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# GLOBAL STABILITY OF AN IMMIGRATING COMMENSAL – HOST MODEL WITH LIMITED RESOURCES

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**ABSTRACT**

In this paper we establish the global stability of an immigrating commensal – host model with limited resources, by constructing a suitable Liapunov's function in case of co-existent equilibrium state.

**AMD Classification**: 92D25, 92D40.

**Keywords**: Equilibrium state, Liapunov's function, Global stability.

## 1. INTRODUCTION

Phani Kumar and N.Ch.Pattabhi Ramacharyulu etc. [6] examined the local stability of an immigrating commensal – host model with limited resources on the quasi-linear basic balancing equations. The present investigation is mainly devoted to establish the global stability of the co-existent equilibrium state of an immigrating commensal – host model with limited resources ,by employing a properly constructed Liapunov's function.

## 2. LIAPUNOV'S STABILITY ANALYSIS

Many approaches are available for the stability analysis of linear,time-invarient systems. However for non-linear systems and/or time-varying systems, stability analysis may be extremely difficult or impossible. Liapunov Stability analysis is one method that may be applied for non-linear systems. In 1892 A.M. Liapunov introduced the direct method to study the global stability of equilibrium states in case of linear and non-linear systems. His method is based on the chief characteristic of constructing a scalar function called Liapunov's function. That is by using the direct method of Liapunov,we can determine the stability of a system without solving the state. This is quite advantageous because solving non-linear and/or time-invarying state equation is very difficult. To day this method is widely recognized as an efficient tool in theory of control systems, dynamical systems, systems with time lag, power system analysis, and time varying non-linear feed back systems, multi species ecological systems and so on.

The stability behaviour of solutions of linear and weakly non-linear system is done by using the techniques of variation of constants formulae and integral inequalities. So this analysis is confined to a small neighborhood of operating point i.e., local stability. Further, the techniques used there in require explicit knowledge of solutions of corresponding linear systems. The stability behavior of a physical system is discussed by several authors like Kapoor [2], Lotka [3], Ogata [4] Bhaskara Rama Sarma and N.Ch.PattabhiRamacharyulu [1], Lakshminarayan and N.Ch.Pattabhi Ramacharyulu [5] etc.

If the total energy of a physical system has a local minimum at a certain equilibrium point, then that point is stable. This idea was generalized by Liapunov to study stability problems in a broader context.

**2.1. STABILITY BY LIAPUNOV'S DIRECT METHOD**

Consider an autonomous system

$$\left.\begin{aligned} \frac{dx}{dt} &= F(x,y) \\ \frac{dy}{dt} &= G(x,y) \end{aligned}\right\} \qquad (1)$$

Assume that this system has an isolated critical point taken as (0, 0). Consider a function E(x, y) possessing continuous partial derivatives along the path of (1). This path is represented by C= [(x (t), y (t)] in the parametric form. E(x, y) can be regarded as a function of't' along C with rate of change

$$\frac{dE}{dt} = \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt}$$

$$\frac{dE}{dt} = \frac{\partial E}{\partial x}F(x,y) + \frac{\partial E}{\partial y}G(x,y) \qquad (2)$$

**2.2. DEFINITIONS**

1. E (x, y) is said to be positive definite if E (x, y)>0 $\forall$ (x,y) $\neq$ (0,0)
2. E (x, y) is said to be positive semi-definite if E (x, y) > 0 & E (0, 0) =0
3. E (x, y) is said to be negative definite if E (x, y) < 0
4. E (x, y) is said to be negative semi-definite if E (x, y) < 0 & E (0, 0) =0

A Positive definite function E (x, y) with the property that (2) is negative semi-definite is called a Liapunov's function for the system (1). The following theorem is the basic discovery.

**Theorem**: If there exists a Liapunov's function E (x, y) for the system (1), then the critical point (0,0) is stable. Further more, if this function has additional property that the function (2) is negative definite, then the critical point (0, 0) is asymptotically stable.

