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AN ENEMY AND TIME DELAYED AMMENSAL SPECIES PAIR-LIAPUNOV'S STABILITY ANALYSIS

K.V.L.N.ACHARYULU

Faculty of Mathematics
Bapatla Engineering College
Bapatla-522101
Email:kvlna@yahoo.com

ABSTRCT

In the paper ,The global stability is proved at co-existence state with the help of a properly Liapunov's function for the mathematical model of a delayed Ammensal and enemy species pair with limited resources.

AMS Classification: 92 D 25, 92 D 40

Key words: Equilibrium states, Stability, Liapunov's function for global stability

1) Introduction:

The present investigation is principally concentrated on the establishment of the global stability of a delayed Ammensal and enemy species pair with limited resources at the co-existent equilibrium state by utilizing a property constructed by Liapunov's function. Acharyulu.K.V.L.N. and Pattabhi Ramacharyulu. N.Ch. [1-7] analyzed the global stability of an Ammensal- enemy system on the quasi-linear basic balancing equations at various resources with Liapunov's criteria. Local stability analysis for an Ammensal- enemy species with various resources in different cases has been also carried out in the author's earlier work. Many authors like Lotka[9], Kapur[8] etc. applied this method in various situations for building the global stability.

Notations adopted

N_1 and N_2 are the populations of the Ammensal and enemy species with natural growth rates a_1 and a_2 respectively.

a_{11} = The rate of decrease of the Ammensal due to insufficient food.

a_{12} = The rate of increase of the Ammensal due to inhibition by the enemy.

a_{22} = The rate of decrease of the enemy due to insufficient food.

$K_i = a_i / a_{ii}$ are the carrying capacities of N_i, i = 1, 2

$\alpha = a_{12} / a_{11}$ is the coefficient of Ammensalism.

ψ = the delay in receiving benefit from the enemy, which is a constant.

The state variables N_1 and N_2 as well as the model parameters a_1, a_2, a_{11}, a_{22}, K_1, $K_{2,}$ α and ψ are assumed to be non-negative constants.

2)Basic equations:

(i). Equation for the growth rate of Ammensal species (S_1)

$$\frac{dN_1}{dt} = a_{11} [K_1 N_1 - N_1^2 - \alpha\left(t-\psi\right) N_1 N_2] \tag{1}$$

(ii) Equation for the growth rate of enemy species (S_2)

$$\frac{dN_2}{dt} = a_{22} [K_2 N_2 - N_2^2] \tag{2}$$

3) Equilibrium states:

The system under investigation has four equilibrium states:

E$_1$: $\overline{N_1} = 0;\ \overline{N_2} = 0$

In this state both Ammensal and enemy are washed out

E$_2$: $\overline{N_1} = 0;\ \overline{N_2} = K_2$

In this state, only enemy survives and the Ammensals are washed out

E$_3$: $\overline{N_1} = K_1;\ \overline{N_2} = 0$

In this state, only Ammensal survives and enemies are washed out

E$_4$: $\overline{N_1} = K_1 - \alpha K_2;\ \overline{N_2} = K_2$ (3)

The state, in which the Ammensal and enemy co-exist, is called "The normal steady state".

4) Global stability by Liapunov's criteria:

The linearized perturbed equations over the perturbations (U_1, U_2) are

$$\frac{dU_1}{dt} = -a_{11}\overline{N_1}U_1 - a_{12}\overline{N_1}\left(1+a_2\psi\right)U_2 \tag{4}$$

$$\frac{dU_2}{dt} = -a_{22}\overline{N_2}U_2 \tag{5}$$

The corresponding characteristic equation is

$$(\lambda + a_{11}\overline{N_1})(\lambda + a_{22}\overline{N_2}) = 0 \tag{6}$$

$$\Rightarrow \lambda^2 + (a_{11}\overline{N_1} + a_{22}\overline{N_2})\lambda + a_{11}a_{22}\overline{N_1}\,\overline{N_2} = 0 \tag{7}$$

Equation (7) is of the form of $\lambda^2 + p\lambda + q = 0$

where

$$p = a_{11}\overline{N_1} + a_{22}\overline{N_2} > 0 \tag{8}$$

$$q = a_{11}a_{22}\overline{N_1}\,\overline{N_2} > 0 \tag{9}$$

$\therefore$ The conditions for the existence of Liapunov's function are satisfied.

