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AN UNSTEADY LAMINAR VISCOUS FLOW IN A STRAIGHT PIPE OF EQUILATERAL TRIANGULAR CROSS SECTION

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ABSTRACT:

The present paper deals with an unsteady laminar viscous flow through a straight tube of an equilateral triangular cross section. The flow is induced by a time dependant periodic pressure gradient f(t) down the tube length, which has been expanded as Fourier trigonometric series. The basic field equation is solved to yield the velocity distribution. Expressions for shear stress on the tube wall and the flow rate across the cross section have been obtained. The cases of flow for large and small frequencies have been derived.

Introduction:

The problem of an unsteady laminar flow a viscous incompressible homogeneous fluid through a long straight pipe of an equilateral triangular cross section is investigated. The flow is influenced by a periodic pressure gradient down the pipe length, which is expressed as a sequence of periodic pulses superposed over a constant pressure gradient. The field equation is solved analytically to obtain the velocity field. Using this, the flow rate across the cross section and the skin friction are computed. The limiting cases of large and small frequencies of the oscillating pressure gradient are also examined. Such a flow through a tube of circular cross section was investigated earlier by Uchida [1].

FORMULATION AND SOLUTION OF THE PROBLEM:

Consider an unsteady laminar viscous flow through a tube of an equilateral triangular cross section under the influence of time dependant pressure gradient f(t) down the tube length. Choosing a system of rectangular Cartesian coordinates (x, y, z) with Z-coordinate parallel to the axis of the tube. The rectilinear flow through tube may be expressed by the velocity (o, o, w(x, y)). Then the equation of continuity shows that w is independent of Z.

The Navier stokes equations in the x, y directions reduce to $\frac{\partial \mathrm{p}}{\partial x} = \frac{\partial p}{\partial y} = 0$ and the equation of motion in the Z – direction reduces to

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \upsilon\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \qquad (1)$$

where p is the pressure, ρ is the density, and $\upsilon = \frac{\mu}{\rho}$ = Kinematic viscousity coefficient of the fluid. We consider here the flow under the time dependant pressure gradient $-\frac{1}{\rho}\frac{\partial p}{\partial z} = f(t)$

This function $f(t)$ can be represented as a fourier series in the interval $\left(0, \frac{2\pi}{\sigma}\right)$.

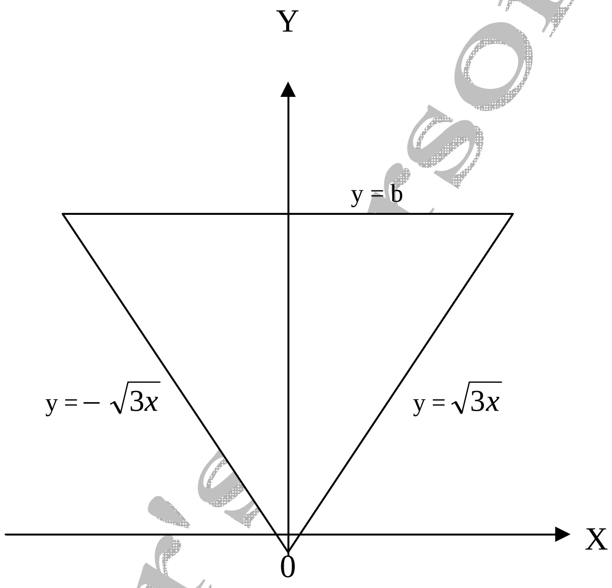
i.e. $-\frac{1}{\rho}\frac{\partial p}{\partial z} = \partial_0 + \sum_{n=1}^{\infty} \left(\alpha_{c_n} \cos n\sigma t + i\alpha_{s_n} \sin n\sigma t\right) = \alpha_0 + \operatorname{Re} \sum_{n=1}^{\infty} \alpha_n e^{in\sigma t}$

where $\alpha_n = \alpha_{c_n} - i\alpha_{sn}$ (2)

σ may be interpreted as the fundamental frequency, nσ is the nth mode frequency of $f(t)$, $\frac{2\pi}{\sigma}$ is the fundamental period and $\frac{2\pi}{n\sigma}$ is the period in the nth mode of vibration of $f(t)$

The tube boundary is given by $\Gamma: (y-b)(y^2 - 3x^2) = 0$ (3)

The velocity field satisfies the equation of motion (1) and vanishes on the boundary Γ.



Let the velocity field expressed as

$$w(x,y,t) = w_0(x,y) + \operatorname{Re}\sum_{n=1}^{\infty} w_n(x,y) e^{in\sigma t} \quad (4)$$

Here $w_0(x,y)$ corresponds to the velocity field under the influence of the constant pressure gradient α_0 and $w_n(x,y)e^{in\sigma t}$ corresponds to the velocity field under the influence of the pressure gradient $\alpha_n e^{in\sigma t}$.