**3. BASIC EQUATIONS OF THE MODEL**

The basic equations for the growth rate of a flourishing commensal and host species with limited resources are given by

$$\left.\begin{aligned} \frac{dN_1}{dt} &= a_{11}\,[\,K_1 N_1 - N_1^2 + C\,N_1 N_2 + H_1] \\ \frac{dN_2}{dt} &= a_{22}\,[\,K_2 N_2 - N_2^2 + H_2\,] \end{aligned}\right\} \qquad (3)$$

## 4. THE EQULIBRIUM STATES

The system has one equilibrium state resulting from $\frac{dN_1}{dt} = 0$; $\frac{dN_2}{dt} = 0$

Co-existence state $E_1$:

$$E_1: \overline{N}_1 = \left(K_1 + C\left(K_2 + \frac{H_2}{K_2}\right)\right) + \frac{H_1}{K_1 + C\left(K_2 + \frac{H_2}{K_2}\right)}; \overline{N}_2 = K_2 + \frac{H_2}{K_2}$$

where Ki= $\frac{a_i}{a_{ii}}$, i = 1, 2 are the carrying capacities of $N_i$.

C = $\frac{a_{12}}{a_{11}}$, the commensal co-efficient.

## 5. LOCAL STABILITY ANALYSIS

The present authors [6] discussed the local stability of the above equilibrium state and which is stable.

## 6. LIAPUNOV'S FUNCTION FOR GLOBAL STABILITY

The linearized perturbed equations over the perturbations ($u_1$, $u_2$) of the system (3) are

$$\frac{du_1}{dt} = -a_{11}\left(\overline{N_1} + \frac{H_1}{K_1 + C\overline{N_2}}\right)u_1 + Ca_{11}\overline{N_1}u_2 \tag{4}$$

$$\frac{du_2}{dt} = -a_{22}\left(\overline{N_2} + \frac{H_2}{K_2}\right)u_2 \tag{5}$$

The characteristic equation is

$$\left(\lambda + a_{11}\left(\overline{N_1} + \frac{H_1}{K_1 + C\overline{N_2}}\right)\right)\left(\lambda + a_{22}\left(\overline{N_2} + \frac{H_2}{K_2}\right)\right) = 0$$

i.e., $\lambda^2 + \left[a_{22}\left(\overline{N}_2 + \frac{H_2}{K_2}\right) + a_{11}\left(\overline{N}_1 + \frac{H_1}{K_1 + C\overline{N_2}}\right)\right]\lambda + a_{22}a_{11}\left(\overline{N}_1 + \frac{H_1}{K_1 + C\overline{N_2}}\right)\left(\overline{N}_2 + \frac{H_2}{K_2}\right) = 0$

This is in the form of $\lambda^2 + p\lambda + q = 0$

where

$$p = a_{22}\left(\overline{N}_2 + \frac{H_2}{K_2}\right) + a_{11}\left(\overline{N}_1 + \frac{H_1}{K_1 + C\overline{N_2}}\right) > 0 \tag{6}$$

$$q = a_{11}a_{22}\left(\overline{N}_1 + \frac{H_1}{K_1 + C\overline{N_2}}\right)\left(\overline{N}_2 + \frac{H_2}{K_2}\right) > 0 \tag{7}$$

Therefore the conditions for Liapunov's function are satisfied.

Now define

$$E(u_1, u_2) = \tfrac{1}{2}(au_1^2 + 2b\, u_1u_2 + cu_2^2) \tag{8}$$

where

$$a = \frac{a_{22}^2\left(\overline{N}_2 + \frac{H_2}{K_2}\right)^2 + \left[a_{11}a_{22}\left(\overline{N}_2 + \frac{H_2}{K_2}\right)\left(\overline{N}_1 + \frac{H_1}{K_1 + C\overline{N_2}}\right)\right]}{D} \tag{9}$$

$$b = \frac{Ca_{11}a_{22}\overline{N}_1\left(\overline{N}_2 + \frac{H_2}{K_2}\right)}{D} \tag{10}$$

$$c = \frac{a_{11}{}^2\left(\overline{N}_1 + \frac{H_1}{K_1 + C\overline{N_2}}\right)^2 + C^2 a_{11}{}^2\overline{N_1}^2 + \left[a_{11}a_{22}\left(\overline{N}_2 + \frac{H_2}{K_2}\right)\left(\overline{N}_1 + \frac{H_1}{K_1 + C\overline{N_2}}\right)\right]}{D} \tag{11}$$

$$D = pq = \left[a_{22}\left(\overline{N}_2 + \frac{H_2}{K_2}\right) + a_{11}\left(\overline{N}_1 + \frac{H_1}{K_1 + C\overline{N_2}}\right)\right]\left[a_{22}a_{11}\left(\overline{N}_1 + \frac{H_1}{K_1 + C\overline{N_2}}\right)\left(\overline{N}_2 + \frac{H_2}{K_2}\right)\right] \tag{12}$$

From (6) and (7) it is clear that D > 0 and a > 0.

