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# AN AMMENSAL SPECIES WITH LIMITED RESOURCES AND ENEMY SPECIES WITH UNLIMITED RESOURCES - GLOBAL STABILITY ANALYSIS

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**ABSTRACT**

The paper devotes to explore the global stability of a Mathematical model of an Ammensal with limited resources and enemy with unlimited resources by Liapunov's stability analysis. It is derived by constructing a suitable Liapunov's function for evaluating the global stability of the model in which Ammensal survives and the enemy is in washed out state where the death rate of the enemy species is greater than its birth rate.

**AMS Classification: 92 D 25, 92 D 40**

**Key words**: Equilibrium states, Stability, Liapunov's function for global stability

## 1) Introduction:

A.M. Liapunov initiated a meritorious method in 1892 to examine the global stability of equilibrium points in case of linear and non-linear systems. This method yields stability information directly without solving the differential equations involved in the system. Hence it is also called Liapunov's direct method to detect the criteria for global stability. This method is based on the chief charactestic of constructing a scalar function called Liapunov's function. The stability behaviour of solutions of linear and weakly non-linear system is done by using the techniques of variation of constants formulae and integral inequalities. So this analysis is confined to a small neighbourhood of operating point i.e., local stability. Further, the techniques used therein require explicit knowledge of solutions of corresponding linear systems. Hence, the stability behaviour of a physical system is curbed by these limitations.In this connection Several autors like Lotka[15], Kapur[13],Pattabhi Ramacharyulu[1-11],Lakiminarayan[14] and Bhaskararama Sarma[12] etc.applied this method in various situations for global stability.

K.V.L.N.Acharyulu and and N.ch.Pattabhi Ramacharyulu [1, 2, 3,] probed the Local stability of an Ammensal- enemy eco-system on the quasi-linear basic balancing equations. Local stability analysis for an Ammensal- enemy eco-system with various resources has been also established in the author's earlier work [4-11].The present investigation is mainly concentrated on the establishment of the global stability in which Ammensal survives and the enemy is in washed out state where the death rate of the enemy species is greater than its birth rate by employing a property constructed by Liapunov's function with Liapunov's criteria for global stability

**2) Basic properties:**

Consider an autonomous system

$$\frac{dx}{dt} = F_1(x, y) \text{ and } \frac{dy}{dt} = F_2(x, y) \quad (1)$$

Assume that this system has an isolated initial point taken as (0, 0). Consider a function E(x,y) possessing continuous partial derivatives along with the path of (1).This path is represented by C= [(x (t), y (t)] in the parametric form. E(x,y) can be regarded as a function of 't' along C with rate of change

**(i)Liapunov's Method for Global stability**:

If the total energy of physical system has a local minimum at a certain equilibrium point then the point is said to be stable .Liapunov generalized this principle by constructing a function $E(N_1, N_2)$ whose rate of change is given by

$$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial N_1}.\frac{\partial N_1}{\partial t} + \frac{\partial E}{\partial N_2}.\frac{\partial N_2}{\partial t} = \frac{\partial E}{\partial N_1}F_1 + \frac{\partial E}{\partial N_2}F_2 \quad (2)$$

corresponding to the system.

**(ii)Theorem(A)**: If there exists a Liapunov's function E (x,y) for the system (1), then the critical point (0,0) is stable. Further, if this function has additional property that the function (2) is negative definite, then the critical point (0, 0) is asymptotically stable.

The following theorem provides to ascertain definiteness of a Liapunov's function.

**(iii)Theorem(B)**: The function $E(x,y) = ax^2+bxy+cy^2$ is positive definite if $a>0$ and $b^2-4ac<0$ and negative definite if $a<0$, $b^2 - 4ac<0$.

**Notation Adopted**

$N_1$ and N2 are the populations of the Ammensal and enemy species with natural growth rates $a_1$ and $a_2$ respectively.

$a_{11}$ is rate of decrease of the Ammensal due to insufficient food.

$a_{12}$ is rate of increase of the Ammensal due to inhibition by the enemy.

$K_i = a_i / a_{ii}$ are the carrying capacities of $N_i$, i = 1, 2

The state variables $N_1$ and $N_2$ as well as the model parameters $a_1$, $a_2$, $a_{11}$ and are assumed to be non-negative constants.

**3 Basic equations of the model:**

The equation for the growth rate of the Ammensal species ($S_1$)

$$\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^2 - a_{12} N_1 N_2 \quad (3)$$

and the equation for the growth rate of enemy species ($S_2$)

$$\frac{dN_2}{dt} = -a_2 N_2 \quad (4)$$

Before going to establish the global stability by Liapunov's criteria, we now state the equilibrium states with respective equilibrium points.

