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PROPAGATION OF MICRO ELASTIC WAVES IN AN ISOTROPIC SOLID OF CIRCULAR CROSS-SECTION

T.Ravi[1] and **Dr.A.Anjaneyulu**[2]

1. Department of mathematics, University College of engineering (K.U.), Kothagudem- 507101, A.P. , India. Mobile: 9701409620, E-Mail address: tandraravi@gmail.com

2. Department of Mathematics, V.S.R. & N.V.R. College, Tenali, (Dist) Guntur, A.P., India. Mobile: 9848535453.

Abstract : The main aim of this article is to study the propagation of micro elastic waves in an isotropic solid of circular cross-section.

Key words: Strain tensor, Stress tensor, Micro-rotation vector, Couple stress tensor, Micro-strain tensor, Stress moments.

Introduction: Suitable boundary conditions are to be taken depending on boundary of media for study the propagation of micro elastic waves in an isotropic and elastic rod of circular cross -section. Usually in bounded media, three different types of waves occur *viz*. longitudinal, torsional and lateral. In longitudinal vibrations, elements of the media are external and contract; but any lateral displacement of the axis of symmetry will not be there. In torsional vibrations, each traverse section of the media remains its own plane and has rotation about its centre; the axis of symmetry of the media is remaining undisturbed. Finally lateral waves correspond to the flexure portion of the media and elements of the central axis moving laterally during the motion.

In this article, propagation of the longitudinal and torsional waves is studied.

1. Equations of motion in cylindrical co-ordinates

Transformation of the rectangular Cartesian coordinate system (x_1, x_2, x_3) into the cylindrical coordinate system (r, θ, z) is given by $x_1 = r\,\mathbf{Cos}\,\theta$; $x_2 = r\,\mathbf{Sin}\,\theta$; $x_3 = z$.

Then $\frac{\partial \vec{\mathbf{x}}}{\partial r}$; $\frac{\partial \vec{\mathbf{x}}}{\partial \theta}$; $\frac{\partial \vec{\mathbf{x}}}{\partial z}$ are tangent vectors to the coordinate curves; and given by

$$\frac{\partial \vec{\mathbf{x}}}{\partial r} = \frac{\partial x_1}{\partial r}\vec{\mathbf{e}}_1 + \frac{\partial x_2}{\partial r}\vec{\mathbf{e}}_2 + \frac{\partial x_3}{\partial r}\vec{\mathbf{e}}_3 = \mathbf{Cos}\,\theta\ \vec{\mathbf{e}}_1 + \mathbf{Sin}\,\theta\ \vec{\mathbf{e}}_2 + 0\,\vec{\mathbf{e}}_3$$

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$$\frac{\partial \vec{\mathbf{x}}}{\partial \theta} = \frac{\partial x_1}{\partial \theta}\vec{\mathbf{e}}_1 + \frac{\partial x_2}{\partial \theta}\vec{\mathbf{e}}_2 + \frac{\partial x_3}{\partial \theta}\vec{\mathbf{e}}_3 = -r\,\mathbf{Sin}\,\theta\ \vec{\mathbf{e}}_1 + r\,\mathbf{Cos}\,\theta\ \vec{\mathbf{e}}_2 + 0\,\vec{\mathbf{e}}_3$$

$$\frac{\partial \vec{\mathbf{x}}}{\partial z} = \frac{\partial x_1}{\partial z}\vec{\mathbf{e}}_1 + \frac{\partial x_2}{\partial z}\vec{\mathbf{e}}_2 + \frac{\partial x_3}{\partial z}\vec{\mathbf{e}}_3 = 0\ \vec{\mathbf{e}}_1 + 0\,\vec{\mathbf{e}}_2 + 1\,\vec{\mathbf{e}}_3 = \vec{\mathbf{e}}_3$$

Let the coordinate curves are mutually orthogonal**;** denote that a right-handed system of orthogonal unit vectors tangent to the coordinate curves is $\vec{\mathbf{e}}_{\mathbf{r}}$; $\vec{\mathbf{e}}_{\boldsymbol{\theta}}$; $\vec{\mathbf{e}}_{\mathbf{z}}$ **.**

$$\Rightarrow \quad \frac{\partial \vec{\mathbf{x}}}{\partial r} = g_1\,\vec{\mathbf{e}}_{\mathbf{r}} \quad ; \quad \frac{\partial \vec{\mathbf{x}}}{\partial \theta} = g_2\,\vec{\mathbf{e}}_{\boldsymbol{\theta}} \quad ; \quad \frac{\partial \vec{\mathbf{x}}}{\partial z} = g_3\,\vec{\mathbf{e}}_{\mathbf{z}}$$

$$\Rightarrow \qquad g_1 = \left|\frac{\partial \vec{\mathbf{x}}}{\partial r}\right| = \sqrt{\mathbf{Cos}^2\,\theta + \mathbf{Sin}^2\,\theta + 0^2} = 1 \quad ;$$

$$g_2 = \left|\frac{\partial \vec{\mathbf{x}}}{\partial \theta}\right| = \sqrt{r^2\,\mathbf{Cos}^2\,\theta + r^2\,\mathbf{Sin}^2\,\theta + 0^2} = r \qquad ; \qquad g_3 = \left|\frac{\partial \vec{\mathbf{x}}}{\partial z}\right| = 1$$

Hence $g_1 = 1$; $g_2 = r$; $g_3 = 1$**.**

Denote that $\text{Grad}\left[\ell_{ij}(r,\theta,z,t)\right] = \vec{\mathbf{\Psi}}(r,\theta,z,t)$

$$\Rightarrow \qquad \vec{\mathbf{e}}_{\mathbf{r}}\,\Psi_r + \vec{\mathbf{e}}_{\boldsymbol{\theta}}\,\Psi_\theta + \vec{\mathbf{e}}_{\mathbf{z}}\,\Psi_z = \vec{\mathbf{\Psi}}(r,\theta,z,t) = \text{Grad}\left[\ell_{ij}(r,\theta,z,t)\right]$$

$$= \vec{\mathbf{e}}_{\mathbf{r}}\,\frac{1}{g_1}\frac{\partial \ell_{ij}}{\partial r} + \vec{\mathbf{e}}_{\boldsymbol{\theta}}\,\frac{1}{g_2}\frac{\partial \ell_{ij}}{\partial \theta} + \vec{\mathbf{e}}_{\mathbf{z}}\,\frac{1}{g_3}\frac{\partial \ell_{ij}}{\partial z} = \vec{\mathbf{e}}_{\mathbf{r}}\,\frac{\partial \ell_{ij}}{\partial r} + \vec{\mathbf{e}}_{\boldsymbol{\theta}}\,\frac{1}{r}\frac{\partial \ell_{ij}}{\partial \theta} + \vec{\mathbf{e}}_{\mathbf{z}}\,\frac{\partial \ell_{ij}}{\partial z}$$

$$\Rightarrow \qquad \text{Div}\left[\text{Grad}\,\ell_{ij}(r,\theta,z,t)\right] = \text{Div}\,\vec{\mathbf{\Psi}}(r,\theta,z,t)$$

$$= \frac{1}{g_1\,g_2\,g_3}\left[\frac{\partial}{\partial r}(g_2\,g_3\Psi_r) + \frac{\partial}{\partial \theta}(g_1\,g_3\Psi_\theta) + \frac{\partial}{\partial z}(g_1\,g_2\Psi_z)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(r\Psi_r) + \frac{\partial \Psi_\theta}{\partial \theta} + \frac{\partial}{\partial z}(r\Psi_z)\right] = \frac{1}{r}\left[\Psi_r + r\frac{\partial \Psi_r}{\partial r} + \frac{\partial \Psi_\theta}{\partial \theta} + r\frac{\partial \Psi_z}{\partial z}\right]$$

$$= \frac{1}{r}\Psi_r + \frac{\partial}{\partial r}\Psi_r + \frac{1}{r}\frac{\partial \Psi_\theta}{\partial \theta} + \frac{\partial \Psi_z}{\partial z} = \frac{1}{r}\frac{\partial \ell_{ij}}{\partial r} + \frac{\partial}{\partial r}\frac{\partial}{\partial r}\ell_{ij} + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial \ell_{ij}}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \ell_{ij}}{\partial z}\right)$$

$$= \frac{1}{r}\frac{\partial \ell_{ij}}{\partial r} + \frac{\partial}{\partial r}\frac{\partial}{\partial r}\ell_{ij} + \frac{1}{r^2}\frac{\partial}{\partial \theta}\frac{\partial \ell_{ij}}{\partial \theta} + \frac{\partial}{\partial z}\frac{\partial \ell_{ij}}{\partial z}$$

Under the absence of external body force**,** Dynamical equation (consisting of the time derivative of micro-strain) from reference [1] is

$$(5)\textbf{:}\ \frac{\varsigma\,\rho}{2}\frac{\partial^2}{\partial t^2}\ell_{ij}(x_1,x_2,x_3,t) = \Lambda\nabla^2\ell_{kk}(x_1,x_2,x_3,t)\delta_{\text{ij}} + (2\aleph + N)\nabla^2\ell_{ij}(x_1,x_2,x_3,t)$$

$$-\,\tilde{\lambda}\,\ell_{kk}(x_1,x_2,x_3,t)\delta_{\text{ij}} - (2\tilde{\mu} + \nu)\ell_{ij}(x_1,x_2,x_3,t)$$

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Let the components of micro-strain tensor in cylindrical coordinate system (r,θ,z) are denoted by $\ell_{ij}(r,\theta,z,t)$. Then the above equation in cylindrical coordinate system (r,θ,z) can be written as

$$\varsigma\rho\frac{1}{2}\frac{\partial^2}{\partial t^2}\ell_{ij}(r,\theta,z,t)=\Lambda\,\mathrm{Div}\,\mathrm{Grad}\left[\ell_{kk}(r,\theta,z,t)\right]\delta_{ij}$$
$$+(2\aleph+N)\,\mathrm{Div}\,\mathrm{Grad}\left[\ell_{ij}(r,\theta,z,t)\right]$$

$$-\tilde{\lambda}\,\ell_{kk}(r,\theta,z,t)\delta_{ij}-(2\tilde{\mu}+\nu)\ell_{ij}(r,\theta,z,t) \qquad \cdots(1)$$

Then from the equations (1), we have

$$\varsigma\rho\frac{1}{2}\frac{\partial^2}{\partial t^2}\ell_{11}(r,\theta,z,t)$$

$$=\Lambda\mathrm{Div}\,\mathrm{Grad}\left[\ell_{kk}(r,\theta,z,t)\right]\delta_{11}+(2\aleph+N)\,\mathrm{Div}\,\mathrm{Grad}\left[\ell_{11}(r,\theta,z,t)\right]$$
$$-\tilde{\lambda}\,\ell_{kk}(r,\theta,z,t)\delta_{11}-(2\tilde{\mu}+\nu)\ell_{11}(r,\theta,z,t)$$

$$=\Lambda\,\mathrm{Div}\mathrm{Grad}\,[\ell_{11}(r,\theta,z,t)+\ell_{22}(r,\theta,z,t)+\ell_{33}(r,\theta,z,t)]1$$
$$+(2\aleph+N)\,\mathrm{Div}\mathrm{Grad}\left[\ell_{11}(r,\theta,z,t)\right]-(2\tilde{\mu}+\nu)\ell_{11}(r,\theta,z,t)$$
$$-\tilde{\lambda}\,[\ell_{11}(r,\theta,z,t)+\ell_{22}(r,\theta,z,t)+\ell_{33}(r,\theta,z,t)]1$$

$$=\Lambda\begin{bmatrix}\mathrm{Div}\,\mathrm{Grad}\ell_{11}(r,\theta,z,t)+\mathrm{Div}\,\mathrm{Grad}\ell_{22}(r,\theta,z,t)\\ +\mathrm{Div}\,\mathrm{Grad}\ell_{33}(r,\theta,z,t)\end{bmatrix}$$
$$+(2\aleph+N)\,\mathrm{Div}\mathrm{Grad}\ell_{11}(r,\theta,z,t)$$
$$-\left(\tilde{\lambda}+2\tilde{\mu}+\nu\right)\ell_{11}(r,\theta,z,t)-\tilde{\lambda}\,\ell_{22}(r,\theta,z,t)-\tilde{\lambda}\,\ell_{33}(r,\theta,z,t)$$

$$\Rightarrow \qquad \frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}\ell_{11}(r,\theta,z,t)=(\Lambda+2\aleph+N)\mathrm{Div}\mathrm{Grad}\ell_{11}(r,\theta,z,t)$$

$$+\Lambda\,\mathrm{Div}\mathrm{Grad}\ell_{22}(r,\theta,z,t)+\Lambda\,\mathrm{Div}\mathrm{Grad}\ell_{33}(r,\theta,z,t)$$
$$-\left(\tilde{\lambda}+2\tilde{\mu}+\nu\right)\ell_{11}(r,\theta,z,t)$$

$$-\tilde{\lambda}\,\ell_{22}(r,\theta,z,t)-\tilde{\lambda}\,\ell_{33}(r,\theta,z,t) \qquad \cdots(1.1)$$