Also,

$D^2\,(ac\text{-}b^2) =$

$$D^2\left\{\left(\frac{a_{22}{}^2\left(\overline{N}_2 + \frac{H_2}{K_2}\right)^2 + a_{11}a_{22}\left(\overline{N}_2 + \frac{H_2}{K_2}\right)\left(\overline{N}_1 + \frac{H_1}{K_1 + C\overline{N_2}}\right)}{D}\right)\left(\frac{a_{11}{}^2\left(\overline{N}_1 + \frac{H_1}{K_1 + C\overline{N_2}}\right)^2 + C^2 a_{11}{}^2\overline{N_1}^2 + a_{11}a_{22}\left(\overline{N}_2 + \frac{H_2}{K_2}\right)\left(\overline{N}_1 + \frac{H_1}{K_1 + C\overline{N_2}}\right)}{D}\right) - \left(\frac{Ca_{11}a_{22}\overline{N_1}\left(\overline{N}_2 + \frac{H_2}{K_2}\right)}{D}\right)^2\right\}$$

$$\Rightarrow D^2\,(ac - b^2) > 0$$

$$\Rightarrow b^2 - ac < 0 \tag{13}$$

$\therefore$ The function $E\,(u_1, u_2)$ at (8) is positive definite.

Further

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} =$$

$$\left(au_1 + bu_2\right)\left(-a_{11}\left(\overline{N_1} + \frac{H_1}{K_1 + C\overline{N_2}}\right)u_1 + Ca_{11}\overline{N_1}u_2\right) + \left(bu_1 + cu_2\right)\left(-a_{22}\left(\overline{N_2} + \frac{H_2}{K_2}\right)u_2\right)$$

$$= -a\,a_{11}\left(\overline{N_1} + \frac{H_1}{K_1 + C\overline{N_2}}\right)u_1{}^2 + \left(a\;Ca_{11}\overline{N_1} - ba_{11}\left(\overline{N_1} + \frac{H_1}{K_1 + C\overline{N_2}}\right) - ba_{22}\left(\overline{N_2} + \frac{H_2}{K_2}\right)\right)u_1u_2$$
$$+\left(bCa_{11}\overline{N_1} - ca_{22}\left(\overline{N_2} + \frac{H_2}{K_2}\right)\right)u_2{}^2 \tag{14}$$

Substituting the values of a, b and c in (14) we get

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt}+\frac{\partial E}{\partial u_2}\frac{du_2}{dt}=$$

$$-\left(\frac{a_{11}a_{22}\left(\overline{N}_1+\frac{H_1}{K_1+C\overline{N_2}}\right)\left(\overline{N}_2+\frac{H_2}{K_2}\right)\left[a_{22}\left(\overline{N}_2+\frac{H_2}{K_2}\right)+a_{11}\left(\overline{N}_1+\frac{H_1}{K_1+C\overline{N_2}}\right)+\right]}{D}\right)u_1^{\,2}-$$

$$\left(\frac{a_{11}a_{22}\left(\overline{N}_1+\frac{H_1}{K_1+C\overline{N_2}}\right)\left(\overline{N}_2+\frac{H_2}{K_2}\right)\left[a_{22}\left(\overline{N}_2+\frac{H_2}{K_2}\right)+a_{11}\left(\overline{N}_1+\frac{H_1}{K_1+C\overline{N_2}}\right)\right]}{D}\right)u_2^{\,2}$$

$$= -\left[\frac{D}{D}\right]u_1^{\,2}-\left[\frac{D}{D}\right]u_2^{\,2} \qquad (15)$$

$$= -\left(u_1^{\,2}+u_2^{\,2}\right) \qquad (16)$$

$$\therefore \frac{\partial E}{\partial u_1}\frac{du_1}{dt}+\frac{\partial E}{\partial u_2}\frac{du_2}{dt} = -\left(u_1^{\,2}+u_2^{\,2}\right)$$

which is clearly negative definite.

So, E $(u_1, u_2)$ is a Liapunov function for the Linear system.