**4) Equilibrium Points and Equilibrium states:**

The system under investigation has *two* equilibrium states given by $\frac{dN_i}{dt}=0$ where i=1,2

**E₁** : $\bar{N}_1 = 0$; $\bar{N}_2 = 0$ [Fully washed out state ] (5)

**E₂** : $\bar{N}_1 = K_1$ ; $\bar{N}_2 = 0$ [The Ammensal survives and enemy is washed out. ] (6)

Among these two equilibrium points, the state in which the Ammensal survives and the enemy species is washed out is stable where the death rate of the enemy species is greater than its birth rate.

5) **Liapunov's function for global stability:**

The linearised perturbed equations for the basic balancing equations (3) and (4) are

$$\frac{dU_1}{dt} = -a_1U_1 - \overline{N_1}a_{12}U_2 \qquad (7)$$

$$and \quad \frac{dU_2}{dt} = -a_2U_2 \qquad (8)$$

The characteristic equation is

$$(\lambda + a_1)\ (\lambda + a_2) = 0 \qquad (9)$$

i.e. $\lambda^2 + (a_1 + a_2)\,\lambda + a_1a_2 = 0$

$$\Rightarrow \lambda^2 + p\lambda + q = 0$$

$$where\, P = a_1 + a_2 > 0,\ q = a_1a_2 > 0 \qquad (10)$$

The required conditions for Liapunov's function are satisfied

Let E $(U_1, U_2) = \frac{1}{2}(aU_1^2 + 2b\, U_1U_2 + CU_2^2)$ (11)

where $a = \frac{a_2^{\,2} + a_1a_2}{D}$

$$b = -\frac{a_{12}a_2\overline{N_1}}{D}$$

and $c = \frac{a_1^2 + a_{12}^2\overline{N_1}^{\,2} + (a_1a_2)}{D}$

Where D = pq = $(a_1 + a_2)(a_1a_2) > 0$ (12)

From (8), it is clear that D>0 and a>0

$$D^2(ac - b^2) = b^2\left[\frac{a_2^2 + a_1a_2}{D}.\frac{a_1^2 + a_{12}^2\overline{N_1}^{\,2} + a_1a_2}{D} - \frac{a_{12}^2a_2^2\overline{N_1}^{\,2}}{D}\right]$$

$$= [a_2^2a_1^2 + a_1a_2^{\,3} + a_1^3a_2 + a_1a_2a_{12}^2\overline{N_1}^{\,2} + a_1^2a_2^2]$$

$\Rightarrow$ $ac - b^2 > 0$ (since $D^2 > 0$)

i.e., $b^2 - ac < 0$

$\therefore$ The function $E(U_1,U_2)$ is positive definite (13)

Further $S = \frac{\partial E}{\partial U_1}.\frac{dU_1}{dt} + \frac{\partial E}{\partial U_2}.\frac{dU_2}{dt}$

$$= (aU_1 + bU_2)\ (-a_1U_1 - a_{12}\overline{N_1}U_2) + (bU_1 + cU_2)(-a_2U_2)$$

$$= -aa_1U_1^2 - aa_{12}\overline{N_1}U_1U_2 - ba_1U_1U_2 - ba_{12}\overline{N_1}U_2{}^2 - ba_2U_1U_2 - ca_2U_2^2$$

$$= -aa_1U_1^2 - (aa_{12}\overline{N_1} + ba_1 + ba_2)U_1U_2 - (ba_{12}\overline{N_1} + ca_2)U_2^2 \qquad (14)$$

Substituting the values of a, b and c, we obtain

$$-aa_1U_1^2 = -(\frac{a_2^2 + a_1a_2}{D})a_1U_1^2 = -\left[\frac{(a_1 + a_2)a_1a_2}{D}\right]U_1^2, = -\left(\frac{D}{D}\right)U_1^2$$

$$-(aa_{12}\overline{N_1} + ba_1 + ba_2)U_1U_2 = \left[\left(\frac{a_2^2 + a_1a_2}{D}\right)a_{12}\overline{N_1} - \frac{a_{12}a_2\overline{N_1}}{D}a_1 + a_2\right]U_1U_2$$

$$= \left[\frac{a_2^2a_{12}\overline{N_1} + a_1a_2a_{12}\overline{N_1} - a_1a_2a_{12}\overline{N_1} - a_2^2a_{12}\overline{N_1}}{D}\right]U_1U_2 = 0 \text{ and}$$

