd method	Absolute error
0	0	0	0	0	0	0	0
0.1	-0.08981668	-0.08981640	0.000000028	-0.089810729	0.000005960	-0.089816689	0.000000260
0.2	-0.15839171	-0.15839145	0.000000026	-0.158370020	0.000021696	-0.158388940	0.000002503
0.3	-0.20408261	-0.20408294	0.000000033	-0.204031710	0.000050902	-0.204075320	0.000007614
0.4	-0.22477448	-0.22477477	0.000000029	-0.224677150	0.000097334	-0.224758120	0.000016659
0.5	-0.21782982	-0.21782997	0.000000015	-0.217669430	0.000160396	-0.217800180	0.000029787
0.6	-0.18003601	-0.18003588	0.000000013	-0.179795150	0.000240862	-0.179989170	0.000046730
0.7	-0.10754061	-0.10754051	0.000000010	-0.107203600	0.000337000	-0.107473720	0.000066801
0.8	0.004220664	0.004220757	0.000000009	0.0046671517	0.000446487	0.0043100547	0.000089339
0.9	0.160609010	0.160609270	0.000000026	0.161166700	0.000556610	0.1607217300	0.000112429
1.0	0.367878850	0.367879188	0.000000338	0.3685381400	0.000659286	0.3680127800	0.000133335
1.1	0.633286000	0.633286957	0.000000957	0.6340272400	0.000741243	0.6334366200	0.000150620
1.2	0.965213360	0.965215676	0.000002316	0.9659779100	0.000764548	0.9653726200	0.000155270
1.3	1.37331680	1.373316108	0.000000692	1.373976700	0.000659942	1.373469100	0.000152300
1.4	1.86867330	1.868666240	0.000007060	1.869059600	0.000386238	1.868796800	0.000012350
1.5	2.46396830	2.463951546	0.000016754	2.463902500	0.000065803	2.464032600	0.000064300
1.6	3.17370990	3.173668331	0.000041569	3.172979400	0.000730514	3.173678200	0.000037193
1.7	4.01443480	4.014352777	0.000082023	4.012793500	0.001641273	4.014252200	0.000454072
1.8	5.00500770	5.004838615	0.000169085	5.002136700	0.002861022	5.004573800	0.000442504
1.9	6.16685580	6.166546632	0.000309168	6.162427900	0.004427909	6.166088100	0.000785350
2.0	7.52437690	7.523809523	0.000567377	7.517968200	0.006408691	7.522314470	0.001246445

Table 4

Solution by DT Method, 4th order Numerov and the Implicit 3rd order (k=4) N.D with h=0.02

x	Exact Solution	Solution by the D.T method	Absolute error	Numerical Solution by the Numerov method	Absolute error	Numerical Solution by Implicit 3rd method	Absolute error
0	0	0	0	0	0	0	0
0.1	-0.08981668	-0.08981640	0.000000028	-0.089810746	0.000000774	-0.089816376	0.000000029
0.2	-0.15839171	-0.15839145	0.000000026	-0.158390880	0.000000834	-0.158390950	0.000000491
0.3	-0.20408261	-0.20408294	0.000000033	-0.204077180	0.000005364	-0.204081710	0.000001221
0.4	-0.22477448	-0.22477477	0.000000029	-0.224675988	0.000015079	-0.224772350	0.000002428
0.5	-0.21782982	-0.21782997	0.000000015	-0.217801570	0.000028133	-0.217825960	0.000004008
0.6	-0.18003601	-0.18003588	0.000000013	-0.179989460	0.000046432	-0.180029910	0.000005990
0.7	-0.10754061	-0.10754051	0.000000010	-0.107472360	0.000068247	-0.107532080	0.000008441
0.8	0.004220664	0.004220757	0.000000009	0.0043136887	0.000093024	0.0042324257	0.000011710
0.9	0.160609010	0.160609270	0.000000026	0.0160726960	0.000117659	0.1606250100	0.000015705
1.0	0.367878850	0.367879188	0.000000338	0.3680189800	0.000139713	0.3679001900	0.000020742
1.1	0.633286000	0.633286957	0.000000957	0.6334417500	0.000155746	0.6333149100	0.000027179
1.2	0.965213360	0.965215676	0.000002316	0.9653668400	0.000150024	0.9652523400	0.000034987
1.3	1.37331680	1.373316108	0.000000692	1.373428300	0.000111579	1.373366200	0.000045657
1.4	1.86867330	1.868666240	0.000007060	1.868697200	0.000023841	1.868735700	0.000058889
1.5	2.46396830	2.463951546	0.000016754	2.463860500	0.000107765	2.464047400	0.000071717
1.6	3.17370990	3.173668331	0.000041569	3.173415200	0.000294685	3.173678200	0.000088930
1.7	4.01443480	4.014352777	0.000082023	4.013891200	0.000553131	4.014546400	0.000100135
1.8	5.00500770	5.004838615	0.000169085	5.004120800	0.000886917	5.005132200	0.000115871
1.9	6.16685580	6.166546632	0.000309168	6.165532100	0.001336097	6.166996500	0.000123023
2.0	7.52437690	7.523809523	0.000567377	7.522479100	0.001897119	7.524518000	0.000126838