Similarly we have

$$\frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}\ell_{22}(r,\theta,z,t)=(\Lambda+2\aleph+N)\mathrm{Div}\mathrm{Grad}\ell_{22}(r,\theta,z,t)$$

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$$+\Lambda\,\text{DivGrad}\ell_{11}(r,\theta,z,t)+\Lambda\,\text{DivGrad}\ell_{33}(r,\theta,z,t)$$
$$-\left(\tilde{\lambda}+2\tilde{\mu}+\nu\right)\ell_{22}(r,\theta,z,t)$$
$$-\tilde{\lambda}\,\ell_{11}(r,\theta,z,t)-\tilde{\lambda}\,\ell_{33}(r,\theta,z,t) \qquad \cdots(1.2)$$

$$\frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}\ell_{33}(r,\theta,z,t)=(\Lambda+2\aleph+N)\text{DivGrad}\ell_{33}(r,\theta,z,t)$$
$$+\text{DivGrad}\ell_{11}(r,\theta,z,t)+\text{DivGrad}\ell_{22}(r,\theta,z,t)$$
$$-\left(\tilde{\lambda}+2\tilde{\mu}+\nu\right)\ell_{33}(r,\theta,z,t)$$
$$-\tilde{\lambda}\,\ell_{11}(r,\theta,z,t)-\tilde{\lambda}\,\ell_{33}(r,\theta,z,t) \qquad \cdots(1.3)$$

Consider

$$\varsigma\rho\frac{1}{2}\frac{\partial^2}{\partial t^2}\ell_{23}(r,\theta,z,t)$$
$$=\Lambda\,\text{Div}\,\text{Grad}\ell_{kk}(r,\theta,z,t)\,\delta_{23}+(2\aleph+N)\,\text{Div}\,\text{Grad}\ell_{23}(r,\theta,z,t)$$
$$-\tilde{\lambda}\,\ell_{kk}(r,\theta,z,t)\delta_{23}-(2\tilde{\mu}+\nu)\ell_{23}(r,\theta,z,t)$$
$$=\Lambda\,\text{Div}\,\text{Grad}\ell_{kk}(r,\theta,z,t)\,0+(2\aleph+N)\,\text{Div}\,\text{Grad}\ell_{23}(r,\theta,z,t)$$
$$-\tilde{\lambda}\,\ell_{kk}(r,\theta,z,t)0-(2\tilde{\mu}+\nu)\ell_{23}(r,\theta,z,t)$$

$$\Rightarrow \quad \frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}\ell_{23}(r,\theta,z,t)=(2\aleph+N)\,\text{DivGrad}\ell_{23}(r,\theta,z,t)$$
$$-(2\tilde{\mu}+\nu)\ell_{23}(r,\theta,z,t) \qquad \cdots\cdots\cdots(1.4)$$

Similarly we have

$$\frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}\ell_{31}(r,\theta,z,t)=(2\aleph+N)\,\text{DivGrad}\ell_{31}(r,\theta,z,t)$$
$$-(2\tilde{\mu}+\nu)\ell_{31}(r,\theta,z,t) \qquad \cdots\cdots\cdots(1.5)$$

$$\frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}\ell_{12}(r,\theta,z,t)=(2\aleph+N)\,\text{DivGrad}\ell_{12}(r,\theta,z,t)$$

$$-(2\tilde{\mu}+\nu)\ell_{12}(r,\theta,z,t) \qquad \cdots\cdots\cdots(1.6)$$

1.2. Longitudinal micro - elastic vibrations in an isotropic elastic solid of circular cross-section

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Consider an isotropic and elastic solid of circular cross-section whose radius is a. Let the direction of propagation of wave is taken to be parallel to x_3 - axis. Since the rod is of circular cross-section, cylindrical polar co-ordinates (r, θ, z) are considered.

And the components of micro-strain tensor are given by

$$\ell_{11}(r,\theta,z,t)=\ell_{11}(r,z,t) \quad ; \quad \ell_{33}(r,\theta,z,t)=\ell_{33}(r,z,t) \quad ;$$

$$\ell_{13}(r,\theta,z,t)=\ell_{13}(r,z,t) \quad ; \quad \ell_{22}(r,\theta,z,t)=0=\ell_{12}(r,\theta,z,t)=\ell_{23}(r,\theta,z,t)$$

$$\Rightarrow \quad \frac{\partial}{\partial\theta}\ell_{11}(r,z,\theta,t)=\frac{\partial}{\partial\theta}\ell_{11}(r,z,t)=0=\frac{\partial}{\partial\theta}\ell_{33}(r,z,t)=\frac{\partial}{\partial\theta}\ell_{33}(r,z,\theta,t)$$

$$\text{and} \quad \frac{\partial}{\partial\theta}\ell_{13}(r,z,\theta,t)=\frac{\partial}{\partial\theta}\ell_{13}(r,z,t)=0$$

$$\Rightarrow \quad \text{Div Grad}\,\ell_{ij}=\frac{1}{r}\frac{\partial\ell_{ij}}{\partial r}+\frac{\partial}{\partial r}\frac{\partial\ell_{ij}}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{\partial\ell_{ij}}{\partial\theta}+\frac{\partial}{\partial z}\frac{\partial\ell_{ij}}{\partial z}$$

$$=\frac{1}{r}\frac{\partial\ell_{ij}}{\partial r}+\frac{\partial}{\partial r}\frac{\partial\ell_{ij}}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\theta}0+\frac{\partial}{\partial z}\frac{\partial\ell_{ij}}{\partial z}=\frac{\partial}{\partial r}\frac{\partial\ell_{ij}}{\partial r}+\frac{1}{r}\frac{\partial\ell_{ij}}{\partial r}+\frac{\partial}{\partial z}\frac{\partial\ell_{ij}}{\partial z}$$

Then equation (1.1) takes the form

$$\frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}\ell_{11}=(\Lambda+2\aleph+N)\text{Div Grad}\,\ell_{11}+\Lambda\text{Div Grad}\,0+\Lambda\text{Div Grad}\,\ell_{33}$$

$$-\left(\tilde{\lambda}+2\tilde{\mu}+\nu\right)\ell_{11}-\tilde{\lambda}0-\tilde{\lambda}\ell_{33}$$

$$\Rightarrow \quad \frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}\ell_{11}=(\Lambda+2\aleph+N)\text{Div Grad}\,\ell_{11}+\Lambda\text{Div Grad}\,\ell_{33}$$

$$-\left(\tilde{\lambda}+2\tilde{\mu}+\nu\right)\ell_{11}-\tilde{\lambda}\ell_{33} \qquad \cdots\cdots\cdots(2a')$$

Equation (1.2) takes the form

$$\frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}0=(\Lambda+2\aleph+N)\text{Div Grad}\,0+\Lambda\text{Div Grad}\,\ell_{11}+\Lambda\text{Div Grad}\,\ell_{33}$$

$$-\left(\tilde{\lambda}+2\tilde{\mu}+\nu\right)0-\tilde{\lambda}\,\ell_{11}-\tilde{\lambda}\,\ell_{33}$$

$$\Rightarrow \quad 0=\Lambda\,\text{Div Grad}\,\ell_{11}+\Lambda\,\text{Div Grad}\,\ell_{33}-\tilde{\lambda}\ell_{11}-\tilde{\lambda}\ell_{33} \qquad \cdots\cdots(2b')$$

Similarly equation (1.3) takes the form

$$\frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}\ell_{33}=(\Lambda+2\aleph+N)\text{Div Grad}\,\ell_{33}+\Lambda\text{Div Grad}\,\ell_{11}+\Lambda\text{Div Grad}\,0$$

$$-\left(\tilde{\lambda}+2\tilde{\mu}+\nu\right)\ell_{33}-\tilde{\lambda}\ell_{11}-\tilde{\lambda}0$$

$$\Rightarrow \quad \frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}\ell_{33}=(\Lambda+2\aleph+N)\text{Div Grad}\,\ell_{33}+\Lambda\text{Div Grad}\,\ell_{11}$$

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$$-\left(\tilde{\lambda}+2\tilde{\mu}+\nu\right)\ell_{33}-\tilde{\lambda}\ell_{11} \qquad \cdots\cdots\cdots(2c')$$

Substituting $(2b')$ in $(2a')$, we get

$$\frac{\varsigma\rho}{2}\frac{\partial^2\ell_{11}}{\partial t^2}=(2\aleph+N)\mathrm{Div\,Grad}\ell_{11}-(2\tilde{\mu}+\nu)\ell_{11}$$

$$\Rightarrow \quad \frac{\varsigma\rho}{2}\frac{\partial^2\ell_{11}}{\partial t^2}=(2\aleph+N)\left[\frac{\partial}{\partial r}\frac{\partial\ell_{11}}{\partial r}+\frac{1}{r}\frac{\partial\ell_{11}}{\partial r}+\frac{\partial}{\partial z}\frac{\partial\ell_{11}}{\partial z}\right]-(2\tilde{\mu}+\nu)\ell_{11} \qquad \cdots(9.2a)$$

Substituting $(2b')$ in $(2c')$, we get

$$\frac{\varsigma\rho}{2}\frac{\partial^2\ell_{33}}{\partial t^2}=(2\aleph+N)\mathrm{Div\,Grad}\ell_{33}-(2\tilde{\mu}+\nu)\ell_{33}$$

$$\Rightarrow \quad \frac{\varsigma\rho}{2}\frac{\partial^2\ell_{33}}{\partial t^2}=(2\aleph+N)\left[\frac{\partial}{\partial r}\frac{\partial\ell_{33}}{\partial r}+\frac{1}{r}\frac{\partial\ell_{33}}{\partial r}+\frac{\partial}{\partial z}\frac{\partial\ell_{33}}{\partial z}\right]-(2\tilde{\mu}+\nu)\ell_{33} \qquad \cdots(2b)$$

Again equation (1.5) takes the form

$$\frac{\varsigma\rho}{2}\frac{\partial^2\ell_{31}}{\partial t^2}=(2\aleph+N)\left[\frac{\partial}{\partial r}\frac{\partial\ell_{31}}{\partial r}+\frac{1}{r}\frac{\partial\ell_{31}}{\partial r}+\frac{\partial}{\partial z}\frac{\partial\ell_{31}}{\partial z}\right]-(2\tilde{\mu}+\nu)\ell_{31} \qquad \cdots(2c)$$

Let us seek the solutions of the equations $(2a),(2b),(2c)$ in the form

$$\ell_{11}(r,z,t)=A\mathbf{J}_0(hr)\,\mathrm{e}^{\iota(\varpi t-nz)} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2.1a)$$

$$\ell_{33}(r,z,t)=B\mathbf{J}_0(hr)\,\mathrm{e}^{\iota(\varpi t-nz)} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2.1b)$$

$$\ell_{13}(r,z,t)=H\mathbf{J}_0(hr)\,\mathrm{e}^{\iota(\varpi t-nz)} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2.1c)$$

where ϖ is the angular frequency, n is the wave number and $\mathbf{J}_0$ is Bessel's function of order zero; A, B, H are arbitrary constants.

