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GLOBAL STABILITY OF A COMMENSAL – HOST ECOLOGICAL MODEL WITH A HOST HARVESTING AT A CONSTANT RATE

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ABSTRACT

In this paper we establish the global stability of a commensal – host ecological model with a host harvesting at a constant rate, by constructing a suitable Liapunov's function in case of co-existent equilibrium state.
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Keywords: Equilibrium state, Liapunov's function, Global stability.

1. INTRODUCTION

Phani Kumar and N.Ch.Pattabhi Ramacharyulu etc. [6, 9] examined the local stability of a host – a flourishing commensal eco-system on the quasi-linear basic balancing equations. The present authors has been also discussed the local stability analysis for a host-commensal eco-system in their earlier work [7, 8].The present investigation is mainly devoted to establish the global stability of the co-existent equilibrium state of the host – a flourishing commensal model by employing a properly constructed Liapunov's function.

2. LIAPUNOV'S STABILITY ANALYSIS

In 1892 A.M. Liapunov introduced the direct method to study the global stability of equilibrium states in case of linear and non-linear systems. His method is based on the chief characteristic of constructing a scalar function called Liapunov's function. That is by using the direct method of Liapunov,we can determine the stability of a system without solving the state. This is quite advantageous because solving non-linear and/or time-invarying state equation is very difficult.

The stability behaviour of solutions of linear and weakly non-linear system is done by using the techniques of variation of constants formulae and integral inequalities. So this analysis is confined to a small neighborhood of operating point i.e., local stability. Further, the techniques used there in require explicit knowledge of solutions of corresponding linear systems. The stability behavior of a physical system is discussed by several authors like Kapoor [2], Lotka [3], Ogata [4] Bhaskara Rama Sarma and N.Ch.PattabhiRamacharyulu [1], Lakshminarayan and N.Ch.Pattabhi Ramacharyulu [5] etc.

If the total energy of a physical system has a local minimum at a certain equilibrium point, then that point is stable. This idea was generalized by Liapunov to study stability problems in a broader context.

2.1. STABILITY BY LIAPUNOV'S DIRECT METHOD

Consider an autonomous system

$$\left.\begin{aligned} \frac{dx}{dt} &= F(x,y) \\ \frac{dy}{dt} &= G(x,y) \end{aligned}\right\} \qquad (1)$$

Assume that this system has an isolated critical point taken as (0, 0). Consider a function E(x, y) possessing continuous partial derivatives along the path of (1). This path is represented by C= [(x (t), y (t)] in the parametric form. E(x, y) can be regarded as a function of't' along C with rate of change

$$\frac{dE}{dt} = \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt}$$

$$\frac{dE}{dt} = \frac{\partial E}{\partial x}F(x,y) + \frac{\partial E}{\partial y}G(x,y) \qquad (2)$$

A Positive definite function E (x, y) with the property that (2) is negative semi-definite is called a Liapunov's function for the system (1).

3. BASIC EQUATIONS OF THE MODEL

The basic equations for the growth rate of a flourishing commensal and host species with limited resources are given by

$$\left.\begin{aligned} \frac{dN_1}{dt} &= a_{11}N_1\left[K_1 - N_1 + CN_2\right] \\ \frac{dN_2}{dt} &= a_{22}\left[K_2N_2 - N_2^2 - H_2\right] \end{aligned}\right\} \qquad (3)$$

4. THE EQULIBRIUM STATES

The system under investigation has six equilibrium states given by $\frac{dN_1}{dt}$ = 0 and $\frac{dN_2}{dt}$ =0 these are classified into two categories A and B.

A) The states in which the host alone survives

(A.1) When $H_2 < \frac{K_2^2}{4}$

E₁ : $\overline{N_1} = 0$; $\overline{N_2} = K_2 - \frac{H_2}{K_2}$ (4)

This arises only when $K_2^2 > H_2$

E₂ : $\overline{N_1} = 0$ $\overline{N_2} = \frac{H_2}{K_2}$ (5)

(A.2) When $H_2 = \frac{K_2^2}{4}$

E$_3$: $\overline{N_1} = 0$; $\overline{N_2} = \dfrac{K_2}{2}$ (6)

B) The co-existent States

(B.1) When $H_2 < \dfrac{K_2^{\,2}}{4}$

E$_4$: $\overline{N_1} = K_1 + C\left(K_2 - \dfrac{H_2}{K_2}\right); \overline{N}_2 = K_2 - \dfrac{H_2}{K_2}$ (7)

This exists only when $K_2^{\,2} > H_2$

E$_5$: $\overline{N_1} = K_1 + \dfrac{CH_2}{K_2}$; $\overline{N}_2 = \dfrac{H_2}{K_2}$ (8)

(B.2) When $H_2 = \dfrac{K_2^{\,2}}{4}$

E$_6$: $\overline{N_1} = K_1 + \dfrac{CK_2}{2}$; $\overline{N_2} = \dfrac{K_2}{2}$**;** (9)

where Ki= $\dfrac{a_i}{a_{ii}}$, i = 1, 2 are the carrying capacities of N_i.

C = $\dfrac{a_{12}}{a_{11}}$, the commensal co-efficient.

5. LOCAL STABILITY ANALYSIS

The present authors [6] discussed the local stability of the above six equilibrium states. From which the equilibrium state E_4 is only stable remaining are unstable.

6. LIAPUNOV'S FUNCTION FOR GLOBAL STABILITY

We now examine the global stability of the dynamical system (3).We have already noted that this system has a unique, stable non-trivial co-existent equilibrium state at

$$\overline{N_1} = K_1 + C\left(K_2 - \frac{H_2}{K_2}\right); \overline{N}_2 = K_2 - \frac{H_2}{K_2}$$

Basic Equations:

$$\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^{\,2} + a_{12} N_1 N_2 \qquad (10)$$

$$\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^{\,2} - a_{22} H_2 \qquad (11)$$

The linearized basic equations are

$$\frac{du_1}{dt} = -a_{11}\overline{N_1}u_1 + Ca_{11}\overline{N_1}u_2 \qquad (12)$$

$$\frac{du_2}{dt} = -a_{22}\left(\overline{N_2} - \frac{H_2}{K_2}\right)u_2 \qquad (13)$$

The corresponding characteristic equation is

$$\left(\lambda + a_{11}\overline{N_1}\right)\left(\lambda + a_{22}\left(\overline{N_2} - \frac{H_2}{K_2}\right)\right) = 0$$

i.e., $\lambda^2 + \left[a_{11}\overline{N_1} + a_{22}\left(\overline{N_2} - \frac{H_2}{K_2} \right) \right]\lambda + a_{11}a_{22}\overline{N}_1\left(\overline{N}_2 - \frac{H_2}{K_2} \right) = 0$

This is in the form of $\lambda^2 + p\lambda + q = 0$

Since $\left[a_{11}\overline{N_1} + a_{22}\left(\overline{N_2} - \frac{H_2}{K_2} \right) \right]^2 > 4a_{11}a_{22}\overline{N}_1\left(\overline{N}_2 - \frac{H_2}{K_2} \right)$

$$p = a_{11}\overline{N}_1 + a_{22}\left(\overline{N}_2 - \frac{H_2}{K_2} \right) > 0 \qquad (14)$$

$$q = a_{11}a_{22}\overline{N}_1\left(N_2 - \frac{H_2}{K_2} \right) > 0 \qquad (15)$$

Therefore the conditions for Liapunov's function are satisfied

Now define

$$E(u_1, u_2) = \tfrac{1}{2}\left(au_1^2 + 2b\, u_1u_2 + cu_2^2 \right) \qquad (16)$$

Where $$a = \frac{a_{22}^{\,2}\left(\overline{N_2} - \frac{H_2}{K_2} \right)^2 + a_{11}a_{22}\overline{N_1}\left(\overline{N_2} - \frac{H_2}{K_2} \right)}{D} \qquad (17)$$

$$b = \frac{Ca_{11}\overline{N_1}a_{22}\left(\overline{N_2} - \frac{H_2}{K_2} \right)}{D} \qquad (18)$$

$$c = \frac{a_{11}^{\,2}\overline{N_1}^2 + C^2a_{11}^{\,2}\overline{N_1}^2 + \left[a_{11}a_{22}\overline{N_1}\left(\overline{N_2} - \frac{H_2}{K_2} \right) \right]}{D} \qquad (19)$$

Where $D = pq = \left[a_{11}\overline{N_1} + a_{22}\left(\overline{N_2} - \frac{H_2}{K_2} \right) \right]\left[a_{11}a_{22}\overline{N_1}\left(\overline{N_2} - \frac{H_2}{K_2} \right) \right]$

Now

$D^2(ac-b^2) =$

$$\left\{ \left(\frac{a_{22}^{\,2}\left(\overline{N_2} - \frac{H_2}{K_2} \right)^2 + a_{11}a_{22}\overline{N_1}\left(\overline{N_2} - \frac{H_2}{K_2} \right)}{D} \right)\left(\frac{a_{11}^{\,2}\overline{N_1}^2 + C^2a_{11}^{\,2}\overline{N_1}^2 + a_{11}a_{22}\overline{N}_1\left(\overline{N}_2 - \frac{H_2}{K_2} \right)}{D} \right) - \frac{C^2a_{11}^{\,2}a_{22}^{\,2}\overline{N_1}^2\left(\overline{N}_2 - \frac{H_2}{K_2} \right)^2}{D^2} \right\}$$

$\Rightarrow D^2(ac - b^2) > 0$ Since $D^2 > 0$ so that $ac - b^2 > 0$

$\Rightarrow b^2 - ac < 0$

Since $a > 0$ and $b^2 - ac < 0$, so that the function $E(u_1, u_2)$ is positive definite.