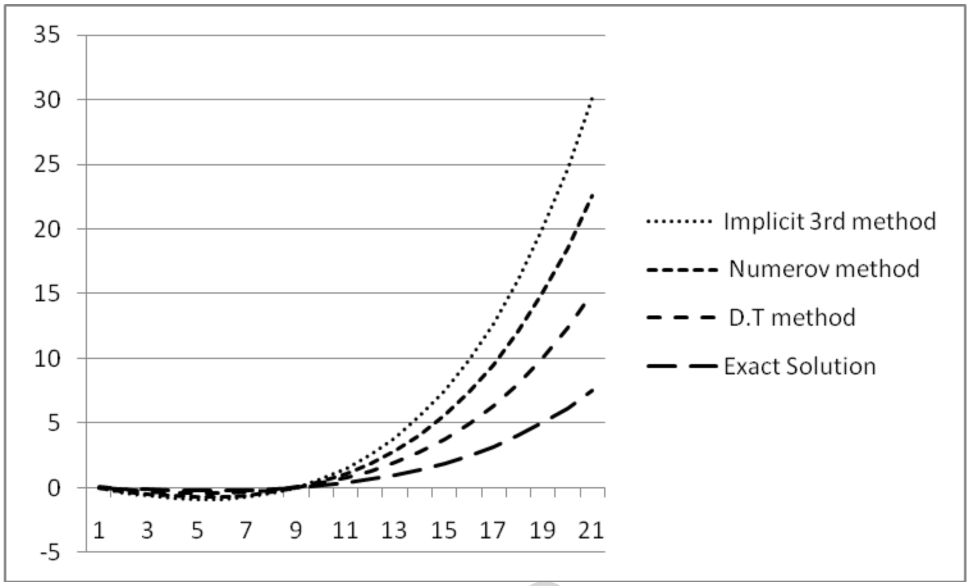

Figure 1

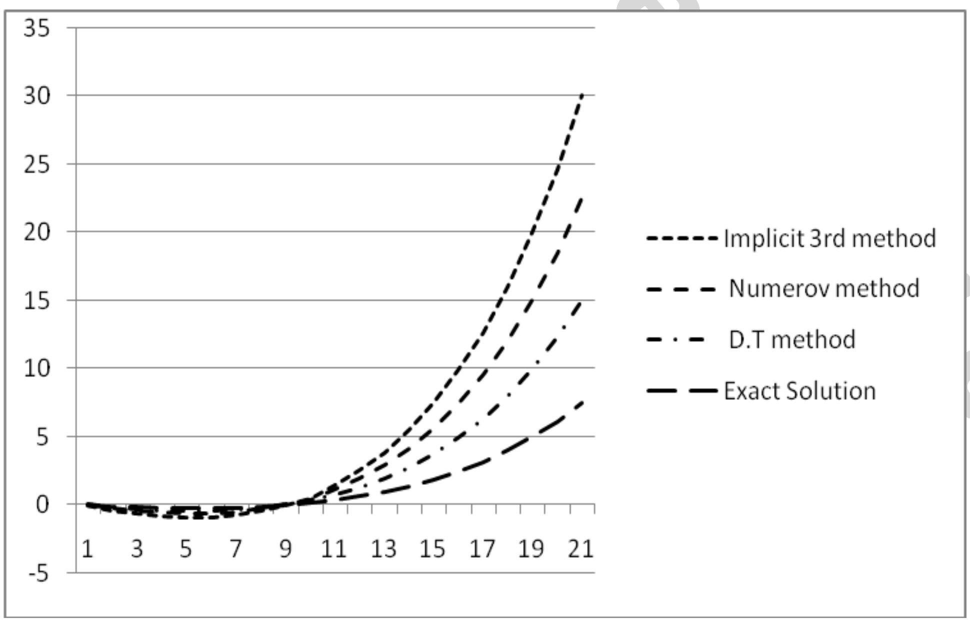

Figure 2

CONCLUSION

In this paper, we have shown that the differential transform method can be used successfully in solving the second order initial value problem and we compare the results with other alternate methods Numerov's method and Implicit method. It may be concluded that the absolute error in the DT Method comparatively with other methods is very small and it can be clearly observed in the graphs.

References

[1] Gear C.W Numerical initial value problems in ordinary differential equations,1971.

[2] Gragg.W.B, Statter.H.J. General Multistep predictor-corrector methods, J.ACM,11(1964), pp. 188-209.

[3] Henrici P. Discrete variable methods in ordinary differential equations, Wiley, 1962.

[4] Jain M K, Numerical solutions of differential equations, Wiley Eastern Limited, 1984.

[5] Klopfenstein R W, Numerical Differentiation Formulas for stiff systems of ordinary differential equations, RCA Review, 32 (1971).pp. 447-462.

[6] Lapidus L, Sein feld J H, Numerical solutions of ordinary differential equations, Academic press, 1971.

[7] Ram Chandra Rao P S. Special Multistep methods based on numerical differentiation for solving the initial value problem,Applied Mathematics and Computation Elsvier, 181,pp 500-510, 2006.

[8] Vishwa Prasad Rao S, Ram Chandra Rao P S. Transform Technique For The Solution Of A Class Of Differential Equation, International Journal of Applied Mathematical Analysis and Applications.Vol 3,No.2,pp. 161-166, 2008.

[9] Vishwa Prasad Rao S, Ram Chandra Rao P S, Prabhakara Rao C. Solution of Differential Equation from the Transform Technique, International Journal Of Computational Science and Mathematics, Vol 3,No.1, pp 121-125, 2011.

[10] Zhou. J. K. Differential transformation and its application for electrical circuits, Huarjung University Press, Wu-uhahn, China,(in Chinese), 1986.