Substituting (2) and (4) in (1) and separating the like terms, we get

$$0 = \alpha_0 + \upsilon\nabla^2 w_0 \quad (5)$$

and $in\sigma w_n = \alpha_n + \upsilon\nabla^2 w_n$ (6)

with the boundary conditions

$$w_0 = 0 \quad (7)$$

and $w_n = 0$ on Γ (8)

Solving the equations we get

$$w_0 = \frac{\alpha_0}{4\upsilon b}(b-y)\left(y^2-3x^2\right) \tag{9}$$

and

$$w_n = \frac{i\alpha_n}{n\sigma}\left[-1+\left\{2\cos\frac{r_n y}{2}\cos\frac{\sqrt{3}r_n x}{2}-\cos rny\right\}-\left\{2\sin\frac{r_n y}{2}\cos\frac{\sqrt{3}r_n x}{2}-\sin r_n y\right\}\frac{\cos\frac{r_n b}{2}}{\sin\frac{r_n b}{2}}\right] \tag{10}$$

Now substituting the equation (10) in (6), we find the value of r_n.

where $r_n = \delta_n(i-1)$ with

$$\delta_n = \sqrt{\frac{n\sigma}{2\upsilon}} \tag{11}$$

Now substituting equations (9) and (10) in equation (4), and simplifying we get

$$w=\frac{\alpha_0}{4\upsilon b}(b-y)\left(y^2-3x^2\right)+\sum_{n=1}^{\infty}\frac{1}{n\sigma}\left[\alpha_{c_n}\{-Q_n\cos n\sigma t-(P_n-1)\sin n\sigma t\}+\alpha_{s_n}\{(P_n-1)\cos n\sigma t-Q_n\sin n\sigma t\}\right] \tag{12}$$

Where P$_n$ and Q$_n$ are given by

$$\begin{aligned}P_n = \frac{1}{2D}\Big[&\cos\frac{\delta_n A_1}{2}\cosh\frac{\delta_n(2b-A_1)}{2}-\cos\frac{\delta_n(2b-A_1)}{2}\cosh\frac{\delta_n A_1}{2}\\ &+\cos\frac{\delta_n A_2}{2}\cosh\frac{\delta_n(2b-A_2)}{2}-\cos\frac{\delta_n(2b-A_2)}{2}\cosh\frac{\delta_n A_2}{2}\\ &-\cos\delta_n\left(\frac{A_1+A_2}{2}\right)\cosh\delta_n\left(\frac{2b-\overline{A_1+A_2}}{2}\right)+\cos\delta_n\left(\frac{2b-\overline{A_1+A_2}}{2}\right)\cosh\delta_n\left(\frac{A_1+A_2}{2}\right)\Big]\end{aligned} \tag{13}$$

And

$$\begin{aligned}Q_n = \frac{1}{2D}\ &\sinh\frac{\delta_n A_1}{2}\sin\frac{\delta_n(2b-A_1)}{2}-\sinh\frac{\delta_n(2b-A_1)}{2}\sin\frac{\delta_n A_1}{2}\\ &+\Big[\sinh\frac{\delta_n A_2}{2}\sin\delta_n\left(\frac{2b-A_2}{2}\right)-\sinh\frac{\delta_n(2b-A_2)}{2}\sin\frac{\delta_n A_2}{2}\\ &-\sinh\delta_n\left(\frac{A_1+A_2}{2}\right)\sin\delta_n\left(\frac{2b-\overline{A_1+A_2}}{2}\right)+\sinh\delta_n\left(\frac{2b-\overline{A_1+A_2}}{2}\right)\sin\delta_n\left(\frac{A_1+A_2}{2}\right)\Big]\end{aligned} \tag{14}$$

$$\left.\begin{aligned}&\text{Here } A_1 = y+\sqrt{3}x \text{ and } A_2 = y-\sqrt{3}x,\\ &A_1+A_2 = 2y \ \ and \ \ D=\frac{1}{2}(\cosh\delta_n b-\cos\delta_n b)\end{aligned}\right\} \tag{15}$$

The flow rate i.e. volume of the fluid that crosses the cross section per unit time is given by

$$Q = \int_{y=0}^{b}\int_{x=-\frac{y}{\sqrt{3}}}^{\frac{y}{\sqrt{3}}} w\,dxdy$$

$$= \frac{\alpha_0 b^4}{60\sqrt{3}\upsilon} + \sum \frac{1}{n\sigma} \Bigg\{ \alpha_{c_n} \Bigg[\frac{\sqrt{3}b}{\delta_n D}(\sin \delta_n b)(\cos n\sigma t + \sin n\sigma t) + \sinh \delta_n b(\cos n\sigma t - \sin n\sigma t)$$

$$- \frac{4\sqrt{3}}{\delta_n{}^2 D}(\cos n\sigma t)\left(\sin^2 \frac{\delta_n b}{2} \cosh^2 \frac{\delta_n b}{2} + \cos^2 \frac{\delta_n b}{2} \sinh^2 \frac{\delta_n b}{2} \right)\Bigg]$$

$$+ \alpha_{s_n} \Bigg[\frac{\sqrt{3}b}{\delta_n D}(\sin \delta_n b)(-\cos n\sigma t + \sin n\sigma t) + \sinh \delta_n b(\cos n\sigma t + \sin n\sigma t)$$

$$- \frac{4\sqrt{3}}{\delta_n{}^2 D}(\sin n\sigma t)\left(\sin^2 \frac{\delta_n b}{2} \cosh^2 \frac{\delta_n b}{2} + \cos^2 \frac{\delta_n b}{2} \sinh^2 \frac{\delta_n b}{2} \right)\Bigg]\Bigg\} \qquad (16)$$

THE SKIN FRICTION ON A TYPICAL SIDE:

The resistance offered by the fluid on the table wall on a typical side is given by s_o, where

$$s_0 = \frac{\alpha_0 \rho b}{4}\left(1 - \frac{3x^2}{b^2}\right) \quad \text{where} \quad \rho = \frac{\mu}{\upsilon}$$

FLOW FOR SMALL FREQUENCIES:

Employing the approximations for small x,

$$\sin x = x - \frac{x^3}{6}, \quad \sinh x = x + \frac{x^3}{6}, \quad \cos x = 1 - \frac{x^2}{2}, \quad \cosh x = 1 + \frac{x^2}{2}$$

For small values of σ, i.e. for small values of δ_n, using all these values in equation (11), we get the velocity field.

$$w = \frac{(y^2 - 3x^2)(b-y)}{4\upsilon b}\left[\alpha_0 + \sum_{n=1}^{\infty}\left(\alpha_{c_n} \cos n\sigma t + \alpha_{s_n} \sin n\sigma t\right)\right]$$

i.e. the velocity for small frequencies $w = \frac{1}{4\upsilon b}\; f(t)\,(y-b)(y^2 - 3x^2)$

where $f(t) = \alpha_0 + \sum_{n=1}^{\infty}\left(\alpha_{c_n} \cos n\sigma t + \alpha_{s_n} \sin n\sigma t\right)$ (17)

It is noticed that the spatial variation of the velocity field under the periodic pressure gradient f(t) is same as that under the constant pressure gradient α_0 is replaced by f(t)

FLOW FOR LARGE FREQUENCIES:

Flow for large σ, i.e. for large δ_n

$$\cosh x = \frac{e^x}{2}, \quad \sinh x = \frac{e^x}{2}$$

using these in equation (12), we get

$$w = \frac{\alpha_0}{4\upsilon b}(b-y)\left(y^2-3x^2\right)+\sum_{n=1}^{\infty}\frac{1}{n\sigma}\left[\alpha_{c_n}\sin n\sigma t-\alpha_{s_n}\cos n\sigma t\right]$$

$$= w_0+\sum_{n=1}^{\infty}\frac{1}{n\sigma}\left[\alpha_{c_n}\cos n\sigma\left(t-\frac{\pi}{2n\sigma}\right)+\alpha_{s_n}\sin n\sigma\left(t-\frac{\pi}{2n\sigma}\right)\right] \quad (18)$$

Flow under the constant pressure gradient

Rigid body flow due to a force equal to the unsteady part of pressure gradient.

where

$$w_0 = \frac{\alpha_0}{4\upsilon b}(b-y)\left(y^2-3x^2\right) \quad (19)$$

The plug flow has a phase lag of $\frac{\pi}{2n\sigma}$

This shows that the velocity in the tube is the velocity under a constant pressure gradient α_0 over which a spatially independent flow is superposed. The plug flow is time dependent following the pressure gradient f(t) with phase length of $\frac{\pi}{2n\sigma}$ together with the amplitude decreasing in the ratio of $\frac{1}{n\sigma}$ in the n^{th} mode of vibration.

References:

1. S.UChida; The pulsing viscous flow superposed on the steady laminar motion of incompressible fluid in a circular pipe, Z angew math physics-7 (1956) 403-422.