ON THE STABILITY OF A FOUR SPECIES: A PREY-PREDATOR-HOST-COMMENSAL-SYN ECO-SYSTEM-II (PREY AND PREDATOR WASHED OUT STATES)

B. Hari Prasad¹, N. Ch. Pattabhi Ramacharyulu²

¹ Department of Mathematics, Chaitanya Degree & P.G. College (Autonomous), Hanamkonda, Warangal, A.P, India. Email: sumathi_prasad73@yahoo.com
² Former Faculty, Department. of Mathematics, NIT Warangal, India.

ABSTRACT: This paper deals with an investigation on a Four Species Syn-Ecological System (Prey and Predator washed out states). The System comprises of a Prey (S₁), a Predator (S₂) that survives upon S₁, two Hosts S₃ and S₄ for which S₁, S₂ are commensal respectively i.e., S₃ and S₄ benefit S₁ and S₂ respectively, without getting effected either positively or adversely. Further S₃ and S₄ are neutral. The model equations of the system constitute a set of four first order non-linear ordinary differential coupled equations. In all, there are sixteen equilibrium points. Criteria for the asymptotic stability of three of the sixteen equilibrium points: the Prey and Predator washed out states only are established in this paper. The system would be stable if all the characteristic roots are negative, in case they are real, and have negative real parts, in case they are complex. The linearized equations for the perturbations over the equilibrium points are analyzed to establish the criteria for stability and the trajectories illustrated.

1. INTRODUCTION:

Mathematical modeling of Eco-System was initiated in 1925 by Lotka [10] and in 1931 by Volterra[14]. The general concepts of modeling have been presented in the treatises of Meyer[11], Kushing[7], Kapur J.N. [5,6] and several others. The ecological interactions can be broadly classified as Prey-Predator, Commensalism, Competition, Neutralism, Mutualism and

Fig. 1 shows the Schematic Sketch of the system under investigation.

![Schematic Sketch of the Syn Eco-System](image)

**Fig. 1 Schematic Sketch of the Syn Eco - System**

2. **BASIC EQUATIONS OF THE MODEL:**

**Notation Adopted:**

- $S_1$: Prey for $S_2$ and commensal for $S_3$.
- $S_2$: Predator surviving upon $S_1$ and commensal for $S_4$.
- $S_3$: Host for the commensal – Prey ($S_1$).
- $S_4$: Host of the commensal – Predator ($S_2$).
- $N_1(t)$: The Population of the Prey ($S_1$).
- $N_2(t)$: The Population of the Predator ($S_2$).
- $N_3(t)$: The Population of the Host ($S_3$) of the Prey ($S_1$).
- $N_4(t)$: The Population of the Host ($S_4$) of the Predator ($S_2$).
- $t$: Time instant.
a₁, a₂, a₃, a₄ : Natural growth rates of S₁, S₂, S₃, S₄
a₁₁, a₂₂, a₃₃, a₄₄ : Self inhibition coefficients of S₁, S₂, S₃, S₄
a₁₂, a₂₁ : Interaction (Prey-Predator) coefficients of S₁ due to S₂ and S₂ due to S₁
a₁₃ : Coefficient for commensal for S₁ due to the Host S₃
a₂₄ : Coefficient for commensal for S₂ due to the Host S₄

\[
\begin{align*}
\frac{dN_1}{dt} &= a_1 N_1 - a_{11} N_1^2 - a_{12} N_1 N_2 + a_{13} N_1 N_3, \quad \ldots \quad (2.1) \\
\frac{dN_2}{dt} &= a_2 N_2 - a_{22} N_2^2 + a_{21} N_1 N_2 + a_{24} N_2 N_4, \quad \ldots \quad (2.2) \\
\frac{dN_3}{dt} &= a_3 N_3 - a_{33} N_3^2, \quad \ldots \quad (2.3) \\
\frac{dN_4}{dt} &= a_4 N_4 - a_{44} N_4^2, \quad \ldots \quad (2.4)
\end{align*}
\]