Substituting $(2.1a)$ in $(2a)$:

$$0=\frac{\varsigma\rho}{2}\frac{\partial^2\ell_{11}}{\partial t^2}-(2\aleph+N)\left[\frac{\partial}{\partial r}\frac{\partial\ell_{11}}{\partial r}+\frac{1}{r}\frac{\partial\ell_{11}}{\partial r}+\frac{\partial}{\partial z}\frac{\partial\ell_{11}}{\partial z}\right]+(2\tilde{\mu}+\nu)\ell_{11}$$

$$=\frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}A\mathbf{J}_0(hr)\,\mathrm{e}^{\iota(\varpi t-nz)}+(2\tilde{\mu}+\nu)A\mathbf{J}_0(hr)\,\mathrm{e}^{\iota(\varpi t-nz)}$$

$$-(2\aleph+N)\left[\frac{\partial}{\partial r}\frac{\partial}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial}{\partial z}\frac{\partial}{\partial z}\right]A\mathbf{J}_0(hr)\,\mathrm{e}^{\iota(\varpi t-nz)}$$

$$=\frac{\varsigma\rho}{2}A\mathbf{J}_0(hr)(\iota\varpi)^2\,\mathrm{e}^{\iota(\varpi t-nz)}+(2\tilde{\mu}+\nu)A\mathbf{J}_0(hr)\,\mathrm{e}^{\iota(\varpi t-nz)}$$

$$-(2\aleph+N)A\left[\mathbf{J}_0''(hr)h^2+\frac{1}{r}\mathbf{J}_0'(hr)h+\mathbf{J}_0(hr)(-n\iota)^2\right]\mathrm{e}^{\iota(\varpi t-nz)}$$

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Hence $$0=\frac{\varsigma\rho}{2}\mathbf{J}_0(hr)\iota^2\varpi^2+(2\tilde{\mu}+\nu)\mathbf{J}_0(hr)$$
$$-(2\aleph+N)\left[\mathbf{J}_0''(hr)h^2+\frac{1}{r}\mathbf{J}_0'(hr)h+\mathbf{J}_0(hr)n^2\iota^2\right]$$
$$=-\varpi^2\frac{\varsigma\rho}{2}\mathbf{J}_0(hr)+(2\tilde{\mu}+\nu)\mathbf{J}_0(hr)$$
$$-\left(\frac{2\aleph+N}{r^2}\right)\left[\mathbf{J}_0''(hr)h^2r^2+\mathbf{J}_0'(hr)hr-r^2n^2\,\mathbf{J}_0(hr)\right]$$

But $h^2r^2\,\mathbf{J}_0''(hr)+hr\,\mathbf{J}_0'(hr)+\left(h^2r^2-0\right)\mathbf{J}_0(hr)=0$

$$\Rightarrow\quad 0=-\varpi^2\frac{\varsigma\rho}{2}\mathbf{J}_0(hr)+(2\tilde{\mu}+\nu)\mathbf{J}_0(hr)-\frac{2\aleph+N}{r^2}\left[-h^2r^2\,\mathbf{J}_0(hr)-r^2n^2\,\mathbf{J}_0(hr)\right]$$

$$\Rightarrow\quad 0=-\varpi^2\frac{\varsigma\rho}{2}+(2\tilde{\mu}+\nu)-\frac{2\aleph+N}{r^2}\left[-h^2r^2-r^2n^2\right]$$

$$\Rightarrow\quad \varpi^2\frac{\varsigma\rho}{2}-(2\tilde{\mu}+\nu)=(2\aleph+N)\left[h^2+n^2\right]$$

$$\Rightarrow\quad \varpi^2\varsigma\rho-2(2\tilde{\mu}+\nu)=2(2\aleph+N)\left[h^2+n^2\right]$$

$$\Rightarrow\quad \frac{\varpi^2\varsigma\rho-2(2\tilde{\mu}+\nu)}{2(2\aleph+N)}=h^2+n^2$$

$$\Rightarrow\quad h^2=\frac{\varpi^2\varsigma\rho-2(2\tilde{\mu}+\nu)}{2(2\aleph+N)}-n^2 \qquad\cdots\cdots\cdots\cdots(2.2)$$

Similarly substituting (2.1b) in (2b): and (2.1c) in (2c): we will get same result (2.2) since are (2a), (2b), (2c) are identical differential equations.

On stress free boundary surface, the required boundary conditions are

$$\tau_{rr}(r,z,t)=0=\tau_{rz}(r,z,t)\quad;\quad m_{r\theta}(r,z,t)=0 \text{ at } r=a$$

Recall (5): $\tau_{ij}=\tilde{\lambda}\,\ell_{kk}\,\delta_{ij}+(2\tilde{\mu}+\nu)\ell_{ij}-\nu\,\omega_{ij}$ from reference [2]

Consider $0=\tau_{rz}(a,z,t)=\tau_{13}(a,z,t)$

$$=\tilde{\lambda}\,\ell_{kk}(a,z,t)\delta_{13}+(2\tilde{\mu}+\nu)\ell_{13}(a,z,t)-\nu\,\omega_{13}(a,z,t)$$
$$=\tilde{\lambda}\,\ell_{kk}(a,z,t)0+(2\tilde{\mu}+\nu)\ell_{13}(a,z,t)-\nu\,\omega_{13}(a,z,t)$$

$$\Rightarrow\quad \nu\,\omega_{13}(a,z,t)=(2\tilde{\mu}+\nu)\ell_{13}(a,z,t)$$

Recall (6.2): $m_{qi}(x_1,x_2,x_3,t)=\varepsilon_{ijk}\,\sigma_{qkj}(x_1,x_2,x_3,t)$

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And (6): $\sigma_{qij} = \Lambda \ell_{kk,q}\,\delta_{ij} + (2\aleph + N)\,\ell_{ij,q} - \nu\,\omega_{ij,q}$ from reference [2]

$$\Rightarrow \quad m_{qi} = \varepsilon_{ijk}\left[\Lambda\ell_{pp,q}\,\delta_{kj} + (2\aleph + N)\ell_{kj,q} - N\,\omega_{kj,q}\right]$$

But
$$\begin{aligned}
m_{12} &= \varepsilon_{2jk}\left[\Lambda\ell_{pp,1}\,\delta_{kj} + (2\aleph + N)\ell_{kj,1} - N\,\omega_{kj,1}\right] \\
&= \varepsilon_{231}\left[\Lambda\ell_{pp,1}\,\delta_{13} + (2\aleph + N)\ell_{13,1} - N\,\omega_{13,1}\right] \\
&\quad + \varepsilon_{213}\left[\Lambda\ell_{pp,1}\,\delta_{31} + (2\aleph + N)\ell_{31,1} - N\,\omega_{31,1}\right] \\
&= 1\left[\Lambda\ell_{pp,1}\,0 + (2\aleph + N)\ell_{13,1} - N\,\omega_{13,1}\right] \\
&\quad - 1\left[\Lambda\ell_{pp,1}\,0 + (2\aleph + N)\ell_{13,1} + N\,\omega_{13,1}\right] = -2N\,\omega_{13,1}
\end{aligned}$$

Thus
$$\begin{aligned}
m_{12}(r,z,t) &= -2N\,\omega_{13,1}(r,z,t) = -2N\frac{\partial}{\partial r}\omega_{13}(r,z,t) \\
&= -2N\frac{\partial}{\partial r}\left(\frac{2\tilde{\mu}+\nu}{\nu}\ell_{13}(r,z,t)\right) = -2N\frac{2\tilde{\mu}+\nu}{\nu}\frac{\partial}{\partial r}\ell_{13}(r,z,t)
\end{aligned}$$

But from (2.1c): $\ell_{13}(r,z,t) = H\,\mathbf{J}_0(hr)\,e^{\iota(\varpi t - nz)}$.

$$\begin{aligned}
\Rightarrow \quad m_{12}(r,z,t) &= -\frac{2\tilde{\mu}+\nu}{\nu}2N\frac{\partial}{\partial r}H\,\mathbf{J}_0(hr)\,e^{\iota(\varpi t - nz)} \\
&= -\frac{2\tilde{\mu}+\nu}{\nu}2N\,H\,\mathbf{J}_0'(hr)\,h\,e^{\iota(\varpi t - nz)}
\end{aligned}$$

Consider $0 = m_{r\theta}(a,z,t) = m_{12}(a,z,t) = -\dfrac{2\tilde{\mu}+\nu}{\nu}2N\,H\,\mathbf{J}_0'(ha)\,h\,e^{\iota(\varpi t - nz)}$

$\Rightarrow \quad 0 = -\dfrac{2\tilde{\mu}+\nu}{\nu}2N\,\mathbf{J}_0'(ha)\,h\,H \quad \Rightarrow \quad 0 = \mathbf{J}_0'(ha)\,h\,H$ where H is arbitrary.

$$\Rightarrow \quad 0 = h\,\mathbf{J}_0'(ha)$$

We have $\mathbf{J}_0(r) = 1 + \sum_{k=1}^{\infty}\dfrac{(-1)^k}{(k!)^2}\left(\dfrac{r}{2}\right)^{2k} = 1 - \left(\dfrac{r}{2}\right)^2 + \sum_{k=2}^{\infty}\dfrac{(-1)^k}{(k!)^2}\left(\dfrac{r}{2}\right)^{2k}$

$$\Rightarrow \quad \mathbf{J}_0'(r) = 0 - 2\left(\frac{r}{2}\right)^2\frac{1}{2} + \sum_{k=2}^{\infty}\frac{(-1)^k}{(k!)^2}2k\left(\frac{r}{2}\right)^{2k-1}\frac{1}{2} = -\left(\frac{r}{2}\right)^2 + \sum_{k=2}^{\infty}\frac{(-1)^k}{(k!)^2}k\left(\frac{r}{2}\right)^{2k-1}$$

$$\Rightarrow \quad \mathbf{J}_0'(ha) = -\left(\frac{ha}{2}\right)^2 + \sum_{k=2}^{\infty}\frac{(-1)^k}{(k!)^2}k\left(\frac{ha}{2}\right)^{2k-1} \cong -\left(\frac{ha}{2}\right)^2 = \frac{-h^2a^2}{4}$$

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Thus $0 = h\,\mathbf{J}_0'(ha) \cong h\left(\dfrac{-h^2a^2}{4}\right) = \dfrac{-h^3a^2}{4} \quad \Rightarrow \quad h^3 = 0 \quad \Rightarrow \quad h = 0$

$\Rightarrow \quad 0 = h^2 = \dfrac{\varpi^2\varsigma\,\rho - 2(2\tilde{\mu}+\nu)}{2(2\aleph+N)} - n^2$ from (2.2). But $\varpi^2 = n^2c^2$ or $\dfrac{\varpi^2}{c^2} = n^2$

$$\Rightarrow \quad 0 = \frac{\varpi^2\varsigma\,\rho - 2(2\tilde{\mu}+\nu)}{2(2\aleph+N)} - \frac{\varpi^2}{c^2} \quad \Rightarrow \quad \frac{\varpi^2}{c^2} = \frac{\varpi^2\varsigma\,\rho - 2(2\tilde{\mu}+\nu)}{2(2\aleph+N)}$$

$$\Rightarrow \quad \frac{\varpi^2\, 2(2\aleph+N)}{\varpi^2\varsigma\,\rho - 2(2\tilde{\mu}+\nu)} = c^2$$

So the micro elastic wave is propagated in an isotropic solid with the speed c given by

$$c^2 = \frac{\varpi^2\, 2(2\aleph+N)}{\varpi^2\varsigma\,\rho - 2(2\tilde{\mu}+\nu)} \qquad \cdots\cdots\cdots\cdots(2.3)$$

and is known as ***micro-elastic longitudinal wave.***

Clearly this micro elastic wave is dispersive and has cut-off frequency ϖ_0 so that

$$\varpi_0^2 = \frac{2(2\tilde{\mu}+\nu)}{\varsigma\rho} < \varpi^2$$

Consider $0 = \tau_{rr}(a,z,t) = \tau_{11}(a,z,t)$

$$= \tilde{\lambda}\,\ell_{kk}(a,z,t)\delta_{11} + (2\tilde{\mu}+\nu)\ell_{11}(a,z,t) - \nu\,\omega_{11}(a,z,t)$$
$$= \tilde{\lambda}\,\ell_{kk}(a,z,t)1 + (2\tilde{\mu}+\nu)\ell_{11}(a,z,t) - \nu\,0$$
$$= \tilde{\lambda}[\ell_{11}(a,z,t) + \ell_{22}(a,z,t) + \ell_{33}(a,z,t)] + (2\tilde{\mu}+\nu)\ell_{11}(a,z,t)$$
$$= \tilde{\lambda}[\ell_{11}(a,z,t) + 0 + \ell_{33}(a,z,t)] + (2\tilde{\mu}+\nu)\ell_{11}(a,z,t)$$
$$= (\tilde{\lambda} + 2\tilde{\mu}+\nu)\ell_{11}(a,z,t) + \tilde{\lambda}\ell_{33}(a,z,t)$$