Now we define

$$E(U_1, U_2) = ½ (aU_1^2 + 2 b U_1 U_2 + c U_2) \tag{10}$$

where

$$a = \frac{(a_{22}\overline{N_2})^2 + (a_{11}a_{22}\overline{N_1}\,\overline{N_2})}{D} \tag{11}$$

$$b = \frac{-a_{12}\, a_{22}\, \overline{N_1}\, \overline{N_2}\left(1+a_2\psi\right)}{D} \qquad (12)$$

$$c = \frac{(a_{11}\overline{N_1})^2 + (a_{12}\overline{N_1}(1+a_2\psi))^2 + (a_{11}a_{22}\overline{N_1}\,\overline{N_2})}{D} \qquad (13)$$

where D = pq = ($a_{11}\overline{N_1} + a_{22}\overline{N_2}$) ($a_{11}\, a_{22}\, \overline{N_1}\, \overline{N_2}$) (14)

From equations (8) and (9) it is clear that D > 0 and a > 0

Also $D^2\,(ac - b^2)$

$$= D^2\left[\left(\frac{(a_{22}\overline{N_2})^2 + a_{11}a_{22}\overline{N_1}\,\overline{N_2}}{D}\right)\left(\frac{(a_{11}\overline{N_1})^2 + (a_{12}\overline{N_1}(1+a_2\psi))^2 + a_{11}a_{22}\overline{N_1}\,\overline{N_2}}{D}\right) - \frac{(a_{12}a_{22}\overline{N_1}\,\overline{N_2})^2}{D^2}\right]$$

$$= 2a_{11}{}^2a_{22}{}^2\overline{N_1}^2\,\overline{N_2}^2 + a_{11}a_{22}\overline{N_1}\,\overline{N_2}\left(a_{22}{}^2\overline{N_2}^2 + a_{11}{}^2\overline{N_1}^2 + a_{12}{}^2\overline{N_1}^2(1+a_2\psi)^2\right) > 0 \qquad (15)$$

$\Rightarrow D^2\,(\,ac - b^2\,) > 0$

$\Rightarrow ac - b^2 > 0$ i.e., $b^2 - ac < 0$ (16)

$\therefore$ The function E (U_1, U_2) at (10) is positive definite.

Further

$$\frac{\partial E}{\partial U_1}\frac{dU_1}{dt} + \frac{\partial E}{\partial U_2}\frac{dU_2}{dt} = \left[(aU_1 + bU_2)(-a_{11}\overline{N_1}U_1 - a_{12}\overline{N_1}(1+a_2\psi)U_2) + (bU_1 + cU_2)(-a_{22}\overline{N_2}U_2)\right]$$

$$= -aa_{11}\overline{N_1}U_1{}^2 - (aa_{12}\overline{N_1}(1+a_2\psi) - ba_{11}\overline{N_1} - ba_{22}\overline{N_2})U_1U_2 + (-ba_{12}\overline{N_1}(1+a_2\psi) - ca_{22}\overline{N_2})U_2{}^2 \qquad (17)$$

Substituting the values of a, b and c from (11) (12) and (13) in (17) we get

$$\frac{\partial E}{\partial U_1}\frac{dU_1}{dt} + \frac{\partial E}{\partial U_2}\frac{dU_2}{dt} = -\left[\frac{a_{22}{}^2\overline{N_2}^2 + a_{11}\, a_{22}\,\overline{N_1}\,\overline{N_2}}{D}\right]a_{11}\overline{N_1}U_1{}^2 +$$

$$\left[-\left(\frac{a_{22}{}^2\overline{N_2}^2 + a_{11}a_{22}\overline{N_1}\,\overline{N_2}}{D}\right)a_{12}\overline{N_1}(1+a_2\psi) + \left(\frac{a_{12}a_{22}\overline{N_1}\,\overline{N_2}(1+a_2\psi)}{D}\right)a_{11}\overline{N_1} + \left(\frac{a_{12}a_{22}\overline{N_1}\,\overline{N_2}(1+a_2\psi)}{D}\right)a_{22}\overline{N_2}\right]U_1U_2$$