Further the variables N₁, N₂, N₃, N₄ are non-negative and the model parameters a₁, a₂, a₃, a₄; a₁₁, a₂₂, a₃₃, a₄₄; a₁₂, a₂₁, a₁₃, a₂₄ are assumed to be non-negative constants.
3 EQUILIBRIUM STATES:

The system under investigation has sixteen equilibrium states defined by

\[ \frac{dN_i}{dt} = 0, \ i = 1, 2, 3, 4 \quad \text{......... (3.1)} \]

are given in the following table.

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Equilibrium States</th>
<th>Equilibrium Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Fully Washed out state</td>
<td>( \tilde{N}_1 = 0, \tilde{N}_2 = 0, \tilde{N}_3 = 0, \tilde{N}_4 = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>Only the Host (S4) of S2 survives</td>
<td>( \tilde{N}_1 = 0, \tilde{N}_2 = 0, \tilde{N}_3 = 0, \tilde{N}<em>4 = \frac{a_4}{a</em>{44}} )</td>
</tr>
<tr>
<td>3</td>
<td>Only the Host (S3) of S1 survives</td>
<td>( \tilde{N}_1 = 0, \tilde{N}_2 = 0, \tilde{N}<em>3 = \frac{a_3}{a</em>{33}}, \tilde{N}_4 = 0 )</td>
</tr>
<tr>
<td>4</td>
<td>Only the Predator S2 survives</td>
<td>( \tilde{N}_1 = 0, \tilde{N}<em>2 = \frac{a_2}{a</em>{22}}, \tilde{N}_3 = 0, \tilde{N}_4 = 0 )</td>
</tr>
<tr>
<td>5</td>
<td>Only the Prey S1 survives</td>
<td>( \tilde{N}<em>1 = \frac{a_1}{a</em>{11}}, \tilde{N}_2 = 0, \tilde{N}_3 = 0, \tilde{N}_4 = 0 )</td>
</tr>
<tr>
<td>6</td>
<td>Prey (S1) and Predator (S2) washed out</td>
<td>( \tilde{N}_1 = 0, \tilde{N}_2 = 0, \tilde{N}<em>3 = \frac{a_3}{a</em>{33}}, \tilde{N}<em>4 = \frac{a_4}{a</em>{44}} )</td>
</tr>
<tr>
<td>7</td>
<td>Prey (S1) and Host (S3) of S1 washed out</td>
<td>( \tilde{N}<em>1 = 0, \tilde{N}<em>2 = \frac{a_3a</em>{44} + a_4a</em>{24}}{a_{22}a_{44}}, \tilde{N}_3 = 0, \tilde{N}<em>4 = \frac{a_4}{a</em>{44}} )</td>
</tr>
<tr>
<td>8</td>
<td>Prey (S1) and Host (S4) of S2 washed out</td>
<td>( \tilde{N}_1 = 0, \tilde{N}<em>2 = \frac{a_2}{a</em>{22}}, \tilde{N}<em>3 = \frac{a_3}{a</em>{33}}, \tilde{N}_4 = 0 )</td>
</tr>
<tr>
<td>9</td>
<td>Predator (S2) and Host (S3) of S1 washed out</td>
<td>( \tilde{N}<em>1 = \frac{a_1}{a</em>{11}}, \tilde{N}_2 = 0, \tilde{N}_3 = 0, \tilde{N}<em>4 = \frac{a_4}{a</em>{44}} )</td>
</tr>
<tr>
<td>10</td>
<td>Predator (S2) and Host (S4) of S2 washed out</td>
<td>( \tilde{N}<em>1 = \frac{a_1a</em>{22} + a_2a_{12}}{a_{11}a_{22}}, \tilde{N}<em>2 = \frac{a_1a</em>{21} + a_2a_{11}}{a_{11}a_{22}}, \tilde{N}_3 = 0, \tilde{N}_4 = 0 )</td>
</tr>
<tr>
<td>11</td>
<td>Prey (S1) and Predator (S2) survives</td>
<td>( \tilde{N}<em>1 = \frac{a_1a</em>{23} - a_2a_{13}}{a_{11}a_{23}}, \tilde{N}<em>2 = \frac{a_1a</em>{21} - a_2a_{11}}{a_{11}a_{22}}, \tilde{N}_3 = 0, \tilde{N}_4 = 0 )</td>
</tr>
<tr>
<td>12</td>
<td>Only the Prey (S1) washed out</td>
<td>( \tilde{N}<em>1 = 0, \tilde{N}<em>2 = \frac{a_3a</em>{44} + a_4a</em>{24}}{a_{22}a_{44}}, \tilde{N}<em>3 = \frac{a_3}{a</em>{33}}, \tilde{N}<em>4 = \frac{a_4}{a</em>{44}} )</td>
</tr>
<tr>
<td>13</td>
<td>Only the predator (S2) washed out</td>
<td>( \tilde{N}<em>1 = \frac{a_1a</em>{23} + a_3a_{13}}{a_{11}a_{23}}, \tilde{N}_2 = 0, \tilde{N}<em>3 = \frac{a_3}{a</em>{33}}, \tilde{N}<em>4 = \frac{a_4}{a</em>{44}} )</td>
</tr>
</tbody>
</table>
14 Only the Host (S₃) of S₁ washed out