But from (2.1a): $\quad \ell_{11}(r,z,t) = A\,\mathbf{J}_0(hr)\,\mathrm{e}^{\iota(\varpi t - nz)}$

And from (2.1b): $\quad \ell_{33}(r,z,t) = B\,\mathbf{J}_0(hr)\,\mathrm{e}^{\iota(\varpi t - nz)}$.

$$\Rightarrow \quad 0 = (\tilde{\lambda} + 2\tilde{\mu}+\nu)A\,\mathbf{J}_0(ha)\,\mathrm{e}^{\iota(\varpi t - nz)} + \tilde{\lambda}\,B\,\mathbf{J}_0(ha)\,\mathrm{e}^{\iota(\varpi t - nz)}$$

$$\Rightarrow \quad 0 = (\tilde{\lambda} + 2\tilde{\mu}+\nu)A + \tilde{\lambda}\,B$$

Thus we have no information about the speed of micro-elastic wave from the boundary condition $0 = \tau_{rr}(a,z,t)$

3. Torsinal micro- elastic waves in an isotropic elastic solid of circular cross-section

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Consider an isotropic and elastic solid of circular cross-section whose radius is a. Let the direction of propagation of wave is taken to be parallel to x_3 - axis. Since the rod is of circular cross-section, cylindrical polar co-ordinates (r, θ, z) are considered.

And the components of micro-strain tensor are given by

$$\ell_{22}(r,\theta,z,t)=\ell_{22}(r,z,t) \quad ; \quad \ell_{12}(r,\theta,z,t)=\ell_{12}(r,z,t) \quad ;$$

$$\ell_{23}(r,\theta,z,t)=\ell_{23}(r,z,t) \quad ; \quad \ell_{11}(r,\theta,z,t)=0=\ell_{13}(r,\theta,z,t)=\ell_{33}(r,\theta,z,t)$$

$$\Rightarrow \quad \frac{\partial}{\partial\theta}\ell_{22}(r,z,\theta,t)=\frac{\partial}{\partial\theta}\ell_{22}(r,z,t)=0=\frac{\partial}{\partial\theta}\ell_{12}(r,z,t)=\frac{\partial}{\partial\theta}\ell_{12}(r,z,\theta,t)$$

and $\frac{\partial}{\partial\theta}\ell_{23}(r,z,\theta,t)=\frac{\partial}{\partial\theta}\ell_{23}(r,z,t)=0$

$$\Rightarrow \quad \text{Div Grad}\,\ell_{ij}=\frac{1}{r}\frac{\partial\ell_{ij}}{\partial r}+\frac{\partial}{\partial r}\frac{\partial\ell_{ij}}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{\partial\ell_{ij}}{\partial\theta}+\frac{\partial}{\partial z}\frac{\partial\ell_{ij}}{\partial z}$$

$$=\frac{1}{r}\frac{\partial\ell_{ij}}{\partial r}+\frac{\partial}{\partial r}\frac{\partial\ell_{ij}}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\theta}0+\frac{\partial}{\partial z}\frac{\partial\ell_{ij}}{\partial z}=\frac{\partial}{\partial r}\frac{\partial\ell_{ij}}{\partial r}+\frac{1}{r}\frac{\partial\ell_{ij}}{\partial r}+\frac{\partial}{\partial z}\frac{\partial\ell_{ij}}{\partial z}$$

Then equation (1.1) takes the form

$$\frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}0=(\Lambda+2\aleph+N)\text{Div Grad}0+\Lambda\text{Div Grad}\,\ell_{22}+\Lambda\text{Div Grad}0$$

$$-(\tilde{\lambda}+2\tilde{\mu}+\nu)0-\tilde{\lambda}\,\ell_{22}-\tilde{\lambda}\,0$$

$$\Rightarrow \quad 0=\Lambda\,\text{Div Grad}\,\ell_{22}-\tilde{\lambda}\ell_{22} \qquad \cdots\cdots\cdots\cdots(3a')$$

Equation (1.2) takes the form

$$\frac{\varsigma\rho}{2}\frac{\partial^2\ell_{22}}{\partial t^2}=(\Lambda+2\aleph+N)\text{Div Grad}\,\ell_{22}+\Lambda\,\text{Div Grad}0+\Lambda\,\text{Div Grad}0$$

$$-(\tilde{\lambda}+2\tilde{\mu}+\nu)\ell_{22}-\tilde{\lambda}0-\tilde{\lambda}0$$

$$\Rightarrow \quad \frac{\varsigma\rho}{2}\frac{\partial^2\ell_{22}}{\partial t^2}=(\Lambda+2\aleph+N)\text{Div Grad}\,\ell_{22}-(\tilde{\lambda}+2\tilde{\mu}+\nu)\ell_{22} \qquad \cdots\cdots(3b')$$

Substituting $(3a')$ in $(3b')$, we have

$$\frac{\varsigma\rho}{2}\frac{\partial^2\ell_{22}}{\partial t^2}=(2\aleph+N)\text{Div Grad}\,\ell_{22}-(2\tilde{\mu}+\nu)\ell_{22}$$

$$\Rightarrow \quad \frac{\varsigma\rho}{2}\frac{\partial^2\ell_{22}}{\partial t^2}=(2\aleph+N)\left(\frac{\partial}{\partial r}\frac{\partial\ell_{22}}{\partial r}+\frac{1}{r}\frac{\partial\ell_{22}}{\partial r}+\frac{\partial}{\partial z}\frac{\partial\ell_{22}}{\partial z}\right)-(2\tilde{\mu}+\nu)\ell_{22} \qquad \cdots(3a)$$

Equation (1.4) takes the form

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$$\frac{\varsigma\rho}{2}\frac{\partial^2 \ell_{23}}{\partial t^2} = (2\aleph + N)\,\mathrm{DivGrad}\ell_{23} - (2\tilde{\mu}+\nu)\ell_{23}$$

$$\Rightarrow \quad \frac{\varsigma\,\rho}{2}\frac{\partial^2 \ell_{23}}{\partial t^2} = (2\aleph + N)\left(\frac{\partial}{\partial r}\frac{\partial \ell_{23}}{\partial r} + \frac{1}{r}\frac{\partial \ell_{23}}{\partial r} + \frac{\partial}{\partial z}\frac{\partial \ell_{23}}{\partial z}\right) - (2\tilde{\mu}+\nu)\ell_{23} \qquad \cdots(3\text{b})$$

Equation (1.6) takes the form

$$\frac{\varsigma\rho}{2}\frac{\partial^2 \ell_{12}}{\partial t^2} = (2\aleph + N)\,\mathrm{DivGrad}\ell_{12} - (2\tilde{\mu}+\nu)\ell_{12}$$

$$\Rightarrow \quad \frac{\varsigma\,\rho}{2}\frac{\partial^2 \ell_{12}}{\partial t^2} = (2\aleph + N)\left(\frac{\partial}{\partial r}\frac{\partial \ell_{12}}{\partial r} + \frac{1}{r}\frac{\partial \ell_{12}}{\partial r} + \frac{\partial}{\partial z}\frac{\partial \ell_{12}}{\partial z}\right) - (2\tilde{\mu}+\nu)\ell_{12} \qquad \cdots(3\text{c})$$

Let us seek the solutions of the equations $(3\text{a}), (3\text{b}), (3\text{c})$ in the form

$$\ell_{22}(r, z, t) = A\mathbf{J}_0(hr)\,\mathrm{e}^{\iota(\varpi t - nz)} \qquad \cdots\cdots\cdots\cdots(3.1\text{a})$$

$$\ell_{23}(r, z, t) = B\mathbf{J}_0(hr)\,\mathrm{e}^{\iota(\varpi t - nz)} \qquad \cdots\cdots\cdots\cdots(3.1\text{b})$$

$$\ell_{12}(r, z, t) = H\,\mathbf{J}_0(hr)\,\mathrm{e}^{\iota(\varpi t - nz)} \qquad \cdots\cdots\cdots\cdots(3.1\text{c})$$

where $\mathbf{J}_0$ is Bessel's function of order zero; A, B, H are arbitrary constants; ϖ is the angular frequency, and n is the wave number

Substituting (3.1a) in (3a):

$$0 = \frac{\varsigma\,\rho}{2}\frac{\partial^2 \ell_{22}}{\partial t^2} - (2\aleph + N)\left(\frac{\partial}{\partial r}\frac{\partial \ell_{22}}{\partial r} + \frac{1}{r}\frac{\partial \ell_{22}}{\partial r} + \frac{\partial}{\partial z}\frac{\partial \ell_{22}}{\partial z}\right) + (2\tilde{\mu}+\nu)\ell_{22}$$

$$= \frac{\varsigma\,\rho}{2}\frac{\partial^2}{\partial t^2}A\mathbf{J}_0(hr)\,\mathrm{e}^{\iota(\varpi t - nz)} + (2\tilde{\mu}+\nu)A\mathbf{J}_0(hr)\,\mathrm{e}^{\iota(\varpi t - nz)}$$
$$- (2\aleph + N)\left[\frac{\partial}{\partial r}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial}{\partial z}\frac{\partial}{\partial z}\right]A\mathbf{J}_0(hr)\,\mathrm{e}^{\iota(\varpi t - nz)}$$

$$= \frac{\varsigma\,\rho}{2}A\mathbf{J}_0(hr)(\iota\varpi)^2\,\mathrm{e}^{\iota(\varpi t - nz)} + (2\tilde{\mu}+\nu)A\mathbf{J}_0(hr)\,\mathrm{e}^{\iota(\varpi t - nz)}$$
$$- (2\aleph + N)A\left[\mathbf{J}_0''(hr)h^2 + \frac{1}{r}\mathbf{J}_0'(hr)h + \mathbf{J}_0(hr)(-n\iota)^2\right]\mathrm{e}^{\iota(\varpi t - nz)}$$

Hence $$0 = \frac{\varsigma\,\rho}{2}\mathbf{J}_0(hr)\iota^2\varpi^2 + (2\tilde{\mu}+\nu)\mathbf{J}_0(hr)$$
$$- (2\aleph + N)\left[\mathbf{J}_0''(hr)h^2 + \frac{1}{r}\mathbf{J}_0'(hr)h + \mathbf{J}_0(hr)n^2\iota^2\right]$$

$$= -\varpi^2\frac{\varsigma\,\rho}{2}\mathbf{J}_0(hr) + (2\tilde{\mu}+\nu)\mathbf{J}_0(hr)$$

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$$-\left(\frac{2\aleph+N}{r^2}\right)\left[\mathbf{J}_0''(hr)h^2r^2+\mathbf{J}_0'(hr)hr-r^2n^2\,\mathbf{J}_0(hr)\right]$$

But $h^2r^2\,\mathbf{J}_0''(hr)+hr\,\mathbf{J}_0'(hr)+\left(h^2r^2-0\right)\mathbf{J}_0(hr)=0$

$$\Rightarrow\quad 0=-\varpi^2\frac{\varsigma\,\rho}{2}\mathbf{J}_0(hr)+(2\tilde{\mu}+\nu)\mathbf{J}_0(hr)-\frac{2\aleph+N}{r^2}\left[-h^2r^2\,\mathbf{J}_0(hr)-r^2n^2\,\mathbf{J}_0(hr)\right]$$

$$\Rightarrow\quad 0=-\varpi^2\frac{\varsigma\,\rho}{2}+(2\tilde{\mu}+\nu)-\frac{2\aleph+N}{r^2}\left[-h^2r^2-r^2n^2\right]$$

$$\Rightarrow\quad \varpi^2\frac{\varsigma\,\rho}{2}-(2\tilde{\mu}+\nu)=(2\aleph+N)\left[h^2+n^2\right]$$

$$\Rightarrow\quad \varpi^2\varsigma\,\rho-2(2\tilde{\mu}+\nu)=2(2\aleph+N)\left[h^2+n^2\right]\qquad\Rightarrow\quad\frac{\varpi^2\varsigma\,\rho-2(2\tilde{\mu}+\nu)}{2(2\aleph+N)}=h^2+n^2$$

$$\Rightarrow\quad h^2=\frac{\varpi^2\varsigma\,\rho-2(2\tilde{\mu}+\nu)}{2(2\aleph+N)}-n^2\qquad\cdots\cdots\cdots\cdots\cdots(3.2)$$

Similarly substituting (3.1b) in (3b): and (3.1c) in (3c): we will get same result (3.2) since are (3a), (3b), (3c) are identical differential equations.