Next we prove that E $(u_1, u_2)$ is also a Liapunov function for the non-linear system

If $f_1$ and $f_2$are two functions in $N_1$ and $N_2$ defined by

$$f_1(N_1, N_2)=a_1N_1-a_{11}N_1^{\,2}+a_{12}N_1N_2+a_{11}H_1 \qquad (17)$$

$$f_2(N_1, N_2)=a_2N_2-a_{22}N_2^{\,2}+a_{22}H_2 \qquad (18)$$

Now we have to show that $\frac{\partial E}{\partial u_1}f_1+\frac{\partial E}{\partial u_2}f_2$ is negative definite

By putting $N_1 = \overline{N}_1 + u_1$ and $N_2 = \overline{N}_2 + u_2$ in (17) and (18) we get

$$\frac{du_1}{dt} = a_{11}(\overline{N}_1+u_1) - a_{11}(\overline{N}_1+u_1)^2 + a_{12}(\overline{N}_1+u_1)(\overline{N}_2+u_2) + a_{11}H_1$$

$$= -a_{11}\left[\frac{H_1}{K_1+C\overline{N}_2}+\overline{N}_1\right]u_1+Ca_{11}\overline{N}_1u_2+F(u_1,u_2)$$

where $F(u_1, u_2) = -a_{11}u_1^2 + a_{12}u_1u_2$

$$\Rightarrow f_1(u_1, u_2) = \frac{du_1}{dt} = - -a_{11}\left[\frac{H_1}{K_1+CN_2}+\overline{N}_1\right]u_1+Ca_{11}\overline{N}_1u_2+F(u_1,u_2) \qquad (19)$$

Similarly

$$\frac{du_2}{dt}=a_2(\overline{N}_2+u_2)-a_{22}(\overline{N}_2+u_2)^2-a_{22}u_2$$

$$=-a_{22}\left(\overline{N}_2+\frac{H_2}{K_2}\right)u_2+G(u_1,u_2)$$

where $G(u_1, u_2) = -a_{22}u_{22}$

$$\Rightarrow f_2(u_1, u_2) = \frac{du_2}{dt}=-a_{22}\left(\overline{N}_2-\frac{H_2}{K_2}\right)u_2+G(u_1,u_2) \qquad (20)$$

From (8)

$$\frac{\partial E}{\partial u_1} = au_1 + bu_2 \tag{21}$$

$$\frac{\partial E}{\partial u_2} = bu_1 + cu_2 \tag{22}$$

Now $\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2 = (au_1 + bu_2)\left\{-a_{11}\left[\overline{N}_1 + \frac{H_1}{K_1 + C\overline{N_2}}\right]u_1 + Ca_{11}\overline{N}_1 u_2 + F(u_1, u_2)\right\}$

$$+ \ (bu_1 + cu_2)\left\{-a_{22}\left(\overline{N}_2 + \frac{H_2}{K_2}\right)u_2 + G(u_1, u_2)\right\}$$

$$= \ (au_1 + bu_2)\left[-a_{11}\left[\overline{N}_1 + \frac{H_1}{K_1 + C\overline{N_2}}\right]u_1 + Ca_{11}\overline{N}_1 u_2\right] + (bu_1 + cu_2)\left(-a_{22}\left(\overline{N}_2 + \frac{H_2}{K_2}\right)\right)u_2 + (au_1 + bu_2)F(u_1, u_2) + \ (bu_1 + cu_2)G(u_1, u_2) \tag{23}$$

$$\frac{\partial E}{du_2} f_1 + \frac{\partial E}{du_1} f_2 = -(u_1^{\ 2} + u_2^{\ 2}) + (au_1 + bu_2)F(u_1, u_2) + (bu_1 + cu_2)G(u_1, u_2) \tag{24}$$

By introducing polar coordinates (24) becomes

$$\frac{\partial E}{du_2} f_1 + \frac{\partial E}{du_1} f_2 = -r^2 + r[(a\cos\theta + b\sin\theta)F(u_1, u_2) + (b\cos\theta + c\sin\theta)G(u_1, u_2) \tag{25}$$

Let us denote the largest of the numbers | $a$|, |$b$|, |$c$| by $K$.

Our assumptions imply that | F ($u_1$, $u_2$)| < $\frac{r}{6K}$ and | G($u_1$, $u_2$)| < $\frac{r}{6K}$, for all sufficiently small r > 0.

So $$\frac{\partial E}{\partial u_1} f_1 + \ \frac{\partial E}{\partial u_2} f_2 < -r^2 + \frac{4Kr^2}{6K} \ = - \frac{r^2}{3} < 0 \tag{26}$$

Thus E ($u_1$, $u_2$) is a positive definite function with the property that

$\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2$ is negative definite. (27)

$\therefore$ The equilibrium point $E_1$ is **asymptotically stable.**

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