\[ \bar{N}_1 = \frac{\delta_1}{\delta_1}, \bar{N}_2 = \frac{\delta_2}{\delta_1}, \bar{N}_3 = 0, \bar{N}_4 = \frac{a_4}{a_{44}} \]
where
\[ \delta_1 = a_{44}(a_{11}a_{22} + a_{12}a_{21}) > 0 \]
\[ \delta_2 = a_2a_{23}a_{44} - a_{21}(a_{14}a_{44} + a_{12}a_{24}) \]
\[ \delta_3 = a_4a_{21}a_{44} - a_{11}(a_{14}a_{44} + a_{12}a_{24}) \]

15 Only the Host (S₄) of S₂ washed out

\[ \bar{N}_1 = \frac{\sigma_2}{\sigma_1}, \bar{N}_2 = \frac{\sigma_1}{\sigma_1}, \bar{N}_3 = \frac{a_4}{a_{33}}, \bar{N}_4 = 0 \]
where
\[ \sigma_1 = a_{33}\left(a_{11}a_{22} + a_{12}a_{21}\right) > 0 \]
\[ \sigma_2 = a_{22}\left(a_{14}a_{44} + a_{12}a_{24}\right) - a_2a_{12}a_{33} \]
\[ \sigma_3 = a_3\left(a_{14}a_{44} + a_{12}a_{24}\right) + a_2a_{12}a_{33} > 0 \]

16 The co-existent state (or) Normal steady state

\[ \bar{N}_1 = \frac{a_2a_{44}\psi_1 - a_{12}a_{33}\psi_2}{\psi_3}, \bar{N}_2 = \frac{a_2a_{44}\psi_1 + a_1a_{33}\psi_2}{\psi_3}, \bar{N}_3 = \frac{a_4}{a_{33}}, \bar{N}_4 = 0 \]
where
\[ \psi_1 = a_4a_{33} + a_4a_{13} > 0 \]
\[ \psi_2 = a_2a_{44} + a_4a_{23} > 0 \]
\[ \psi_3 = a_{34}a_{44}(a_{11}a_{22} + a_{12}a_{21}) > 0 \]

The present paper deals with the Prey and Predator washed out states only. The stability of the other equilibrium states will be presented in the forthcoming communications.