On stress free boundary surface, the required boundary conditions are

$$\tau_{r\theta}(r,z,t)=0\quad;\quad m_{rr}(r,z,t)=0=m_{rz}(r,z,t)\ \text{ at }\ r=a$$

Recall (5): $\tau_{ij}=\tilde{\lambda}\,\ell_{kk}\,\delta_{ij}+(2\tilde{\mu}+\nu)\ell_{ij}-\nu\,\omega_{ij}$ from reference [2]

Consider $0=\tau_{r\theta}(a,z,t)=\tau_{12}(a,z,t)$

$$=\tilde{\lambda}\,\ell_{kk}(a,z,t)\delta_{12}+(2\tilde{\mu}+\nu)\ell_{12}(a,z,t)-\nu\,\omega_{12}(a,z,t)$$

$$=\tilde{\lambda}\,\ell_{kk}(a,z,t)0+(2\tilde{\mu}+\nu)\ell_{12}(a,z,t)-\nu\,\omega_{12}(a,z,t)$$

$$\Rightarrow\quad \nu\,\omega_{12}(a,z,t)=(2\tilde{\mu}+\nu)\ell_{12}(a,z,t)$$

Recall (6.2): $m_{qi}(x_1,x_2,x_3,t)=\varepsilon_{ijk}\,\sigma_{qkj}(x_1,x_2,x_3,t)$

And (6): $\sigma_{qij}=\Lambda\ell_{kk,q}\,\delta_{ij}+(2\aleph+N)\,\ell_{ij,q}-\nu\,\omega_{ij,q}$ from reference [2]

$$\Rightarrow\quad m_{qi}=\varepsilon_{ijk}\left[\Lambda\ell_{pp,q}\delta_{kj}+(2\aleph+N)\ell_{kj,q}-N\,\omega_{kj,q}\right]$$

But $m_{13}=\varepsilon_{3jk}\left[\Lambda\ell_{pp,1}\delta_{kj}+(2\aleph+N)\ell_{kj,1}-N\,\omega_{kj,1}\right]$

$$=\varepsilon_{312}\left[\Lambda\ell_{pp,1}\delta_{12}+(2\aleph+N)\ell_{12,1}-N\,\omega_{12,1}\right]$$

$$+\varepsilon_{321}\left[\Lambda\ell_{pp,1}\delta_{21}+(2\aleph+N)\ell_{21,1}-N\,\omega_{21,1}\right]$$

$$=1\left[\Lambda\ell_{pp,1}0+(2\aleph+N)\ell_{12,1}-N\,\omega_{12,1}\right]$$

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$$-1\left[\Lambda\ell_{pp,1}0+(2\aleph+N)\ell_{12,1}+N\,\omega_{12,1}\right]=-2N\,\omega_{12,1}$$

$$\Rightarrow \quad m_{13}(r,z,t)=-2N\,\omega_{12,1}(r,z,t)=-2N\frac{\partial}{\partial r}\omega_{12}(r,z,t)$$

$$=-2N\frac{\partial}{\partial r}\left(\frac{2\tilde{\mu}+\nu}{\nu}\ell_{12}(r,z,t)\right)=-2N\frac{2\tilde{\mu}+\nu}{\nu}\frac{\partial}{\partial r}\ell_{12}(r,z,t)$$

But from

$$(3.1c): \quad \ell_{12}(r,z,t)=H\,\mathbf{J}_0(hr)\,e^{\iota(\varpi t-nz)}.$$

$$\Rightarrow \quad m_{13}(r,z,t)=-\frac{2\tilde{\mu}+\nu}{\nu}2N\frac{\partial}{\partial r}H\,\mathbf{J}_0(hr)\,e^{\iota(\varpi t-nz)}$$

$$=-\frac{2\tilde{\mu}+\nu}{\nu}2N\,H\,\mathbf{J}_0'(hr)h\,e^{\iota(\varpi t-nz)}$$

Consider $0=m_{rz}(a,z,t)=m_{13}(a,z,t)=-\frac{2\tilde{\mu}+\nu}{\nu}2N\,H\,\mathbf{J}_0'(ha)h\,e^{\iota(\varpi t-nz)}$

$$\Rightarrow \quad 0=-\frac{2\tilde{\mu}+\nu}{\nu}2N\,\mathbf{J}_0'(ha)hH \quad \Rightarrow \quad 0=\mathbf{J}_0'(ha)hH \text{ where } H \text{ is arbitrary.}$$

$$\Rightarrow \quad 0=h\,\mathbf{J}_0'(ha)$$

We have $\mathbf{J}_0(r)=1+\sum_{k=1}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{r}{2}\right)^{2k}=1-\left(\frac{r}{2}\right)^2+\sum_{k=2}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{r}{2}\right)^{2k}$

$$\Rightarrow \quad \mathbf{J}_0'(r)=0-2\left(\frac{r}{2}\right)^2\frac{1}{2}+\sum_{k=2}^{\infty}\frac{(-1)^k}{(k!)^2}2k\left(\frac{r}{2}\right)^{2k-1}\frac{1}{2}=-\left(\frac{r}{2}\right)^2+\sum_{k=2}^{\infty}\frac{(-1)^k}{(k!)^2}k\left(\frac{r}{2}\right)^{2k-1}$$

Thus $\mathbf{J}_0'(ha)=-\left(\frac{ha}{2}\right)^2+\sum_{k=2}^{\infty}\frac{(-1)^k}{(k!)^2}k\left(\frac{ha}{2}\right)^{2k-1}\cong-\left(\frac{ha}{2}\right)^2=\frac{-h^2a^2}{4}$

$$\Rightarrow \quad 0=h\,\mathbf{J}_0'(ha)\cong h\left(\frac{-h^2a^2}{4}\right)=\frac{-h^3a^2}{4} \quad \Rightarrow \quad h^3=0 \quad \Rightarrow \quad h=0$$

$$\Rightarrow \quad 0=h^2=\frac{\varpi^2\varsigma\,\rho-2(2\tilde{\mu}+\nu)}{2(2\aleph+N)}-n^2 \text{ from }(3.2)\text{. But } \varpi^2=n^2c^2 \text{ or } \frac{\varpi^2}{c^2}=n^2$$

$$\Rightarrow \quad 0=\frac{\varpi^2\varsigma\,\rho-2(2\tilde{\mu}+\nu)}{2(2\aleph+N)}-\frac{\varpi^2}{c^2} \quad \Rightarrow \quad \frac{\varpi^2}{c^2}=\frac{\varpi^2\varsigma\,\rho-2(2\tilde{\mu}+\nu)}{2(2\aleph+N)}$$

$$\Rightarrow \quad \frac{\varpi^2\,2(2\aleph+N)}{\varpi^2\varsigma\,\rho-2(2\tilde{\mu}+\nu)}=c^2$$

So the micro elastic wave is propagated in an isotropic solid with the speed c given by

$$c^2=\frac{\varpi^2\,2(2\aleph+N)}{\varpi^2\varsigma\,\rho-2(2\tilde{\mu}+\nu)} \qquad \cdots\cdots\cdots\cdots(3.3)$$

and is ***micro-elastic torsional wave***.

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Clearly this micro elastic wave is dispersive and has cut-off frequency ϖ_0 so that

$$\varpi_0^2 = \frac{2(2\tilde{\mu}+\nu)}{\varsigma\rho} < \varpi^2$$

Observe that the speed of micro-elastic torsional wave is same as the speed of micro-elastic longitudinal wave.

Consider $m_{11} = \varepsilon_{1jk}\left[\Lambda\ell_{pp,1}\delta_{kj} + (2\aleph+N)\ell_{kj,1} - N\,\omega_{kj,1}\right]$

$$= \varepsilon_{123}\left[\Lambda\ell_{pp,1}\delta_{23} + (2\aleph+N)\ell_{23,1} - N\,\omega_{23,1}\right]$$

$$+\varepsilon_{132}\left[\Lambda\ell_{pp,1}\delta_{32} + (2\aleph+N)\ell_{32,1} - N\,\omega_{32,1}\right]$$

$$= 1\left[\Lambda\ell_{pp,1}0 + (2\aleph+N)\ell_{23,1} - N\,\omega_{23,1}\right]$$

$$-1\left[\Lambda\ell_{pp,1}0 + (2\aleph+N)\ell_{23,1} + N\,\omega_{23,1}\right] = -2N\,\omega_{23,1}$$

$$\Rightarrow \quad 0 = m_{rr}(a,z,t) = m_{11}(a,z,t) = -2N\omega_{23,1}(a,z,t) \quad \Rightarrow \quad 0 = \omega_{23,1}(a,z,t)$$

Thus we have no information about the speed of micro-elastic wave from the boundary condition $0 = m_{rr}(a,z,t)$

4. Propagation of one dimensional longitudinal micro - elastic wave in an isotropic elastic solid of circular cross-section

In section **2,** longitudinal micro-elastic wave propagation in an isotropic elastic solid of circular cross-section is studied, in which the components of micro-strain tensor are functions of r, z, t.

For the one dimensional longitudinal micro-elastic wave propagation, the components of micro-strain tensor are given by

$$\ell_{11}(r,\theta,z,t) = \ell_{11}(r,t) \quad ; \quad \ell_{33}(r,\theta,z,t) = \ell_{33}(r,t) \quad ;$$

$$\ell_{13}(r,\theta,z,t) = \ell_{13}(r,t) \quad ; \quad \ell_{22}(r,\theta,z,t) = 0 = \ell_{12}(r,\theta,z,t) = \ell_{23}(r,\theta,z,t)$$

$$\Rightarrow \quad \frac{\partial}{\partial\theta}\ell_{11}(r,z,\theta,t) = \frac{\partial}{\partial\theta}\ell_{11}(r,t) = 0 = \frac{\partial}{\partial\theta}\ell_{33}(r,t) = \frac{\partial}{\partial\theta}\ell_{33}(r,z,\theta,t)$$

$$\text{and} \quad \frac{\partial}{\partial\theta}\ell_{13}(r,z,\theta,t) = \frac{\partial}{\partial\theta}\ell_{13}(r,t) = 0 = \frac{\partial}{\partial z}\ell_{13}(r,t) = \frac{\partial}{\partial z}\ell_{13}(r,z,\theta,t)$$

$$\text{and} \quad \frac{\partial}{\partial z}\ell_{11}(r,z,\theta,t) = \frac{\partial}{\partial z}\ell_{11}(r,t) = 0 = \frac{\partial}{\partial z}\ell_{33}(r,t) = \frac{\partial}{\partial z}\ell_{13}(r,z,\theta,t)$$

$$\Rightarrow \quad \text{Div Grad}\,\ell_{ij} = \frac{1}{r}\frac{\partial\ell_{ij}}{\partial r} + \frac{\partial}{\partial r}\frac{\partial\ell_{ij}}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{\partial\ell_{ij}}{\partial\theta} + \frac{\partial}{\partial z}\frac{\partial\ell_{ij}}{\partial z}$$

$$= \frac{1}{r}\frac{\partial\ell_{ij}}{\partial r} + \frac{\partial}{\partial r}\frac{\partial\ell_{ij}}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta}0 + \frac{\partial}{\partial z}0 = \frac{\partial}{\partial r}\frac{\partial\ell_{ij}}{\partial r} + \frac{1}{r}\frac{\partial\ell_{ij}}{\partial r}$$