$$+\left[\left(\frac{a_{12}a_{22}\overline{N_1}\,\overline{N_2}(1+a_2\psi)}{D}\right)a_{12}\overline{N_1}(1+a_2\psi) - \left(\frac{a_{11}{}^2N_1{}^2 + a_{12}{}^2\overline{N_1}^2(1+a_2\psi)^2 + a_{11}a_{22}N_1\overline{N_2}}{D}\right)a_{22}\overline{N_2}\right]U_2{}^2$$

$$\frac{\partial E}{\partial U_1}\frac{dU_1}{dt} + \frac{\partial E}{\partial U_2}\frac{dU_2}{dt} = -\left[\left(\frac{a_{11}a_{22}\overline{N_1}\,\overline{N_2}(a_{11}\overline{N_1} + a_{22}\overline{N_2})}{D}\right)U_1^2 + \left(\frac{a_{11}a_{22}\overline{N_1}\,\overline{N_2} + (a_{11}N_1 + a_{22}\overline{N_2})}{D}\right)U_2{}^2\right]$$

$$= -\frac{1}{D}\left[DU_1{}^2 + DU_2{}^2\right] \qquad (18)$$

$$= -\left(\,U_1{}^2 + U_2{}^2\,\right) \qquad (19)$$

$$\therefore \frac{\partial E}{\partial U_1}\frac{dU_1}{dt} + \frac{\partial E}{\partial U_2}\frac{dU_2}{dt} = -(U_1{}^2 + U_2{}^2) \qquad (20)$$

which is clearly negative definite

So, E (U_1, U_2) at (10) is a Liapunov function for the Linear system.

Next we have prove that E (U_1, U_2) is also a Liapunov function for the non-linear system.

If f_1 and $f_2$are two functions in N_1 and N_2 are defined by

$$f_1 (N_1, N_2) = N_1 [a_1 - a_{11} N_1 - a_{12} N_2(t- \psi)] \tag{21}$$

$$f_2(N_1, N_2) = N_2 [a_2 - a_{22} N_2] \tag{22}$$

Now we have to show that $\frac{\partial E}{\partial U_1} f_1 + \frac{\partial E}{\partial U_2} f_2$ is negative definite

Putting $N_1 = \overline{N_1} + U_1$; $N_2 = \overline{N_2} + U_2$ in (1) and (2) we get

$$\frac{dU_1}{dt} = \left(\overline{N_1} + U_1\right)\left[a_1 - a_{11}\left(\overline{N_1} + U_1\right) - a_{12}\left[\left(1 - a_2\psi\right)\left(\overline{N_2} + U_2\right) + a_{22}\left(\overline{N_2} + U_2\right)^2 \psi\right]\right]$$

$$= a_1\overline{N_1} - a_{11}\overline{N_1}^2 - a_{11}\overline{N_1}U_1 - a_{12}\left(1 - a_2\psi\right)\overline{N_1}\,\overline{N_2} - a_{12}\left(1 - a_2\psi\right)\overline{N_1}\psi_2$$

$$-a_{12}a_{22}\overline{N_2}^2\overline{N_1}\psi - 2a_{12}a_{22}\overline{N_1}\,\overline{N_2}U_2\psi - a_{12}a_{22}\overline{N_1}U_2^2\psi$$

$$+a_1U_1 - a_{11}\overline{N_1}U_1 - a_{12}\left(1 - a_2\psi\right)\overline{N_2}U_1 - a_{12}\left(1 - a_2\psi\right)U_1U_2 - a_{11}U_1^2 - a_{12}a_{22}\overline{N_2}^2U_1\psi$$

$$-2a_{12}a_{22}\overline{N_2}U_1U_2\psi - a_{12}a_{22}U_2^2U_1\psi$$

$$\Rightarrow f_1 (u_1, u_2) = \frac{dU_1}{dt} - a_{11}\overline{N_1}U_1 - a_{12}\left(1 - a_2\psi\right)\overline{N_1}\, U_2 + F(U_1, U_2) \tag{23}$$

where $F(U_1, U_2)$

$$= -2a_{12}a_{22}\overline{N_1}\,\overline{N_2}U_2\psi - a_{12}a_{22}\overline{N_1}U_2^2\psi - \left(a_{12}\left(1 - a_2\psi\right) + 2a_{12}a_{22}\overline{N_2}\psi\right)U_1U_2 - a_{11}U_1^2 - a_{12}a_{22}U_2^2U_1\psi$$