4. STABILITY OF THE PREY AND PREDATOR WASHED OUT EQUILIBRIUM STATES: (Sl. Nos. 2, 3, 6 in the above table)

4.1 Equilibrium point \( \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0, \bar{N}_4 = \frac{a_4}{a_{44}} \)

Let us consider small deviations from the steady state
i.e. \( \bar{N}_i(t) = \bar{N}_i + u_i(t), i = 1,2,3,4 \) ................. (4.1.1)
where \( u_i(t) \) is a small perturbation in the species \( S_i \).

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of \( u_1, u_2, u_3, u_4 \).

We get
\[ \frac{du_1}{dt} = a_1u_1 \] .................. (4.1.2) \[ \frac{du_2}{dt} = p_2u_2 \] ......(4.1.3)
\[ \frac{du_3}{dt} = a_3u_3 \] .................. (4.1.4) \[ \frac{du_4}{dt} = -a_4u_4 \] ......(4.1.5)
Here \[ p_2 = \left( a_2 + \frac{a_3 a_{24}}{a_{44}} \right) > 0 \] ........................ (4.1.6)

The characteristic equation of which is 
\[ (\lambda - a_1)(\lambda - p_2)(\lambda - a_3)(\lambda + a_4) = 0 \] ........................ (4.1.7)

The roots \( a_1,p_2,a_3 \) are positive and \(-a_4\) is negative.
Hence the steady state is **unstable**.

The solutions of the equations (4.1.2), (4.1.3), (4.1.4), (4.1.5) are
\[ u_1 = u_{10}e^{a_1 t} \] ........................ (4.1.8) \[ u_2 = u_{20}e^{p_2 t} \] ........................ (4.1.9) \[ u_3 = u_{30}e^{a_3 t} \] ........................ (4.1.10) \[ u_4 = u_{40}e^{a_4 t} \] ........................ (4.1.11)
where \( u_{10}, u_{20}, u_{30}, u_{40} \) are the initial values of \( u_1,u_2,u_3,u_4 \) respectively.

In the three equilibrium states, there would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates \( a_1, a_2, a_3, a_4 \) and the initial values of the perturbations \( u_{10}(t), u_{20}(t) \) \( u_{30}(t), u_{40}(t) \) of the species \( S_1,S_2,S_3,S_4 \) of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations.

**Case (i):** If \( u_{10} < u_{20} < u_{30} < u_{40} \) and \( a_1 < p_2 < a_3 < a_4 \)
In this case the Host \( (S_4) \) of \( S_2 \) has the least natural birth rate. Initially it is dominated over by the Prey \( (S_1) \), Predator \( (S_2) \), Host \( (S_3) \) of \( S_1 \) till the time instant \( t_{14}^{*}, t_{24}^{*}, t_{34}^{*} \) respectively and thereafter the dominance is reversed.
Here
\[ t_{14}^{*} = \frac{1}{a_1 + a_4} \log \left( \frac{u_{40}}{u_{10}} \right) \]
\[ t_{24}^{*} = \frac{1}{p_2 + a_4} \log \left( \frac{u_{20}}{u_{20}} \right) \]
\[ t_{34}^{*} = \frac{1}{a_3 + a_4} \log \left( \frac{u_{40}}{u_{30}} \right) \] ........................ (4.1.12)

**Case (ii):** If \( u_{20} < u_{10} < u_{40} < u_{30} \) and \( a_4 < a_2 < p_2 < a_3 \)
In this case the Host \( (S_4) \) of \( S_2 \) has the least natural birth rate. Initially it is dominated over by the Prey \( (S_1) \), Predator \( (S_2) \) till the time instant \( t_{14}^{*}, t_{24}^{*} \) respectively and thereafter the dominance is reversed. Also the Prey \( (S_1) \) dominates over the
Predator (S₂) till the time instant \( t^{*}_{21} \) and the dominance gets reversed there after.