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Equation (1.1) takes the form

$$\frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}\ell_{11} = (\Lambda + 2\aleph + N)\operatorname{Div}\operatorname{Grad}\ell_{11} + \Lambda\operatorname{Div}\operatorname{Grad}0 + \Lambda\operatorname{Div}\operatorname{Grad}\ell_{33}$$

$$-\left(\tilde{\lambda} + 2\tilde{\mu} + \nu\right)\ell_{11} - \tilde{\lambda}0 - \tilde{\lambda}\ell_{33}$$

$$\Rightarrow \quad \frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}\ell_{11} = (\Lambda + 2\aleph + N)\operatorname{Div}\operatorname{Grad}\ell_{11} + \Lambda\operatorname{Div}\operatorname{Grad}\ell_{33} - \left(\tilde{\lambda} + 2\tilde{\mu} + \nu\right)\ell_{11} - \tilde{\lambda}\ell_{33} \quad \cdots\cdots\cdots(4a')$$

Equation (1.2) takes the form

$$\frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}0 = (\Lambda + 2\aleph + N)\operatorname{Div}\operatorname{Grad}0 + \Lambda\operatorname{Div}\operatorname{Grad}\ell_{11} + \Lambda\operatorname{Div}\operatorname{Grad}\ell_{33} - \left(\tilde{\lambda} + 2\tilde{\mu} + \nu\right)0 - \tilde{\lambda}\ell_{11} - \tilde{\lambda}\ell_{33}$$

$$\Rightarrow \quad 0 = \Lambda\operatorname{Div}\operatorname{Grad}\ell_{11} + \Lambda\operatorname{Div}\operatorname{Grad}\ell_{33} - \tilde{\lambda}\ell_{11} - \tilde{\lambda}\ell_{33} \quad \cdots\cdots(4b')$$

Similarly equation (1.3) takes the form

$$\frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}\ell_{33} = (\Lambda + 2\aleph + N)\operatorname{Div}\operatorname{Grad}\ell_{33} + \Lambda\operatorname{Div}\operatorname{Grad}\ell_{11} + \Lambda\operatorname{Div}\operatorname{Grad}0 - \left(\tilde{\lambda} + 2\tilde{\mu} + \nu\right)\ell_{33} - \tilde{\lambda}\ell_{11} - \tilde{\lambda}0$$

$$\Rightarrow \quad \frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}\ell_{33} = (\Lambda + 2\aleph + N)\operatorname{Div}\operatorname{Grad}\ell_{33} + \Lambda\operatorname{Div}\operatorname{Grad}\ell_{11} - \left(\tilde{\lambda} + 2\tilde{\mu} + \nu\right)\ell_{33} - \tilde{\lambda}\ell_{11} \quad \cdots\cdots\cdots(4c')$$

Substituting $(4b')$ in $(4a')$, we get

$$\frac{\varsigma\rho}{2}\frac{\partial^2\ell_{11}}{\partial t^2} = (2\aleph + N)\operatorname{Div}\operatorname{Grad}\ell_{11} - (2\tilde{\mu} + \nu)\ell_{11}$$

$$\Rightarrow \quad \frac{\varsigma\rho}{2}\frac{\partial^2\ell_{11}}{\partial t^2} = (2\aleph + N)\left[\frac{\partial}{\partial r}\frac{\partial\ell_{11}}{\partial r} + \frac{1}{r}\frac{\partial\ell_{11}}{\partial r}\right] - (2\tilde{\mu} + \nu)\ell_{11} \quad \cdots\cdots(4a)$$

Substituting $(4b')$ in $(4c')$, we get

$$\frac{\varsigma\rho}{2}\frac{\partial^2\ell_{33}}{\partial t^2} = (2\aleph + N)\operatorname{Div}\operatorname{Grad}\ell_{33} - (2\tilde{\mu} + \nu)\ell_{33}$$

$$\Rightarrow \quad \frac{\varsigma\rho}{2}\frac{\partial^2\ell_{33}}{\partial t^2} = (2\aleph + N)\left[\frac{\partial}{\partial r}\frac{\partial\ell_{33}}{\partial r} + \frac{1}{r}\frac{\partial\ell_{33}}{\partial r}\right] - (2\tilde{\mu} + \nu)\ell_{33} \quad \cdots\cdots(4b)$$

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Again equation (1.5) takes the form

$$\frac{\varsigma\rho}{2}\frac{\partial^2\ell_{31}}{\partial t^2}=(2\aleph+N)\left[\frac{\partial}{\partial r}\frac{\partial\ell_{31}}{\partial r}+\frac{1}{r}\frac{\partial\ell_{31}}{\partial r}\right]-(2\tilde{\mu}+\nu)\ell_{31} \qquad \cdots\cdots(4c)$$

Let us seek the solutions of the equations $(4a),(4b),(4c)$ in the form

$$\ell_{11}(r,t)=A\mathbf{J}_0(hr)\, e^{\iota\varpi t} \qquad \cdots\cdots(4.1a)$$

$$\ell_{33}(r,t)=B\mathbf{J}_0(hr)e^{\iota\varpi t} \qquad \cdots\cdots(4.1b)$$

$$\ell_{13}(r,t)=H\,\mathbf{J}_0(hr)e^{\iota\varpi t} \qquad \cdots\cdots(4.1c)$$

where ϖ is the angular frequency, n is the wave number and $\mathbf{J}_0$ is Bessel's function of order zero; A, B, H are arbitrary constants.

Substituting $(4.1a)$ in $(4a)$:

$$0=\frac{\varsigma\rho}{2}\frac{\partial^2\ell_{11}}{\partial t^2}-(2\aleph+N)\left[\frac{\partial}{\partial r}\frac{\partial\ell_{11}}{\partial r}+\frac{1}{r}\frac{\partial\ell_{11}}{\partial r}\right]+(2\tilde{\mu}+\nu)\ell_{11}$$

$$=\frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}A\mathbf{J}_0(hr)e^{\iota\varpi t}+(2\tilde{\mu}+\nu)A\mathbf{J}_0(hr)e^{\iota\varpi t}$$

$$-(2\aleph+N)\left[\frac{\partial}{\partial r}\frac{\partial}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}\right]A\mathbf{J}_0(hr)e^{\iota\varpi t}$$

$$=\frac{\varsigma\rho}{2}A\mathbf{J}_0(hr)(\iota\varpi)^2e^{\iota\varpi t}+(2\tilde{\mu}+\nu)A\mathbf{J}_0(hr)e^{\iota\varpi t}$$

$$-(2\aleph+N)A\left[\mathbf{J}_0''(hr)h^2+\frac{1}{r}\mathbf{J}_0'(hr)h\right]e^{\iota\varpi t}$$

Thus $$0=\frac{\varsigma\rho}{2}\mathbf{J}_0(hr)\iota^2\varpi^2+(2\tilde{\mu}+\nu)\mathbf{J}_0(hr)-(2\aleph+N)\left[\mathbf{J}_0''(hr)h^2+\frac{1}{r^2}\mathbf{J}_0'(hr)h\right]$$

Or $$0=\varpi^2\frac{\varsigma\rho}{2}\mathbf{J}_0(hr)+(2\tilde{\mu}+\nu)\mathbf{J}_0(hr)-\frac{2\aleph+N}{r^2}\left[\mathbf{J}_0''(hr)h^2r^2+\mathbf{J}_0'(hr)hr\right]$$

But $h^2r^2\,\mathbf{J}_0''(hr)+hr\mathbf{J}_0'(hr)+(h^2r^2-0)\mathbf{J}_0(hr)=0$

$$\Rightarrow \quad 0=-\varpi^2\frac{\varsigma\rho}{2}\mathbf{J}_0(hr)+(2\tilde{\mu}+\nu)\mathbf{J}_0(hr)-\frac{2\aleph+N}{r^2}\left[-h^2r^2\,\mathbf{J}_0(hr)\right]$$

$$\Rightarrow \quad 0=-\varpi^2\frac{\varsigma\rho}{2}+(2\tilde{\mu}+\nu)-\frac{2\aleph+N}{r^2}\left[-h^2r^2\right]$$

$$\Rightarrow \quad \varpi^2\frac{\varsigma\rho}{2}-(2\tilde{\mu}+\nu)=(2\aleph+N)h^2$$

$$\Rightarrow \quad \varpi^2\varsigma\rho-2(2\tilde{\mu}+\nu)=2(2\aleph+N)\,h^2$$

$$\Rightarrow \quad h^2=\frac{\varpi^2\varsigma\rho-2(2\tilde{\mu}+\nu)}{2(2\aleph+N)} \qquad \cdots\cdots(4.2)$$

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Similarly substituting $(4.1b)$ in $(4b)$**:** and $(4.1c)$ in $(4c)$**:** we will get same result (4.2) since $(4a), (4b), (4c)$ are identical differential equations.

On stress free boundary surface, the required boundary conditions are

$$\tau_{rr}(r,t)=0=\tau_{rz}(r,t) \quad ; \quad m_{r\theta}(r,t)=0 \text{ at } r=a$$

Recall (5)**:** $\tau_{ij}=\tilde{\lambda}\,\ell_{kk}\,\delta_{ij}+(2\tilde{\mu}+\nu)\ell_{ij}-\nu\,\omega_{ij}$ from reference [2]

Consider $0=\tau_{rz}(a,t)=\tau_{13}(a,t)=\tilde{\lambda}\,\ell_{kk}(a,t)\delta_{13}+(2\tilde{\mu}+\nu)\ell_{13}(a,t)-\nu\,\omega_{13}(a,t)$

$$=\tilde{\lambda}\,\ell_{kk}(a,t)0+(2\tilde{\mu}+\nu)\ell_{13}(a,t)-\nu\,\omega_{13}(a,t)$$

$$\Rightarrow \quad \nu\,\omega_{13}(a,t)=(2\tilde{\mu}+\nu)\ell_{13}(a,t)$$

Recall (6.2)**:** $m_{qi}(x_1,x_2,x_3,t)=\varepsilon_{ijk}\,\sigma_{qkj}(x_1,x_2,x_3,t)$

And (6)**:** $\sigma_{qij}=\Lambda\ell_{kk,q}\,\delta_{ij}+(2\aleph+N)\;\ell_{ij,q}-\nu\,\omega_{ij,q}$ from reference [2]

$$\Rightarrow \quad m_{qi}=\varepsilon_{ijk}\left[\Lambda\ell_{pp,q}\,\delta_{kj}+(2\aleph+N)\ell_{kj,q}-N\,\omega_{kj,q}\right]$$

But $m_{12}=\varepsilon_{2jk}\left[\Lambda\ell_{pp,1}\,\delta_{kj}+(2\aleph+N)\ell_{kj,1}-N\,\omega_{kj,1}\right]$

$$=\varepsilon_{231}\left[\Lambda\ell_{pp,1}\,\delta_{13}+(2\aleph+N)\ell_{13,1}-N\,\omega_{13,1}\right]$$

$$+\varepsilon_{213}\left[\Lambda\ell_{pp,1}\,\delta_{31}+(2\aleph+N)\ell_{31,1}-N\,\omega_{31,1}\right]$$

$$=1\left[\Lambda\ell_{pp,1}\,0+(2\aleph+N)\ell_{13,1}-N\,\omega_{13,1}\right]$$

$$-1\left[\Lambda\ell_{pp,1}\,0+(2\aleph+N)\ell_{13,1}+N\,\omega_{13,1}\right]=-2N\,\omega_{13,1}$$

Thus $m_{12}(r,t)=-2N\,\omega_{13,1}(r,t)=-2N\dfrac{\partial}{\partial r}\omega_{13}(r,t)$

$$=-2N\frac{\partial}{\partial r}\left(\frac{2\tilde{\mu}+\nu}{\nu}\ell_{13}(r,t)\right)=-2N\,\frac{2\tilde{\mu}+\nu}{\nu}\frac{\partial}{\partial r}\ell_{13}(r,t)$$

But from $(4.1c)$**:** $\ell_{13}(r,t)=H\,\mathbf{J}_0(hr)\,\mathrm{e}^{\iota\varpi t}$.

$$\Rightarrow \quad m_{12}(r,t)=-\frac{2\tilde{\mu}+\nu}{\nu}2N\frac{\partial}{\partial r}H\,\mathbf{J}_0(hr)\,\mathrm{e}^{\iota\varpi t}=-\frac{2\tilde{\mu}+\nu}{\nu}2N\,H\,\mathbf{J}_0'(hr)h\,\mathrm{e}^{\iota\varpi t}$$

Consider $0=m_{r\theta}(a,t)=m_{12}(a,t)=-\dfrac{2\tilde{\mu}+\nu}{\nu}2N\,H\,\mathbf{J}_0'(ha)h\,\mathrm{e}^{\iota\varpi t}$

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$\Rightarrow \quad 0 = -\frac{2\tilde{\mu}+\nu}{\nu} 2N\, \mathbf{J}_0'(ha)hH \quad \Rightarrow \quad 0 = \mathbf{J}_0'(ha)hH$ where H is arbitrary**.**

$\Rightarrow \quad 0 = h\,\mathbf{J}_0'(ha)$

We have

$$\mathbf{J}_0(r) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{r}{2}\right)^{2k} = 1 - \left(\frac{r}{2}\right)^2 + \sum_{k=2}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{r}{2}\right)^{2k}$$

$$\Rightarrow \quad \mathbf{J}_0'(r) = 0 - 2\left(\frac{r}{2}\right)^2 \frac{1}{2} + \sum_{k=2}^{\infty} \frac{(-1)^k}{(k!)^2} 2k \left(\frac{r}{2}\right)^{2k-1} \frac{1}{2} = -\left(\frac{r}{2}\right)^2 + \sum_{k=2}^{\infty} \frac{(-1)^k}{(k!)^2} k \left(\frac{r}{2}\right)^{2k-1}$$

$$\Rightarrow \quad \mathbf{J}_0'(ha) = -\left(\frac{ha}{2}\right)^2 + \sum_{k=2}^{\infty} \frac{(-1)^k}{(k!)^2} k \left(\frac{ha}{2}\right)^{2k-1} \cong -\left(\frac{ha}{2}\right)^2 = \frac{-h^2a^2}{4}$$

Thus $0 = h\,\mathbf{J}_0'(ha) \cong h\left(\frac{-h^2a^2}{4}\right) = \frac{-h^3a^2}{4} \quad \Rightarrow \quad h^3 = 0 \quad \Rightarrow \quad h = 0$

$\Rightarrow \quad 0 = h^2 = \frac{\varpi^2 \varsigma\, \rho - 2(2\tilde{\mu}+\nu)}{2(2\aleph + N)}$ from (4.2)**.** But $\varpi^2 = n^2c^2$

$\Rightarrow \quad 0 = n^2c^2\varsigma\,\rho - 2(2\tilde{\mu}+\nu) \quad \Rightarrow \quad 2(2\tilde{\mu}+\nu) = n^2c^2\varsigma\,\rho$

$\Rightarrow \quad \frac{2(2\tilde{\mu}+\nu)}{n^2\varsigma\,\rho} = c^2$

So the micro elastic wave is propagated in an isotropic solid with the speed c given by

$$c^2 = \frac{2(2\tilde{\mu}+\nu)}{n^2\varsigma\,\rho} \qquad \cdots\cdots\cdots\cdots(4.3)$$

and is known as one dimensional ***micro-elastic longitudinal wave.***

Clearly this micro elastic wave is dispersive**.**

Consider $\quad 0 = \tau_{rr}(a,t) = \tau_{11}(a,t) = \tilde{\lambda}\,\ell_{kk}(a,t)\delta_{11} + (2\tilde{\mu}+\nu)\ell_{11}(a,t) - \nu\,\omega_{11}(a,t)$

$$= \tilde{\lambda}\,\ell_{kk}(a,t)1 + (2\tilde{\mu}+\nu)\ell_{11}(a,t) - \nu\, 0$$

$$= \tilde{\lambda}\,[\ell_{11}(a,t) + \ell_{22}(a,t) + \ell_{33}(a,t)] + (2\tilde{\mu}+\nu)\ell_{11}(a,t)$$

$$= \tilde{\lambda}\,[\ell_{11}(a,t) + 0 + \ell_{33}(a,t)] + (2\tilde{\mu}+\nu)\ell_{11}(a,t)$$

$$= (\tilde{\lambda} + 2\tilde{\mu}+\nu)\ell_{11}(a,t) + \tilde{\lambda}\ell_{33}(a,t)$$

But from (4.1a)**:** $\quad \ell_{11}(r,t) = A\,\mathbf{J}_0(hr)\mathrm{e}^{\iota\varpi t}$

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And from (4.1b): $\ell_{33}(r,t) = B\mathbf{J}_0(hr)\,\mathrm{e}^{\iota\varpi t}$.

$$\Rightarrow \qquad 0 = \left(\tilde{\lambda} + 2\tilde{\mu} + \nu\right) A\mathbf{J}_0(ha)\,\mathrm{e}^{\iota\varpi t} + \tilde{\lambda}\, B\mathbf{J}_0(ha)\,\mathrm{e}^{\iota\varpi t}$$

$$\Rightarrow \qquad 0 = \left(\tilde{\lambda} + 2\tilde{\mu} + \nu\right) A + \tilde{\lambda}\, B$$

Thus we have no information about the speed of micro-elastic wave from the boundary condition $0 = \tau_{rr}(a,t)$

5. Propagation of one dimensional torsinal micro - elastic wave in an isotropic elastic solid of circular cross-section:

In section **3,** torsinal micro-elastic wave propagation in an isotropic elastic solid of circular cross-section is studied, in which the components of micro-strain tensor are functions of r, z, t.

For the propagation of one dimensional torsinal micro-elastic wave, the components of micro-strain tensor are given by

$$\ell_{22}(r,\theta,z,t) = \ell_{22}(r,t) \quad ; \quad \ell_{12}(r,\theta,z,t) = \ell_{12}(r,t) \quad ;$$

$$\ell_{23}(r,\theta,z,t) = \ell_{23}(r,t) \quad ; \quad \ell_{11}(r,\theta,z,t) = 0 = \ell_{13}(r,\theta,z,t) = \ell_{33}(r,\theta,z,t)$$

$$\Rightarrow \qquad \frac{\partial}{\partial\theta}\ell_{22}(r,z,\theta,t) = \frac{\partial}{\partial\theta}\ell_{22}(r,t) = 0 = \frac{\partial}{\partial\theta}\ell_{12}(r,t) = \frac{\partial}{\partial\theta}\ell_{12}(r,z,\theta,t)$$

$$\text{And } \frac{\partial}{\partial\theta}\ell_{23}(r,z,\theta,t) = \frac{\partial}{\partial\theta}\ell_{23}(r,t) = 0 = \frac{\partial}{\partial z}\ell_{23}(r,t) = \frac{\partial}{\partial z}\ell_{23}(r,z,\theta,t)$$

$$\Rightarrow \qquad \operatorname{Div}\operatorname{Grad}\ell_{ij} = \frac{1}{r}\frac{\partial\ell_{ij}}{\partial r} + \frac{\partial}{\partial r}\frac{\partial\ell_{ij}}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{\partial\ell_{ij}}{\partial\theta} + \frac{\partial}{\partial z}\frac{\partial\ell_{ij}}{\partial z}$$

$$= \frac{1}{r}\frac{\partial\ell_{ij}}{\partial r} + \frac{\partial}{\partial r}\frac{\partial\ell_{ij}}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta}0 + \frac{\partial}{\partial z}0 = \frac{\partial}{\partial r}\frac{\partial\ell_{ij}}{\partial r} + \frac{1}{r}\frac{\partial\ell_{ij}}{\partial r}$$

Then equation (1.1) takes the form

$$\frac{\varsigma\rho}{2}\frac{\partial^2}{\partial t^2}0 = (\Lambda + 2\aleph + N)\operatorname{Div}\operatorname{Grad}0 + \Lambda\operatorname{Div}\operatorname{Grad}\ell_{22} + \Lambda\operatorname{Div}\operatorname{Grad}0$$

$$-\left(\tilde{\lambda} + 2\tilde{\mu} + \nu\right)0 - \tilde{\lambda}\ell_{22} - \tilde{\lambda}0$$

$$\Rightarrow \qquad 0 = \Lambda\operatorname{Div}\operatorname{Grad}\ell_{22} - \tilde{\lambda}\ell_{22} \qquad \cdots\cdots\cdots\cdots(5\text{a}')$$

Equation (1.2) takes the form

$$\frac{\varsigma\rho}{2}\frac{\partial^2\ell_{22}}{\partial t^2} = (\Lambda + 2\aleph + N)\operatorname{Div}\operatorname{Grad}\ell_{22} + \Lambda\operatorname{Div}\operatorname{Grad}0 + \Lambda\operatorname{Div}\operatorname{Grad}0$$

$$-\left(\tilde{\lambda} + 2\tilde{\mu} + \nu\right)\ell_{22} - \tilde{\lambda}0 - \tilde{\lambda}0$$

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$$\Rightarrow \quad \frac{\varsigma\,\rho}{2}\frac{\partial^2 \ell_{22}}{\partial t^2} = (\Lambda + 2\aleph + N)\text{Div Grad}\,\ell_{22} - \left(\tilde{\lambda} + 2\tilde{\mu} + \nu\right)\ell_{22} \qquad \cdots\cdots(5b')$$

Substituting $(5a')$ in $(5b')$, we have

$$\frac{\varsigma\,\rho}{2}\frac{\partial^2 \ell_{22}}{\partial t^2} = (2\aleph + N)\text{Div Grad}\,\ell_{22} - (2\tilde{\mu} + \nu)\ell_{22}$$

$$\Rightarrow \quad \frac{\varsigma\,\rho}{2}\frac{\partial^2 \ell_{22}}{\partial t^2} = (2\aleph + N)\left(\frac{\partial}{\partial r}\frac{\partial \ell_{22}}{\partial r} + \frac{1}{r}\frac{\partial \ell_{22}}{\partial r}\right) - (2\tilde{\mu} + \nu)\ell_{22} \qquad \cdots\cdots(5a)$$

Equation (1.4) takes the form

$$\frac{\varsigma\rho}{2}\frac{\partial^2 \ell_{23}}{\partial t^2} = (2\aleph + N)\,\text{DivGrad}\,\ell_{23} - (2\tilde{\mu} + \nu)\ell_{23}$$

$$\Rightarrow \quad \frac{\varsigma\,\rho}{2}\frac{\partial^2 \ell_{23}}{\partial t^2} = (2\aleph + N)\left(\frac{\partial}{\partial r}\frac{\partial \ell_{23}}{\partial r} + \frac{1}{r}\frac{\partial \ell_{23}}{\partial r}\right) - (2\tilde{\mu} + \nu)\ell_{23} \qquad \cdots\cdots(5b)$$

Equation (1.6) takes the form

$$\frac{\varsigma\rho}{2}\frac{\partial^2 \ell_{12}}{\partial t^2} = (2\aleph + N)\,\text{DivGrad}\,\ell_{12} - (2\tilde{\mu} + \nu)\ell_{12}$$

$$\Rightarrow \quad \frac{\varsigma\,\rho}{2}\frac{\partial^2 \ell_{12}}{\partial t^2} = (2\aleph + N)\left(\frac{\partial}{\partial r}\frac{\partial \ell_{12}}{\partial r} + \frac{1}{r}\frac{\partial \ell_{12}}{\partial r}\right) - (2\tilde{\mu} + \nu)\ell_{12} \qquad \cdots\cdots(5c)$$

Let us seek the solutions of the equations $(5a)$, $(5b)$, $(5c)$ in the form

$$\ell_{22}(r,t) = A\mathbf{J}_0(hr)\,\mathrm{e}^{\iota\varpi t} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(5.1a)$$

$$\ell_{23}(r,t) = B\,\mathbf{J}_0(hr)\,\mathrm{e}^{\iota\varpi t} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(5.1b)$$

$$\ell_{12}(r,t) = H\,\mathbf{J}_0(hr)\,\mathrm{e}^{\iota\varpi t} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(5.1c)$$

where ϖ is the angular frequency, n is the wave number and $\mathbf{J}_0$ is Bessel's function of order zero; A, B, H are arbitrary constants.

Substituting $(5.1a)$ in $(5a)$:

$$0 = \frac{\varsigma\,\rho}{2}\frac{\partial^2 \ell_{22}}{\partial t^2} - (2\aleph + N)\left(\frac{\partial}{\partial r}\frac{\partial \ell_{22}}{\partial r} + \frac{1}{r}\frac{\partial \ell_{22}}{\partial r}\right) + (2\tilde{\mu} + \nu)\ell_{22}$$

$$= \frac{\varsigma\,\rho}{2}\frac{\partial^2}{\partial t^2}A\,\mathbf{J}_0(hr)\,\mathrm{e}^{\iota\varpi t} + (2\tilde{\mu} + \nu)A\mathbf{J}_0(hr)\,\mathrm{e}^{\iota\varpi t}$$

$$- (2\aleph + N)\left[\frac{\partial}{\partial r}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}\right]A\,\mathbf{J}_0(hr)\,\mathrm{e}^{\iota\varpi t}$$

$$= \frac{\varsigma\,\rho}{2}A\mathbf{J}_0(hr)(\iota\varpi)^2\,\mathrm{e}^{\iota(\varpi t - nz)} + (2\tilde{\mu} + \nu)A\mathbf{J}_0(hr)\,\mathrm{e}^{\iota\varpi t}$$

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$$-(2\aleph+N)A\left[\mathbf{J}_0''(hr)h^2+\frac{1}{r}\mathbf{J}_0'(hr)h\right]e^{\iota\varpi t}$$

$$\Rightarrow \qquad 0=\frac{\varsigma\rho}{2}\mathbf{J}_0(hr)\iota^2\varpi^2+(2\tilde{\mu}+\nu)\mathbf{J}_0(hr)-(2\aleph+N)\left[\mathbf{J}_0''(hr)h^2+\frac{1}{r}\mathbf{J}_0'(hr)h\right]$$

$$=-\varpi^2\frac{\varsigma\rho}{2}\mathbf{J}_0(hr)+(2\tilde{\mu}+\nu)\mathbf{J}_0(hr)-\frac{2\aleph+N}{r^2}\left[\mathbf{J}_0''(hr)h^2r^2+\mathbf{J}_0'(hr)hr\right]$$

But $h^2r^2\,\mathbf{J}_0''(hr)+hr\,\mathbf{J}_0'(hr)+\left(h^2r^2-0\right)\mathbf{J}_0(hr)=0$

$$\Rightarrow \qquad 0=-\varpi^2\frac{\varsigma\rho}{2}\mathbf{J}_0(hr)+(2\tilde{\mu}+\nu)\mathbf{J}_0(hr)-\frac{2\aleph+N}{r^2}\left[-h^2r^2\,\mathbf{J}_0(hr)\right]$$

$$\Rightarrow \qquad 0=-\varpi^2\frac{\varsigma\rho}{2}+(2\tilde{\mu}+\nu)-\frac{2\aleph+N}{r^2}\left[-h^2r^2\right]$$

$$\Rightarrow \qquad \varpi^2\frac{\varsigma\rho}{2}-(2\tilde{\mu}+\nu)=(2\aleph+N)h^2$$

$$\Rightarrow \qquad \varpi^2\varsigma\rho-2(2\tilde{\mu}+\nu)=2(2\aleph+N)h^2$$

$$\Rightarrow \qquad h^2=\frac{\varpi^2\varsigma\rho-2(2\tilde{\mu}+\nu)}{2(2\aleph+N)} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(5.2)$$

Similarly substituting $(5.1\mathrm{b})$ in $(5\mathrm{b})$: and $(5.1\mathrm{c})$ in $(5\mathrm{c})$: we will get same result (5.2) since are $(5\mathrm{a})$, $(5\mathrm{b})$, $(5\mathrm{c})$ are identical differential equations.

On stress free boundary surface, the required boundary conditions are

$$\tau_{r\theta}(r,t)=0 \quad ; \quad m_{rr}(r,t)=0=m_{rz}(r,t) \text{ at } r=a$$

Recall (5): $\tau_{ij}=\tilde{\lambda}\,\ell_{kk}\,\delta_{ij}+(2\tilde{\mu}+\nu)\ell_{ij}-\nu\,\omega_{ij}$ from reference [2]

Consider $0=\tau_{r\theta}(a,t)=\tau_{12}(a,t)=\tilde{\lambda}\,\ell_{kk}(a,t)\delta_{12}+(2\tilde{\mu}+\nu)\ell_{12}(a,t)-\nu\,\omega_{12}(a,t)$

$$=\tilde{\lambda}\,\ell_{kk}(a,t)0+(2\tilde{\mu}+\nu)\ell_{12}(a,t)-\nu\,\omega_{12}(a,t)$$

$$\Rightarrow \qquad \nu\,\omega_{12}(a,t)=(2\tilde{\mu}+\nu)\ell_{12}(a,t)$$

Recall (6.2): $m_{qi}(x_1,x_2,x_3,t)=\varepsilon_{ijk}\,\sigma_{qkj}(x_1,x_2,x_3,t)$

And (6): $\sigma_{qij}=\Lambda\,\ell_{kk,q}\,\delta_{ij}+(2\aleph+N)\;\ell_{ij,q}-\nu\,\omega_{ij,q}$ from reference [2]

$$\Rightarrow \qquad m_{qi}=\varepsilon_{ijk}\left[\Lambda\ell_{pp,q}\,\delta_{kj}+(2\aleph+N)\ell_{kj,q}-N\,\omega_{kj,q}\right]$$

But $m_{13}=\varepsilon_{3jk}\left[\Lambda\ell_{pp,1}\,\delta_{kj}+(2\aleph+N)\ell_{kj,1}-N\,\omega_{kj,1}\right]$

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$$= \varepsilon_{312}\left[\Lambda\ell_{pp,1}\,\delta_{12} + (2\aleph + N)\ell_{12,1} - N\,\omega_{12,1}\right]$$

$$+ \varepsilon_{321}\left[\Lambda\ell_{pp,1}\,\delta_{21} + (2\aleph + N)\ell_{21,1} - N\,\omega_{21,1}\right]$$

$$= 1\left[\Lambda\ell_{pp,1}\,0 + (2\aleph + N)\ell_{12,1} - N\,\omega_{12,1}\right]$$

$$-1\left[\Lambda\ell_{pp,1}\,0 + (2\aleph + N)\ell_{12,1} + N\,\omega_{12,1}\right] = -2N\,\omega_{12,1}$$

$$\Rightarrow \quad m_{13}(r,t) = -2N\,\omega_{12,1}(r,t) = -2N\frac{\partial}{\partial r}\omega_{12}(r,t)$$

$$= -2N\frac{\partial}{\partial r}\left(\frac{2\tilde{\mu}+\nu}{\nu}\ell_{12}(r,t)\right) = -2N\frac{2\tilde{\mu}+\nu}{\nu}\frac{\partial}{\partial r}\ell_{12}(r,t)$$

But from $(5.1c)$: $\ell_{12}(r,t) = H\,\mathbf{J}_0(hr)\,e^{\iota\varpi t}$.

$$\Rightarrow \quad m_{13}(r,t) = -\frac{2\tilde{\mu}+\nu}{\nu}2N\frac{\partial}{\partial r}H\,\mathbf{J}_0(hr)\,e^{\iota\varpi t} = -\frac{2\tilde{\mu}+\nu}{\nu}2N\,H\,\mathbf{J}_0'(hr)h\,e^{\iota\varpi t}$$

Consider $0 = m_{rz}(a,t) = m_{13}(a,t) = -\dfrac{2\tilde{\mu}+\nu}{\nu}2N\,H\,\mathbf{J}_0'(ha)h\,e^{\iota\varpi t}$

$$\Rightarrow \quad 0 = -\frac{2\tilde{\mu}+\nu}{\nu}2N\,\mathbf{J}_0'(ha)hH \quad \Rightarrow \quad 0 = \mathbf{J}_0'(ha)hH \quad \text{where } H \text{ is arbitrary.}$$

$$\Rightarrow \quad 0 = h\,\mathbf{J}_0'(ha)$$

We have $\mathbf{J}_0(r) = 1 + \sum_{k=1}^{\infty}\dfrac{(-1)^k}{(k!)^2}\left(\dfrac{r}{2}\right)^{2k} = 1 - \left(\dfrac{r}{2}\right)^2 + \sum_{k=2}^{\infty}\dfrac{(-1)^k}{(k!)^2}\left(\dfrac{r}{2}\right)^{2k}$

$$\Rightarrow \quad \mathbf{J}_0'(r) = 0 - 2\left(\frac{r}{2}\right)^2\frac{1}{2} + \sum_{k=2}^{\infty}\frac{(-1)^k}{(k!)^2}2k\left(\frac{r}{2}\right)^{2k-1}\frac{1}{2} = -\left(\frac{r}{2}\right)^2 + \sum_{k=2}^{\infty}\frac{(-1)^k}{(k!)^2}k\left(\frac{r}{2}\right)^{2k-1}$$

$$\Rightarrow \quad \mathbf{J}_0'(ha) = -\left(\frac{ha}{2}\right)^2 + \sum_{k=2}^{\infty}\frac{(-1)^k}{(k!)^2}k\left(\frac{ha}{2}\right)^{2k-1} \cong -\left(\frac{ha}{2}\right)^2 = \frac{-h^2a^2}{4}$$

Thus $0 = h\,\mathbf{J}_0'(ha) \cong h\left(\dfrac{-h^2a^2}{4}\right) = \dfrac{-h^3a^2}{4} \quad \Rightarrow \quad h^3 = 0 \quad \Rightarrow \quad h = 0$

$$\Rightarrow \quad 0 = h^2 = \frac{\varpi^2\varsigma\rho - 2(2\tilde{\mu}+\nu)}{2(2\aleph + N)} \text{ from } (5.2)\text{. But } \varpi^2 = n^2c^2$$

$$\Rightarrow \quad 0 = n^2c^2\varsigma\rho - 2(2\tilde{\mu}+\nu) \quad \Rightarrow \quad 2(2\tilde{\mu}+\nu) = n^2c^2\varsigma\rho$$

So $\dfrac{2(2\tilde{\mu}+\nu)}{n^2\varsigma\rho} = c^2$

So the micro elastic wave is propagated in an isotropic solid with the speed c given by

$$c^2 = \frac{2(2\tilde{\mu}+\nu)}{n^2\varsigma\rho} \qquad \cdots\cdots\cdots\cdots(5.3)$$

and is one dimensional ***micro-elastic torsional wave***.

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Clearly this micro elastic wave is dispersive.

Observe that the speed of one dimensional micro-elastic torsional wave is same as the speed of one dimensional micro-elastic longitudinal wave.

Consider $m_{11} = \varepsilon_{1jk}\left[\Lambda\ell_{pp,1}\delta_{kj} + (2\aleph + N)\ell_{kj,1} - N\,\omega_{kj,1}\right]$

$$= \varepsilon_{123}\left[\Lambda\ell_{pp,1}\delta_{23} + (2\aleph + N)\ell_{23,1} - N\,\omega_{23,1}\right] + \varepsilon_{132}\left[\Lambda\ell_{pp,1}\delta_{32} + (2\aleph + N)\ell_{32,1} - N\,\omega_{32,1}\right]$$

$$= 1\left[\Lambda\ell_{pp,1}0 + (2\aleph + N)\ell_{23,1} - N\,\omega_{23,1}\right] - 1\left[\Lambda\ell_{pp,1}0 + (2\aleph + N)\ell_{23,1} + N\,\omega_{23,1}\right] = -2N\,\omega_{23,1}$$

$$\Rightarrow \quad 0 = m_{rr}(a,t) = m_{11}(a,t) = -2N\omega_{23,1}(a,t) \quad \Rightarrow \quad 0 = \omega_{23,1}(a,t) = \frac{\partial}{\partial r}\omega_{23}(a,t)$$

Thus we have no information about the speed of micro-elastic wave from the boundary condition $0 = m_{rr}(a,t)$

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