(24) Similarly $\frac{dU_2}{dt} = (\overline{N_2} + U_2)(a_2 - a_{22}(\overline{N_2} + U_2))$

$$= a_2\overline{N_2} - a_{22}\overline{N_2}^2 - a_{22}\overline{N_2}U_2 + a_2U_2 - a_{22}\overline{N_2}U_2 - a_{22}U_2^2$$

$$= - a_{22}\overline{N_2}\, U_2 + U_2 (a_2 - a_{22}\overline{N_2}) - a_{22} U_2^2$$

$$\Rightarrow f_2 (U_1, U_2) = \frac{dU_2}{dt} = - a_{22}\overline{N_2}\, U_2 + G(U_1, U_2) \tag{25}$$

where, $G(U_1, U_2) = - a_{22} U_2^2$ (26)

From (10)

$$\frac{\partial E}{\partial u_1} = a U_1 + bU_2 \tag{27}$$

$$\frac{\partial E}{\partial u_2} = bU_1 + cU_2 \tag{28}$$

Now

$$\frac{\partial E}{\partial U_1} f_1 + \frac{\partial E}{\partial U_2} f_2 = (aU_1 + bU_2) [- a_{11}\overline{N_1}\, U_1 - a_{12}\left(1 - a_2\psi\right)\overline{N_1}\, U_2 + F(U_1, U_2)]$$

$$+ (bU_1 + cU_2) [- a_{22}\overline{N_2}\, U_2 + G(U_1, U_2)]$$

$$= [(aU_1 + bU_2) (- a_{11}\overline{N_1}\, U_1 - a_{12}\left(1 - a_2\psi\right)\overline{N_1}\, U_2) + (bU_1 + cU_2) (- a_{22}\overline{N_2}\, U_2)]$$

$$+ (aU_1 + bU_2) F(U_1, U_2) + (bU_1 + cU_2) G (U_1, U_2) \tag{29}$$

$$\frac{\partial E}{\partial U_1} f_1 + \frac{\partial E}{\partial U_2} f_1 = - (U_1^2 + U_2^2) + (aU_1 + bU_2) F(U_1, U_2) + (bU_1 + cU_2) G (U_1, U_2) \tag{30}$$

By introducing the polar Co-ordinates $U_1 = r \cos\theta$ $U_2 = r \sin\theta$, equation (30) can be written as

$$\frac{\partial E}{\partial U_1} f_1 + \frac{\partial E}{\partial U_2} f_2 = -r^2 + r\,[(a\cos\theta + b\sin\theta) + F(U_1, U_2) + (b\cos\theta + c\sin\theta)\,G(U_1, U_2) \qquad (31)$$

Let us denote the largest of the numbers |a|, |b|, |c| by M

Our assumptions imply that $|F(U_1, U_2)| < \frac{r}{6M}$ and $|G(U_1, U_2)| < \frac{r}{6M}$, for all sufficiently small r > 0.

So, $$\frac{\partial E}{\partial U_1} f_1 + \frac{\partial E}{\partial U_2} f_2 < -r^2 + \frac{4Mr^2}{6M} = -\frac{r^2}{3} < 0. \qquad (32)$$

Thus E (U_1, U_2) is a positive definite function with the condition

$$\frac{\partial E}{\partial U_1} f_1 + \frac{\partial E}{\partial U_2} f_2 \text{ is negative definite} \qquad (33)$$

Hence, it is **asymptotically stable.**

Conclusion: The global stability is proved with the help of a properly Liapunov's function for the mathematical model of a delayed Ammensal and enemy species pair with limited resources.

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