Here

\[
t^{*}_{21} = \frac{1}{a_i-p_2} \log \left( \frac{u_{30}}{u_{10}} \right)
\]  

…………(4.1.13)

**Case (iii)**: If \( u_{30} < u_{40} < u_{10} < u_{20} \) and \( a_4 < a_1 < p_2 < a_i \)
In this case the Host (S₄) of S₂ has the least natural birth rate. Initially it is dominated over by the Host (S₃) of S₁ till the time instant \( t^{*}_{34} \) and there after the dominance is reversed. Also the Predator (S₂) dominates over the Prey (S₁) till the time instant \( t^{*}_{12} \) and the dominance gets reversed there after.

Here

\[
t^{*}_{12} = \frac{1}{a_i-p_2} \log \left( \frac{u_{20}}{u_{10}} \right)
\]  

…………(4.1.14)

**Case (iv)**: If \( u_{40} < u_{10} < u_{20} < u_{30} \) and \( a_4 < a_1 < p_2 < a_i \)
In this case the Host (S₄) of S₂ has the least natural birth rate. And the Host (S₃) of S₁ dominates the Predator (S₂), Prey (S₁), Host (S₄) of S₂ in natural growth rate as well as in its initial population strength.

4.1.A Trajectories of perturbations:
The trajectories in the \( u_1 - u_2 \) plane given by

\[
\left( \frac{u_1}{u_{10}} \right)^{p_2} = \left( \frac{u_2}{u_{20}} \right)^{a_1} \]  

...(4.1.15)

and are shown in Fig. 2
Also the trajectories in the $u_1 - u_4$ plane given by
\[
\left( \frac{u_1}{u_{10}} \right)^{a_1} = \left( \frac{u_4}{u_{40}} \right)^{a_4} \quad \ldots \ldots (4.1.16)
\]
and are shown in Fig. 3

Similarly the trajectories in the $u_1 - u_3$, $u_2 - u_3$, $u_2 - u_4$, $u_3 - u_4$ planes are
\[
\left( \frac{u_1}{u_{20}} \right)^{a_1} = \left( \frac{u_3}{u_{30}} \right)^{a_1}, \quad \left( \frac{u_2}{u_{20}} \right)^{a_2} = \left( \frac{u_3}{u_{30}} \right)^{a_2}, \quad \left( \frac{u_3}{u_{40}} \right)^{a_3} = \left( \frac{u_4}{u_{40}} \right)^{a_4} \quad \ldots \ldots (4.1.17)
\]
respectively.

**4.2 Equilibrium Point**

\[ N_1 = 0, \quad N_2 = 0, \quad N_3 = \frac{d_3}{a_{33}}, \quad N_4 = 0 \]

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of $u_1, u_2, u_3, u_4$.

We get
\[
\begin{align*}
\frac{du_1}{dt} &= p_1 u_1 \quad \ldots \ldots (4.2.1) \\
\frac{du_2}{dt} &= a_2 u_2 \quad \ldots \ldots (4.2.2) \\
\frac{du_3}{dt} &= -a_3 u_3 \quad \ldots \ldots (4.2.3) \\
\frac{du_4}{dt} &= a_4 u_4 \quad \ldots \ldots (4.2.4)
\end{align*}
\]

Here \[ p_1 = \left( \frac{a_1 + a_3 a_{12}}{a_{33}} \right) > 0 \quad \ldots \ldots (4.2.5) \]

The characteristic equation of which is
\[ (\lambda - p_1)(\lambda - a_2)(\lambda + a_3)(\lambda - a_4) = 0 \quad \ldots \ldots (4.2.6) \]
the roots $p_1, a_2, a_4$ are positive and $-a_3$ is negative.