$$-\left[ba_{12}\overline{N_1} + ca_2\right]U_2^2 = -\left[\frac{(-a_2a_{12}\overline{N_1})a_{12}\overline{N_1} + (a_1{}^2a_{12}{}^2\overline{N_1}^2 + a_1a_2)a_2}{D}\right]U_2^2$$

$$= -\left[\frac{(-a_2a_{12}{}^2\overline{N_1}^2 + a_1{}^2a_2 + a_1{}^2a_2\overline{N_1}^2 + a_1a_2{}^2}{D}\right]U_2^2$$

$$= -\left[\frac{a_1a_2(a_1 + a_2)}{D}\right]U_2^2 = -(\frac{D}{D})U_2^2$$

$$S = \frac{\partial E}{\partial U_1}.\frac{dU_1}{dt} + \frac{\partial E}{\partial U_2}.\frac{dU_2}{dt}$$

$$= -\frac{D}{D}U_1^2 - \frac{D}{D}U_2^2 = -(U_1^2 + U_2^2) \qquad (15)$$

which is negative definite.

So E ($U_1$, $U_2$) is a Liapunov function for the linear system.

Next we will prove that E ($U_1$,$U_2$) is also a Liapunov function for the non-Linear system

Define

$F_1(N_1N_2) = N_1[a_1 - a_{11}N_1 - a_{12}N_2]$ and $F_2(N_1N_2) = -a_2N_2$

By putting $N_1 = \overline{N}_1 + U_1$ and $N_2 = \overline{N}_2 + U_2$ in (3) and (4)

$$\frac{dU_1}{dt} = \overline{N_1} + U_1[a_1 - a_{11}(\overline{N_1} + U_1) - a_{12}(\overline{N_2} + U_2)]$$

$$= (a_1\overline{N_1} - a_{11}\overline{N_1}^2 - a_{12}\overline{N_1N_2}) - a_{11}\overline{N_1}U_1 - a_{12}\overline{N_1}U_2$$

$$+a_1U_1 - a_{11}\overline{N_1}U_1 - a_{11}U_1{}^2 - a_{12}\overline{N_2}U_1 - a_{12}U_1U_2$$

$$= -(a_{11}\overline{N_1})U_1 + (a_1 - a_{11}\overline{N_1} - a_{12}\overline{N_2})U_1 - a_{12}\overline{N_1}U_2 - a_{12}U_1U_2 - a_{11}U_1{}^2$$

$$= -a_{11}\overline{N_1}U_1 - a_{12}\overline{N_1}U_2 - a_{11}U_1{}^2 - a_{12}U_1U_2$$

$$\Rightarrow F_1(U_1,U_2) = \frac{dU_1}{dt} = \; = -a_{11}\overline{N_1}U_1 - a_{12}\overline{N_1}U_2 + f_1(U_1,U_2) \qquad (16)$$

where $f_1(U_1, U_2) = -a_{11}U_1{}^2 - a_{12}U_1U_2$

similarly $\frac{dU_2}{dt} = -a_2\overline{N_2} - a_2U_2$

$\Rightarrow$ $F_2(U_1,U_2) = \dfrac{dU_2}{dt} = -a_2U_2 + f_2(U_1,U_2)$ (17)

Where $f_2(U_1,U_2) = 0$

we have

$$\frac{\partial E}{\partial U_1}F_1 + \frac{\partial E}{\partial U_2}F_2 = -(U_1^2 + U_2^2) + (aU_1 + bU_2)\, f_1(U_1, U_2)$$

By introducing polar coordinates, we get

$$\frac{\partial E}{\partial U_1}F_1 + \frac{\partial E}{\partial U_2}F_2 = -r^2 + r\,[(a\cos\theta + b\sin\theta)\, f_1(U_1, U_2)]$$

Denote largest of the numbers |a|, |b|, |c| by M

Then $|f_1(U_1, U_2)| < \dfrac{r}{6M}$ for all satisfying small r > 0

$$\text{so}\ \frac{\partial E}{\partial U_1}F_1 + \frac{\partial E}{\partial U_2}F_2 < -r^2 + \frac{2Mr^2}{6M} = \frac{-2r^2}{3} < 0 \qquad (18)$$

Thus $E(U_1, U_2)$ is a positive definite in with the property that

$\dfrac{\partial E}{\partial U_1}F_1 + \dfrac{\partial E}{\partial U_2}F_2$ is negative definite.

$\therefore$ The equilibrium point is asymptotically **stable.**

**Conclusion:** The Global stability of mathematical model of an Ammensal with limited resources and enemy with unlimited resources in which Ammensal survives and the enemy is in washed out state where the death rate of the enemy species is greater than it's birth rate is explained.

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