Further

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt}+\frac{\partial E}{\partial u_2}\frac{du_2}{dt}=\left\{(au_1+bu_2)\left(-a_{11}\overline{N_1}u_1+Ca_{11}\overline{N_1}u_2\right)+(bu_1+cu_2)\left(-a_{22}\left(\overline{N}2-\frac{H2}{K2}\right)u_2\right)\right\}$$

$$=-a\,a_{11}\,\overline{N_1}u_1^2+\left[a\;a_{11}C\overline{N_1}-ba_{11}\overline{N_1}-ba_{22}\left(\overline{N2}-\frac{H2}{K2}\right)\right]u_1u_2$$

$$+\left[bCa_{11}\overline{N_1}-ca_{22}\left(\overline{N2}-\frac{H2}{K2}\right)\right]u_2^2 \tag{20}$$

Substituting the values of *a*, *b* and *c* in (20) we get

$$=-\left(\frac{a_{11}a_{22}\overline{N_1}\left(\overline{N2}-\frac{H2}{K2}\right)\left[a_{22}\left(\overline{N2}-\frac{H2}{K2}\right)+a_{11}\overline{N_1}\right]}{D}\right)u_1^2-\left\{\frac{a_{11}a_{22}\overline{N_1}\left(\overline{N2}-\frac{H2}{K2}\right)\left[a_{11}\overline{N_1}+a_{22}\left(\overline{N2}-\frac{H2}{K2}\right)\right]}{D}\right\}u_2^2$$

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt}+\frac{\partial E}{\partial u_2}\frac{du_2}{dt}=-\left[\frac{D}{D}\right]u_1^2+\left[\frac{D}{D}\right]u_2^2=-\left(u_1^2+u_2^2\right)$$

Which is clearly negative definite

∴ E (u_1, u_2) is a Liapunov function.

We prove that E (u_1, u_2) is also a Liapunov's function for the non-linear system also.

If f_1 and f_2 are two functions in N_1 and N_2 defined by

$$f_1\,(N_1, N_2)=a_1N_1-a_{11}N_1^2+a_{12}N_1N_2-a_{11}H_1 \tag{21}$$

$$f_2\,(N_1, N_2)=a_2N_2-a_{22}N_2^2-a_{22}H_2 \tag{22}$$

We have to show that $\frac{\partial E}{\partial u_1}f_1+\frac{\partial E}{\partial u_2}f_2$ is negative definite

Putting $N_1 = \overline{N}_1 + u_1$ and $N_2 = \overline{N}_2 + u_2$ is (10) and (11) we get

$$\frac{du_1}{dt}=a_1(\overline{N}_1+u_1)-a_{11}(\overline{N}_1+u_1)^2+a_{12}(\overline{N}_1+u_1)(\overline{N}_2+u_2)$$

$$=u_1\left[a_1-2a_{11}\left(K_1+C\overline{N_2}\right)+a_{12}\left(K2-\frac{H2}{K2}\right)\right]+a_{12}\overline{N_1}u_2-a_{11}u_1^2+a_{12}u_1u_2$$

$$\frac{du_1}{dt}=-a_{11}\overline{N}_1u_1+Ca_{11}\overline{N}_1u_2+F(u_1,u_2)$$

$$\Rightarrow f_1(u_1,u_2)=\frac{du_1}{dt}=-a_{11}\overline{N}_1u_1+C\,a_{11}\overline{N}_1u_2+F(u_1,u_2)$$

where $F(u_1, u_2) = - a_{11}u_1^2 + a_{12}u_1u_2$

$$\frac{du_2}{dt}=a_2(\overline{N}_2+u_2)-a_{22}(\overline{N}_2+u_2)^2-a_{22}H_2=u_2\,[a_2-2a_{22}\overline{N}_2]-a_{22}u_2^2$$

$$\frac{du_2}{dt}=-a_{22}\overline{N_2}u_2-a_{22}u_2^2$$

$$\Rightarrow f_2(u_1,u_2)=\frac{du_2}{dt}=-a_{22}\overline{N}_2u_2+G(u_1,u_2)$$

where $G(u_1, u_2) = - a_{22}u_2^2$

Now

$$\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2 = (au_1 + bu_2)\left[-a_{11}\overline{N_1}u_1 + Ca_{11}\overline{N_1}u_2\right] + (bu_1 + cu_2)\left(-a_{22}\overline{N_2}\,u_2\right)$$

$$+ (au_1 + bu_2)F(u_1, u_2) + (bu_1 + cu_2)G(u_1, u_2)$$

$$= -(u_1^2 + u_2^2) + (au_1 + bu_2)F(u_1, u_2) + (bu_1 + cu_2)G(u_1, u_2) \qquad (23)$$

By introducing polar coordinates we can write this as

$$= -r^2 + r[(a\cos\theta + b\sin\theta)F(u_1, u_2) + (b\cos\theta + c\sin\theta)G(u_1, u_2) \qquad (24)$$

Denote largest of the numbers | a|, |b|, |c| by *K*,

Our assumptions imply that $|F(u_1, u_2)| < \frac{r}{6K}$ and $|G(u_1, u_2)| < \frac{r}{6K}$,

for all sufficiently small r > 0.

So $\quad \frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2 < -r^2 + \frac{4Kr^2}{6Kr^2} = -\frac{r^2}{3} < 0$

Thus $E(u_1, u_2)$ is a positive definite function with the property that

$\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2$ is negative definite.

$\therefore$ The equilibrium state E_4 is **asymptotically stable** also**.**

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