Hence the steady state is **unstable**.
The solutions of the equations (4.2.1), (4.2.2), (4.2.3), (4.2.4) are

\[ u_1 = u_{10} e^{p_1 t} \quad \cdots \cdots \text{(4.2.7)} \]
\[ u_2 = u_{20} e^{a_2 t} \quad \cdots \cdots \text{(4.2.8)} \]
\[ u_3 = u_{30} e^{-a_3 t} \quad \cdots \cdots \text{(4.2.9)} \]
\[ u_4 = u_{40} e^{a_4 t} \quad \cdots \cdots \text{(4.2.10)} \]

**Case (i) :** If \( u_{10} < u_{40} < u_{30} < u_{20} \) and \( p_1 < a_3 < a_2 < a_4 \)
In this case the Host (S₃) of S₁ has the least natural birth rate. Initially it is dominated over by the Prey (S₁). Host (S₄) of S₂ till the time instant \( t^{*}_{13} t^{*}_{43} \) respectively and there after the dominance is reversed. Also the Predator (S₂) dominates its Host till the time instant \( t^{*}_{42} \) and the dominance gets reversed there after.

Here
\[ t^{*}_{13} = \frac{1}{p_1 + a_3} \log \left( \frac{u_{30}}{u_{10}} \right) \]
\[ t^{*}_{43} = \frac{1}{a_3 + a_4} \log \left( \frac{u_{40}}{u_{30}} \right) \]
\[ t^{*}_{42} = \frac{1}{a_2 - a_4} \log \left( \frac{u_{40}}{u_{30}} \right) \quad \cdots \cdots \text{(4.2.11)} \]

**Case (ii) :** If \( u_{20} < u_{30} < u_{10} < u_{40} \) and \( p_1 < a_2 < a_4 < a_3 \)
In this case the Host (S₃) of S₁ has the least natural birth rate. Initially it is dominated over by the Prey (S₂) till the time instant \( t^{*}_{23} \) and there after the dominance is reversed. Also the Prey (S₁) dominates over the Predator (S₂) till the time instant \( t^{*}_{21} \) and the dominance gets reversed there after.

Here
\[ t^{*}_{23} = \frac{1}{a_2 + a_3} \log \left( \frac{u_{30}}{u_{20}} \right) \]
\[ t^{*}_{21} = \frac{1}{a_1 - a_2} \log \left( \frac{u_{20}}{u_{10}} \right) \quad \cdots \cdots \text{(4.2.12)} \]
**Case (iii)**: If $u_{30} < u_{40} < u_{10}$ and $a_3 < p_1 < a_4 < a_2$  
In this case the Host ($S_3$) of $S_1$ has the least natural birth rate. Initially the Prey ($S_1$) dominates over by the Predator ($S_2$), Host ($S_4$) of $S_2$ till the time instant $t^*_{21}, t^*_{41}$ respectively and thereafter the dominance is reversed.

Here

$$t^*_{41} = \frac{1}{a_1 - a_4} \log \left( \frac{u_{40}}{u_{10}} \right) \quad \ldots (4.2.13)$$

**Case (iv)**: If $u_{40} < u_{20} < u_{30} < u_{10}$ and $p_1 < a_3 < a_4 < a_2$
In this the Host ($S_3$) of $S_1$ has the least natural birth rate. Initially it is dominated over by the Predator ($S_2$), Host ($S_4$) of $S_2$ till the time instant $t^*_{21}, t^*_{43}$ respectively and thereafter the dominance is reversed.

Also the Prey ($S_1$) dominates over the Predator ($S_2$), Host ($S_4$) of $S_2$ till the time instant $t^*_{21}, t^*_{41}$ respectively and the dominance gets reversed thereafter.

### 4.2.A. Trajectories of perturbations:

The trajectories in the $u_1 - u_2$ plane given by

$$\left( \frac{u_1}{u_{10}} \right)^{a_2} = \left( \frac{u_2}{u_{20}} \right)^{p_1} \quad \ldots (4.2.14)$$

and are shown in Fig. 4.
Also the trajectories in the \( u_1 - u_3 \) plane given by

\[
\left( \frac{u_1}{u_{10}} \right)^{a_1} = \left( \frac{u_3}{u_{30}} \right)^{p_1} \quad \text{.........(4.2.15)}
\]

and are shown in Fig. 5

Similarly the trajectories in the \( u_1 - u_4, u_2 - u_4, u_2 - u_3, u_3 - u_4 \) planes are

\[
\left( \frac{u_1}{u_{10}} \right)^{a_1} = \left( \frac{u_4}{u_{40}} \right)^{p_1}, \quad \left( \frac{u_2}{u_{20}} \right)^{a_2} = \left( \frac{u_4}{u_{40}} \right)^{p_2} \quad \text{.........(4.2.16)}
\]

\[
\left( \frac{u_2}{u_{20}} \right)^{a_2} = \left( \frac{u_3}{u_{30}} \right)^{a_4}, \quad \left( \frac{u_3}{u_{30}} \right)^{a_4} = \left( \frac{u_4}{u_{40}} \right)^{p_4} \quad \text{.........(4.2.17)}
\]

respectively.

**4.3 Equilibrium Point** \( \overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = \frac{a_3}{a_{33}}, \overline{N}_4 = \frac{a_4}{a_{44}} \).

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of \( u_1, u_2, u_3, u_4 \).

We get

\[
\frac{du_1}{dt} = s_1 u_1 \quad \text{.........(4.3.1)} \quad \frac{du_2}{dt} = s_2 u_2 \quad \text{.........(4.3.2)}
\]

\[
\frac{du_3}{dt} = -a_3 u_3 \quad \text{.........(4.3.3)} \quad \frac{du_4}{dt} = -a_4 u_4 \quad \text{.........(4.3.4)}
\]

Here

\[
s_1 = \left( a_1 + \frac{a_2 a_4}{a_{33}} \right) > 0, \quad s_2 = \left( a_2 + \frac{a_3 a_4}{a_{44}} \right) > 0 \quad \text{.........(4.3.5)}
\]

The characteristic equation of which is

\[
(\lambda - s_1)(\lambda - s_2)(\lambda + a_3)(\lambda + a_4) = 0 \quad \text{.........(4.3.6)}
\]

the roots \( s_1, s_2 \) are positive and \( -a_3, -a_4 \) are negative.

Hence the steady state is **unstable**.

The solutions of the equations (4.3.1), (4.3.2), (4.3.3), (4.3.4) are

\[
u_1 = u_{10} e^{s_1 t} \quad \text{.........(4.3.7)} \quad u_2 = u_{20} e^{s_2 t} \quad \text{.........(4.3.8)}
\]

\[
u_3 = u_{30} e^{-a_3 t} \quad \text{.........(4.3.9)} \quad u_4 = u_{40} e^{-a_4 t} \quad \text{.........(4.3.10)}
\]
Case (i): If \( u_{30} < u_{20} < u_{30} < u_{40} \) and \( s_1 < s_2 < a_3 < a_4 \)
In this case the Host (S\(_3\)) of S\(_1\) has the least natural birth rate. Initially it is dominated over by the Prey (S\(_1\)), Predator (S\(_2\)) till the time instant \( t^*_{13}, t^*_{23} \) respectively and there after the dominance is reversed. Also the Host (S\(_4\)) of S\(_2\) dominates over the Prey (S\(_1\)), Predator (S\(_2\)) till the time instant \( t^*_{14}, t^*_{24} \) respectively and the dominance gets reversed there after.

\[
T^*_{13} = \frac{1}{s_1 + a_3} \log \left( \frac{u_{30}}{u_{10}} \right), \quad T^*_{23} = \frac{1}{s_2 + a_3} \log \left( \frac{u_{30}}{u_{20}} \right)
\]

Here

\[
T^*_{14} = \frac{1}{s_1 + a_4} \log \left( \frac{u_{40}}{u_{10}} \right), \quad T^*_{24} = \frac{1}{s_2 + a_4} \log \left( \frac{u_{40}}{u_{20}} \right)
\]

…………(4.3.11)

Case (ii): If \( u_{20} < u_{40} < u_{30} < u_{10} \) and \( a_4 < a_3 < s_1 < s_2 \)
In this case the Host (S\(_4\)) of S\(_2\) has the least natural birth rate. Initially it is dominated over by the Predator (S\(_2\)) till the time instant \( t^*_{24} \) and there after the dominance is reversed. Also the Host (S\(_3\)) of S\(_1\) dominates over the Prey (S\(_1\)), Predator (S\(_2\)) till the time instant \( t^*_{23} \) and there after the dominance is reversed. Similarly the Prey (S\(_1\)) dominates over the Predator (S\(_2\)) till the time instant \( t^*_{21} \) and the dominance gets reversed there after.

Here \( T^*_{21} = \frac{1}{s_1 - s_2} \log \left( \frac{u_{20}}{u_{10}} \right) \)

…………(4.3.12)

Case (iii): If \( u_{30} < u_{20} < u_{40} < u_{10} \) and \( s_2 < s_1 < a_4 < a_3 \)
In this case the Host (S\(_4\)) of S\(_2\) has the least natural birth rate. Initially it is dominated over by the Predator (S\(_2\)), Host (S\(_3\)) of S\(_1\) till the time instant \( t^*_{23}, t^*_{14} \) respectively and there after the dominance is reversed.

Here \( T^*_{34} = \frac{1}{a_4 - a_3} \log \left( \frac{u_{40}}{u_{30}} \right) \)

………(4.3.13)
Case (iv) : If $u_{40} < u_{30} < u_{10} < u_{20}$ and $a_3 < s_2 < a_4 < s_1$

In this the Host ($S_3$) of $S_1$ has the least natural birth rate. Initially it is dominated over by the Host ($S_4$) of $S_2$ till the time instant $t^*_{43}$ and there after the dominance is reversed.

Also the Predator ($S_2$) dominates over the Prey ($S_1$) till the time instant $t^*_{12}$ respectively and the dominance gets reversed there after.

Here $$t^*_{43} = \frac{1}{a_4 - a_3} \log \left( \frac{u_{40}}{u_{30}} \right)$$

$$t^*_{12} = \frac{1}{s_1 - s_2} \log \left( \frac{u_{30}}{u_{10}} \right) \ldots\ldots\ldots(4.3.14)$$

4.3.A. Trajectories of perturbations :

The trajectories in the $u_1 - u_2$ plane given by

$$\left( \frac{u_1}{u_{10}} \right)^{s_2} = \left( \frac{u_2}{u_{20}} \right)^{s_1} \ldots\ldots(4.3.15)$$

and are shown in Fig. 6

Also the trajectories in the $u_1 - u_3$ plane given by

$$\left( \frac{u_1}{u_{10}} \right)^{a_2} = \left( \frac{u_3}{u_{30}} \right)^{a_1} \ldots\ldots(4.3.16)$$

and are shown in Fig. 7
Similarly the trajectories in the $u_1 - u_4, u_2 - u_4, u_2 - u_4, u_3 - u_4$ planes are

\[
\left( \frac{u_1}{u_{10}} \right)^{-a_1} = \left( \frac{u_4}{u_{40}} \right)^{e_1}, \left( \frac{u_2}{u_{20}} \right)^{-a_2} = \left( \frac{u_3}{u_{30}} \right)^{e_2} \quad \ldots (4.3.17)
\]

\[
\left( \frac{u_2}{u_{20}} \right)^{-a_3} = \left( \frac{u_4}{u_{40}} \right)^{e_3}, \left( \frac{u_3}{u_{30}} \right)^{-a_4} = \left( \frac{u_4}{u_{40}} \right)^{e_4} \quad \ldots (4.3.18)
\]

respectively.

REFERENCES:


