

**ON THE STABILITY OF A FOUR SPECIES: A PREY-PREDATOR-HOST-COMMENSAL-SYN ECO-SYSTEM-VI
(PREDATOR WASHED OUT STATES)****B. Hari Prasad¹, N. Ch. Pattabhi Ramacharyulu²**

- 1 Department of Mathematics, Chaitanya Degree & P. G. College (Autonomous), Hanamkonda, Warangal, A. P, India. Email: sumathi_prasad73@yahoo.com
 2 Former Faculty, Department of Mathematics, NIT, Warangal, India.
 E-mail: pattabhi1933@yahoo.com

ABSTRACT: This paper deals with an investigation on a Four Species Syn-Ecological System (Predator washed out states). The System comprises of a Prey (S_1), a Predator (S_2) that survives upon S_1 , two Hosts S_3 and S_4 for which S_1 , S_2 are commensal respectively i.e., S_3 and S_4 benefit S_1 and S_2 respectively, without getting effected either positively or adversely. Further S_3 and S_4 are neutral. The model equations of the system constitute a set of four first order non-linear ordinary differential coupled equations. In all, there are sixteen equilibrium points. Criteria for the asymptotic stability of four of the sixteen equilibrium points : the Predator washed out states only are established in this paper. The system would be stable if all the characteristic roots are negative, in case they are real, and have negative real parts, in case they are complex. The linearized equations for the perturbations over the equilibrium points are analyzed to establish the criteria for stability and the trajectories illustrated.

Key words: Equilibrium point, Host, Prey, Predator, Unstable.

1. INTRODUCTION:

Mathematical modeling of Eco-System was initiated in 1925 by Lotka [11]. The general concepts of modeling have been presented in the treatises of Meyer[12], Kushing[8], Kapur J.N. [6,7] and several others. The ecological interactions can be broadly classified as Prey-Predator, Commensalism, Competition, Neutralism, Mutualism and so on. N.C. Srinivas [14] studied competitive eco-systems of two species and three species with limited and unlimited resources. Later Lakshminarayan [9], Lakshminarayan and Pattabhi Ramacharyulu [10] studied Prey-Predator ecological models with partial cover for the Prey and alternate food for the Predator. Recently, Archana Reddy [1] and Bhaskara Rama Sharma [2] investigated diverse problems related to two species competitive systems with time delay, employing analytical and numerical techniques. Further Phani Kumar, Seshagiri Rao and Pattabhi Ramacharyulu [13] studied the stability of a Host-A flourishing commensal species pair with limited resources. The present

authors Hari Prasad.B and Pattabhi Ramacharyulu.N.Ch studied the stability of the fully washed out state [3], co-existent state [5], Prey and Predator washed out states [4]. Continuation of this criteria for the stability of the Predator only washed out states of the system are presented in this paper.

2. BASIC EQUATIONS OF THE MODEL:

Notation Adopted:

- S_1 : Prey for S_2 and commensal for S_3 .
 S_2 : Predator surviving upon S_1 and commensal for S_4 .
 S_3 : Host for the commensal – Prey (S_1).
 S_4 : Host of the commensal – Predator (S_2)
 $N_1(t)$: The Population of the Prey (S_1)
 $N_2(t)$: The Population of the Predator (S_2)
 $N_3(t)$: The Population of the Host (S_3) of the Prey (S_1)
 $N_4(t)$: The Population of the Host (S_4) of the Predator (S_2)
 t : Time instant

a_1, a_2, a_3, a_4 : Natural growth rates of S_1, S_2, S_3, S_4

$a_{11}, a_{22}, a_{33}, a_{44}$: Self inhibition coefficients of S_1, S_2, S_3, S_4

a_{12}, a_{21} : Interaction (Prey-Predator) coefficients of S_1 due to S_2 and S_2 due to S_1

a_{13} : Coefficient for commensal for S_1 due to the Host S_3

a_{24} : Coefficient for commensal for S_2 due to the Host S_4

$k_1 = \frac{a_1}{a_{11}}, k_2 = \frac{a_2}{a_{22}}, k_3 = \frac{a_3}{a_{33}}, k_4 = \frac{a_4}{a_{44}}$: Carrying capacities of S_1, S_2, S_3, S_4

Further the variables N_1, N_2, N_3, N_4 are non-negative and the model parameters $a_1, a_2, a_3, a_4;$
 $a_{11}, a_{22}, a_{33}, a_{44}; a_{12}, a_{21}, a_{13}, a_{24}$ are assumed to be non-negative constants.

The model equations for the growth rates of S_1, S_2, S_3, S_4 are

$$\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^2 - a_{12} N_1 N_2 + a_{13} N_1 N_3 \quad \dots \quad (2.1)$$

$$\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^2 + a_{21} N_1 N_2 + a_{24} N_2 N_4 \quad \dots \quad (2.2)$$

$$\frac{dN_3}{dt} = a_3 N_3 - a_{33} N_3^2 \quad \dots \quad (2.3)$$

$$\frac{dN_4}{dt} = a_4 N_4 - a_{44} N_4^2 \quad \dots \quad (2.4)$$

3 EQUILIBRIUM STATES:

The system under investigation has sixteen equilibrium states defined by

$$\frac{dN_i}{dt} = 0, i = 1, 2, 3, 4 \quad \dots\dots\dots (3.1)$$

are given in the following table.

S.No.	Equilibrium State	Equilibrium Point
1	Fully Washed out state	$\bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0, \bar{N}_4 = 0$
2	Only the Host (S ₄)of S ₂ survives	$\bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0, \bar{N}_4 = \frac{a_4}{a_{44}}$
3	Only the Host (S ₃)of S ₁ survives	$\bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = \frac{a_3}{a_{33}}, \bar{N}_4 = 0$
4	Only the Predator S ₂ survives	$\bar{N}_1 = 0, \bar{N}_2 = \frac{a_2}{a_{22}}, \bar{N}_3 = 0, \bar{N}_4 = 0$
5	Only the Prey S ₁ survives	$\bar{N}_1 = \frac{a_1}{a_{11}}, \bar{N}_2 = 0, \bar{N}_3 = 0, \bar{N}_4 = 0$
6	Prey (S ₁) and Predator (S ₂) washed out	$\bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = \frac{a_3}{a_{33}}, \bar{N}_4 = \frac{a_4}{a_{44}}$
7	Prey (S ₁) and Host (S ₃) of S ₁ washed out	$\bar{N}_1 = 0, \bar{N}_2 = \frac{a_2 a_{44} + a_4 a_{24}}{a_{22} a_{44}}, \bar{N}_3 = 0, \bar{N}_4 = \frac{a_4}{a_{44}}$
8	Prey (S ₁) and Host (S ₄) of S ₂ washed out	$\bar{N}_1 = 0, \bar{N}_2 = \frac{a_2}{a_{22}}, \bar{N}_3 = \frac{a_3}{a_{33}}, \bar{N}_4 = 0$
9	Predator (S ₂) and Host (S ₃) of S ₁ washed out	$\bar{N}_1 = \frac{a_1}{a_{11}}, \bar{N}_2 = 0, \bar{N}_3 = 0, \bar{N}_4 = \frac{a_4}{a_{44}}$
10	Predator (S ₂) and Host (S ₄) of S ₂ washed out	$\bar{N}_1 = \frac{a_1 a_{33} + a_3 a_{13}}{a_{11} a_{13}}, \bar{N}_2 = 0, \bar{N}_3 = \frac{a_3}{a_{33}}, \bar{N}_4 = 0$
11	Prey (S ₁) and Predator (S ₂) survives	$\bar{N}_1 = \frac{a_1 a_{22} - a_2 a_{12}}{a_{11} a_{22} + a_{12} a_{21}}, \bar{N}_2 = \frac{a_1 a_{21} + a_2 a_{11}}{a_{11} a_{22} + a_{12} a_{21}}, \bar{N}_3 = 0, \bar{N}_4 = 0$
12	Only the Prey (S ₁) washed out	$\bar{N}_1 = 0, \bar{N}_2 = \frac{a_2 a_{44} + a_4 a_{24}}{a_{22} a_{44}}, \bar{N}_3 = \frac{a_3}{a_{33}}, \bar{N}_4 = \frac{a_4}{a_{44}}$

13	Only the predator (S_2) washed out	$\bar{N}_1 = \frac{a_1 a_{23} + a_3 a_{13}}{a_{11} a_{13}}, \bar{N}_2 = 0, \bar{N}_3 = \frac{a_3}{a_{33}}, \bar{N}_4 = \frac{a_4}{a_{44}}$
14	Only the Host (S_3) of S_1 washed out	$\bar{N}_1 = \frac{\delta_2}{\delta_1}, \bar{N}_2 = \frac{\delta_3}{\delta_1}, \bar{N}_3 = 0, \bar{N}_4 = \frac{a_4}{a_{44}}$ <i>where</i> $\delta_1 = a_{44}(a_{11}a_{22} + a_{12}a_{21}) > 0$ $\delta_2 = a_1 a_{22} a_{44} - a_{12}(a_2 a_{44} + a_4 a_{24})$ $\delta_3 = a_1 a_{21} a_{44} - a_{11}(a_2 a_{44} + a_4 a_{24})$
15	Only the Host (S_4) of S_2 washed out	$\bar{N}_1 = \frac{\sigma_2}{\sigma_1}, \bar{N}_2 = \frac{\sigma_3}{\sigma_1}, \bar{N}_3 = \frac{a_3}{a_{33}}, \bar{N}_4 = 0$ <i>where</i> $\sigma_1 = a_{33}(a_{11}a_{22} + a_{12}a_{21}) > 0$ $\sigma_2 = a_{22}(a_1 a_{33} + a_3 a_{13}) - a_2 a_{12} a_{33}$ $\sigma_3 = a_{21}(a_1 a_{33} + a_3 a_{13}) + a_2 a_{11} a_{33} > 0$
16	The co-existent state (or) Normal steady state	$\bar{N}_1 = \frac{a_{22} a_{44} \psi_1 - a_{12} a_{33} \psi_2}{\psi_3}, \bar{N}_2 = \frac{a_{21} a_{44} \psi_1 + a_{11} a_{33} \psi_2}{\psi_3},$ $\bar{N}_3 = \frac{a_3}{a_{33}}, \bar{N}_4 = \frac{a_4}{a_{44}}$ <i>where</i> $\psi_1 = a_1 a_{33} + a_3 a_{13} > 0$ $\psi_2 = a_2 a_{44} + a_4 a_{24} > 0$ $\psi_3 = a_{33} a_{44} (a_{11} a_{22} + a_{12} a_{21}) > 0$

The present paper deals with the Predator washed out states only. The stability of the other equilibrium states will be presented in the forth coming communications.

4. Stability of the Predator Washed out Equilibrium States only:

(Sl. No.s 5, 9, 10, 13 in the above table)

The equilibrium point $\bar{N}_1 = \frac{a_1}{a_{11}}, \bar{N}_2 = 0, \bar{N}_3 = 0, \bar{N}_4 = 0$ (Sl. No. 5) was already discussed in the paper "On the Stability of a Four Species: A Prey – Predator – Host – Commensal – Syn Eco-system – IV" communicated to "Internation eJournal of Mathematics and Engineering".

4.1 Equilibrium point $\bar{N}_1 = \frac{a_1}{a_{11}}, \bar{N}_2 = 0, \bar{N}_3 = 0, \bar{N}_4 = \frac{a_4}{a_{44}}$:

Let us consider small deviations from the steady state

$$\text{ie, } \bar{N}_i + u_i(t), i = 1, 2, 3, 4 \quad \dots (4.1.1)$$

where $u_i(t)$ is a small perturbations in the species S_i .

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4 .

we get

$$\frac{du_1}{dt} = -a_1 u_1 - k_1 a_{12} u_2 + k_1 a_{13} u_3 \quad \dots (4.1.2)$$

$$\frac{du_2}{dt} = \alpha_2 u_2 \quad \dots (4.1.3), \quad \frac{du_3}{dt} = a_3 u_3 \quad \dots (4.1.4)$$

$$\frac{du_4}{dt} = -a_4 u_4 \quad \dots (4.1.5)$$

$$\text{Here } \alpha_2 = a_2 + k_1 a_{21} + k_4 a_{24} > 0 \quad \dots (4.1.6)$$

The characteristic equation of which is

$$(\lambda + a_1)(\lambda - \alpha_2)(\lambda - a_3)(\lambda + a_4) = 0 \quad \dots (4.1.7)$$

The roots $-a_1, -a_4$ are negative and α_2, a_3 are positive. Hence the steady state is **unstable** and the solutions of the equations (4.1.2), (4.1.3), (4.1.4), (4.1.5) are

$$u_1 = (u_{10} + \alpha_1 - \alpha_3) e^{-a_1 t} - \alpha_1 e^{\alpha_2 t} + \alpha_3 e^{a_3 t} \quad \dots (4.1.8)$$

$$u_2 = u_{20} e^{\alpha_2 t} \quad \dots (4.1.9), \quad u_3 = u_{30} e^{a_3 t} \quad \dots (4.1.10)$$

$$u_4 = u_{40} e^{-a_4 t} \quad \dots (4.1.11)$$

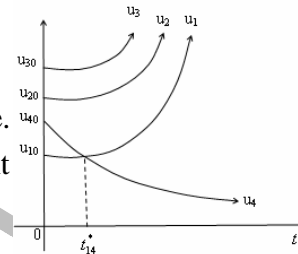
$$\text{Here } \alpha_1 = \frac{k_1 a_{12} u_{20}}{\alpha_2 + a_1}, \quad \alpha_3 = \frac{k_1 a_{13} u_{30}}{\alpha_2 + a_3} \quad \dots (4.1.12)$$

and $u_{10}, u_{20}, u_{30}, u_{40}$ are the initial values of u_1, u_2, u_3, u_4 respectively

In the three equilibrium states, there would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates a_1, a_2, a_3, a_4 and the initial values of the perturbations $u_{10}(t), u_{20}(t), u_{30}(t), u_{40}(t)$ of the species S_1, S_2, S_3, S_4 . Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations.

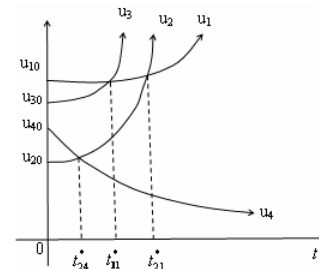
Case (i): If $u_{10} < u_{40} < u_{20} < u_{30}$ and $a_1 < a_4 < \alpha_2 < a_3$

In this case the Host (S_4) of S_2 has the least natural birth rate. Initially it is dominated over by the Prey (S_1) till the time instant t_{14}^* and there after the dominance is reversed.



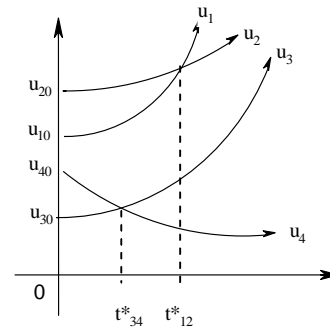
Case (ii): If $u_{20} < u_{40} < u_{30} < u_{10}$ and $a_4 < a_1 < \alpha_2 < a_3$

In this case the Host (S_4) of S_2 has the least natural birth rate. Initially it is dominated over by the Predator (S_2) till the time instant t_{24}^* and thereafter the dominance is reversed. Also the Prey (S_1) dominates over the Predator (S_2), Host (S_3) of S_1 till the time instant t_{21}^*, t_{31}^* respectively and the dominance gets reversed thereafter.



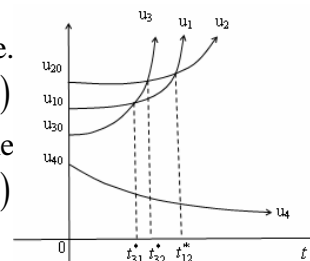
Case (iii): If $u_{30} < u_{40} < u_{10} < u_{20}$ and $a_3 < a_4 < \alpha_2 < a_1$

In this case the Host (S_4) of S_2 has the least natural birth rate. Initially it is dominated over by the Host (S_3) of S_1 till the time instant t_{34}^* and thereafter the dominance is reversed. Also the Predator (S_2) dominates over the Prey (S_1) till the time instant t_{12}^* and the dominance gets reversed thereafter.



Case (iv): If $u_{40} < u_{30} < u_{10} < u_{20}$ and $\alpha_2 < a_1 < a_3 < a_4$

In this case the Host (S_4) of S_2 has the least natural birth rate. Initially the Predator (S_2) dominate over the Prey (S_1), Host (S_3) of S_1 till the time instant t_{12}^*, t_{32}^* respectively and thereafter the dominance is reversed. Also the Prey (S_1) dominates its Host (S_3) till the time instant t_{31}^* and the dominance gets reversed thereafter.



Trajectories of perturbations:

The trajectories in the $u_1 - u_2, u_1 - u_3, u_1 - u_4, u_2 - u_3, u_2 - u_4, u_3 - u_4$ planes are

$$x = A y_1^{\frac{-a_1}{\alpha_2}} - B y_1 + C y_1^{\frac{\alpha_3}{\alpha_2}} \dots (4.1.13)$$

$$x = A y_2^{\frac{-a_1}{\alpha_3}} - B y_2^{\frac{\alpha_2}{\alpha_3}} + C y_2 \dots (4.1.14)$$

$$x = A y_3^{\frac{a_1}{\alpha_4}} - B y_3^{\frac{-\alpha_2}{\alpha_4}} + C y_3^{\frac{-\alpha_3}{\alpha_4}} \dots (4.1.15)$$

$$\left(\frac{u_2}{u_{20}}\right)^{\alpha_3} = \left(\frac{u_3}{u_{30}}\right)^{\alpha_2}, \quad \left(\frac{u_2}{u_{20}}\right)^{-\alpha_4} = \left(\frac{u_4}{u_{40}}\right)^{\alpha_2}, \quad \left(\frac{u_3}{u_{30}}\right)^{-\alpha_4} = \left(\frac{u_4}{u_{40}}\right)^{\alpha_3} \dots (4.1.16)$$

where $A = \frac{u_{10} + \alpha_1 - \alpha_3}{u_{10}}, \quad B = \frac{\alpha_1}{u_{10}}, \quad C = \frac{\alpha_3}{u_{10}} \dots (4.1.17)$

$$x = \frac{u_1}{u_{10}}, \quad y_1 = \frac{u_2}{u_{20}}, \quad y_2 = \frac{u_3}{u_{30}}, \quad y_3 = \frac{u_4}{u_{40}} \dots (4.1.18)$$

4.2 Equilibrium point $\bar{N}_1 = \frac{a_1 a_{33} + a_3 a_{13}}{a_{11} a_{13}}, \bar{N}_2 = 0, \bar{N}_3 = \frac{a_3}{a_{33}}, \bar{N}_4 = 0$:

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4 , we get

$$\frac{du_1}{dt} = \beta_1 u_1 - a_{12} u_2 + \beta_3 u_3 \dots (4.2.1), \quad \frac{du_2}{dt} = \gamma_2 u_2 \dots (4.2.2)$$

$$\frac{du_3}{dt} = -a_3 u_3 \dots (4.2.3), \quad \frac{du_4}{dt} = a_4 u_4 \dots (4.2.4)$$

Here $\beta_1 = a_1 - \frac{2a_1 a_{33} + a_3 a_{13}}{a_{13}} + \frac{a_3 a_{13}}{a_{33}}, \quad \gamma_2 = a_2 + a_{21} \beta_2 > 0, \dots (4.2.5)$

$$\beta_2 = \frac{a_1 a_{33} + a_3 a_{13}}{a_{11} a_{13}} > 0, \quad \beta_3 = \frac{a_1 a_{33} + a_3 a_{13}}{a_{11}} > 0 \dots (4.2.6)$$

The characteristic equation of which is

$$(\lambda - \beta_1)(\lambda - \gamma_2)(\lambda + a_3)(\lambda - a_4) = 0 \dots (4.2.7)$$

Case (A) : If $\beta_1 < 0$ $\left(ie, a_1 < \frac{2a_1 a_{33} + a_3 a_{13}}{a_{13}} + \frac{a_3 a_{13}}{a_{33}} \right)$

The roots γ_2, a_4 are positive and $\beta_1, -a_3$ are negative. Hence the state is **unstable** and the solutions of the equations (4.2.1), (4.2.2), (4.2.3), (4.2.4) are

$$u_1 = (u_{10} + \gamma_3 - \gamma_1) e^{\beta_1 t} + \gamma_1 e^{\gamma_2 t} - \gamma_3 e^{-a_3 t} \quad \dots (4.2.8)$$

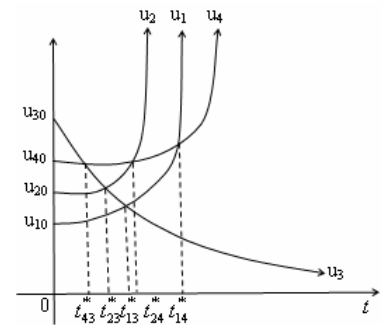
$$u_2 = u_{20} e^{\gamma_2 t} \quad \dots (4.2.9), \quad u_3 = u_{30} e^{-a_3 t} \quad \dots (4.2.10)$$

$$u_4 = u_{40} e^{a_4 t} \quad \dots (4.2.11)$$

$$\text{Here } \gamma_1 = \frac{a_{12} \beta_2 u_{20}}{\beta_1 - \gamma_2}, \quad \gamma_3 = \frac{\beta_3 u_{30}}{\beta_1 + a_3} \quad \dots (4.2.12)$$

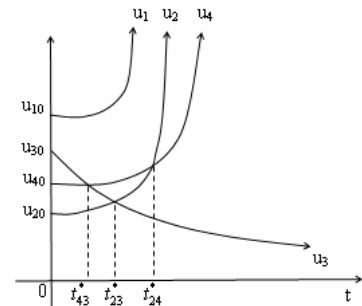
Case (i): If $u_{10} < u_{20} < u_{40} < u_{30}$ and $\alpha_4 < a_3 < \beta_1 < \gamma_2$

In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominated over by the Host (S_4) of S_2 , Predator (S_2), Prey (S_1) till the time instant $t_{43}^*, t_{23}^*, t_{13}^*$ respectively and thereafter the dominance is reversed. Also the Host (S_4) of S_2 dominates over the Predator (S_2), Prey (S_1) till the time instant t_{24}^*, t_{14}^* respectively and the dominance gets reversed thereafter.



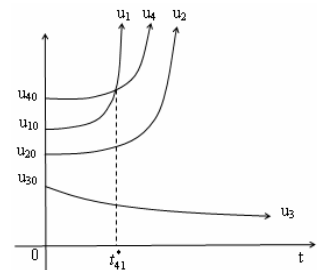
Case (ii): If $u_{20} < u_{40} < u_{30} < u_{10}$ and $a_4 < \gamma_2 < a_3 < \beta_1$

In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominated over by the Host (S_4) of S_2 , Predator (S_2) till the time instant t_{43}^*, t_{23}^* respectively and thereafter the dominance is reversed. Also the Host (S_4) of S_2 dominates over the Predator (S_2) till the time instant t_{24}^* and the dominance gets reversed thereafter.



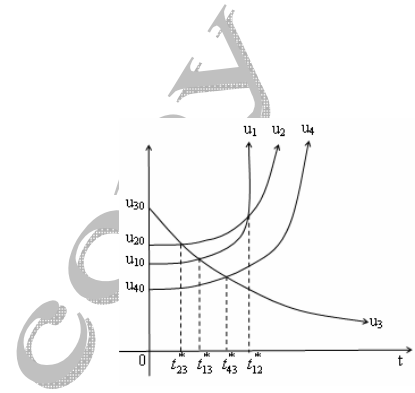
Case (iii): If $u_{30} < u_{20} < u_{10} < u_{40}$ and $a_3 < \gamma_2 < a_4 < \beta_1$

In this case the Host (S_3) of S_1 has the least natural birth rate. Initially the Host (S_4) of S_2 dominates over the Prey (S_1) till the time instant t_{41}^* and thereafter the dominance is reversed.



Case (iv): If $u_{40} < u_{10} < u_{20} < u_{30}$ and $a_4 < \gamma_2 < \beta_1 < a_3$

In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominated over by the Predator (S_2), Prey (S_1), Host (S_4) of S_2 till the time instant $t_{23}^*, t_{13}^*, t_{43}^*$ respectively and thereafter the dominance is reversed. Also the Predator (S_2) dominates over the Prey (S_1) till the time instant t_{12}^* and the dominance gets reversed thereafter.

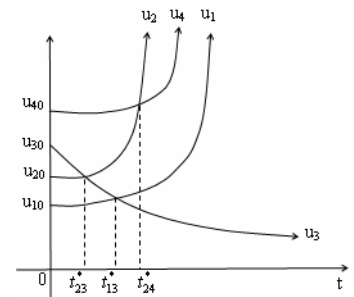


Case (B): If $\beta_1 > 0$ (ie, $a_1 > \frac{2a_1 a_{33} + a_3 a_{13}}{a_{13}} + \frac{a_3 a_{13}}{a_{33}}$)

The roots β_1, γ_2, a_4 are positive and $-a_3$ is negative. Hence the state is **unstable** and the solutions in this case are same as in case (A).

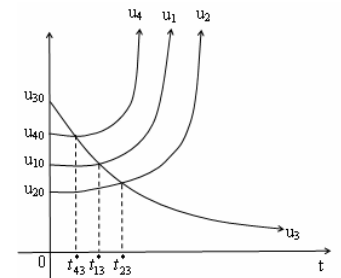
Case (i): If $u_{10} < u_{20} < u_{30} < u_{40}$ and $\beta_1 < a_4 < a_3 < \gamma_2$

In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominated over by the Predator (S_2), Prey (S_1) till the time instant t_{23}^*, t_{13}^* respectively and thereafter the dominance is reversed. Also the Host (S_4) of S_2 dominates over the Predator (S_2) till the time instant t_{24}^* and the dominance gets reversed thereafter.



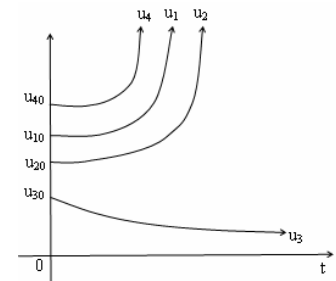
Case (ii): If $u_{20} < u_{10} < u_{40} < u_{30}$ and $\gamma_2 < \beta_1 < a_3 < a_4$

In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominated over by the Host (S_4) of S_2 , Prey (S_1), Predator (S_2) till the time instant $t_{43}^*, t_{13}^*, t_{23}^*$ respectively and thereafter the dominance is reversed.



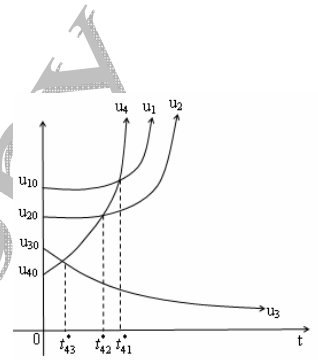
Case (iii): If $u_{30} < u_{20} < u_{10} < u_{40}$ and $a_3 < \gamma_2 < \beta_1 < a_4$

In this case the Host (S_3) of S_1 has the least natural birth rate. And the Host (S_4) of S_2 dominates the Prey (S_1), Predator (S_2), Host (S_3) of S_1 in natural growth rate as well as in its initial population strength.



Case (iv): If $u_{40} < u_{30} < u_{20} < u_{10}$ and $\gamma_2 < a_3 < \beta_1 < a_4$

In this case the Host (S_3) of S_1 has the least natural birth rate. Initially the Prey (S_1), Predator (S_2), Host (S_3) of S_1 dominates over the Host (S_4) of S_2 till the time instant $t_{41}^*, t_{42}^*, t_{43}^*$ respectively and thereafter the dominance is reversed.



Trajectories of Perturbations:

The trajectories in the $u_1 - u_2, u_1 - u_3, u_1 - u_4, u_2 - u_3, u_2 - u_4, u_3 - u_4$ planes are

$$x = A_1 y_1^{\frac{\beta_1}{\gamma_2}} + B_1 y_1 - C_1 y_1^{\frac{-a_3}{\gamma_2}}, \quad x = A_1 y_2^{\frac{-\beta_1}{a_3}} + B_1 y_1^{\frac{-\gamma_2}{a_3}} - C_1 y_2 \quad \dots (4.2.14)$$

$$x = A_1 y_3^{\frac{-\beta_1}{a_4}} + B_1 y_3^{\frac{-\gamma_2}{a_4}} - C_1 y_3^{\frac{a_3}{a_4}} \quad \dots (4.2.15)$$

$$\left(\frac{u_2}{u_{20}}\right)^{-a_3} = \left(\frac{u_3}{u_{30}}\right)^{\gamma_2}, \quad \left(\frac{u_2}{u_{20}}\right)^{a_4} = \left(\frac{u_4}{u_{40}}\right)^{\gamma_2}, \quad \left(\frac{u_3}{u_{30}}\right)^{a_4} = \left(\frac{u_4}{u_{40}}\right)^{-a_3}, \quad \dots (4.2.16)$$

respectively

where $A_1 = \frac{u_{10} + \gamma_3 - \gamma_1}{u_{10}}, B_1 = \frac{\gamma_1}{u_{10}}, C_1 = \frac{\gamma_3}{u_{10}} \quad \dots (4.2.17)$

4.3 Equilibrium state $\bar{N}_1 = \frac{a_1 a_{23} + a_3 a_{13}}{a_{11} a_{13}}, \bar{N}_2 = 0, \bar{N}_3 = \frac{a_3}{a_{33}}, \bar{N}_4 = \frac{a_4}{a_{44}} :$

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher power of u_1, u_2, u_3, u_4 , we get

$$\frac{du_1}{dt} = -\rho_1 u_1 - a_{12} \beta_2 u_2 + a_{13} \beta_2 u_3 \quad \dots (4.3.1)$$

$$\frac{du_2}{dt} = \rho_2 u_2 \quad \dots (4.3.2), \quad \frac{du_3}{dt} = -a_3 u_3 \quad \dots (4.3.3)$$

$$\frac{du_4}{dt} = -a_4 u_4 \quad \dots (4.3.4)$$

Here $\rho_1 = a_1 + k_3 a_{13} > 0, \quad \rho_2 = a_2 + a_{21} \rho_3 + k_4 a_{24} > 0 \quad \dots (4.3.5)$

The characteristic equation of which is

$$(\lambda + \rho_1)(\lambda - \rho_2)(\lambda + a_3)(\lambda + a_4) = 0 \quad \dots (4.3.6)$$

The roots $-\rho_1, -a_3, -a_4$ are negative and ρ_2 is positive. Hence the steady state is **unstable** and the solutions of the equations (4.3.1), (4.3.2), (4.3.3), (4.3.4) are

$$u_1 = (u_{10} + U - V) e^{-\rho_1 t} - U e^{\rho_2 t} - V e^{-a_3 t} \quad \dots (4.3.7)$$

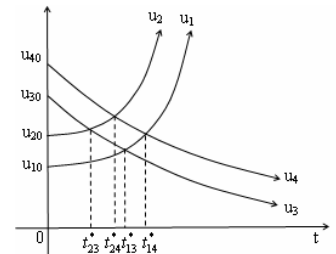
$$u_2 = u_{20} e^{\rho_2 t} \quad \dots (4.3.8) \quad u_3 = u_{30} e^{-a_3 t} \quad \dots (4.3.9)$$

$$u_4 = u_{40} e^{-a_4 t} \quad \dots (4.3.10)$$

$$\text{Here } U = \frac{a_{12} \beta_2 u_{20}}{\rho_2 + \rho_1}, \quad V = \frac{a_{13} \beta_2 u_{30}}{a_3 - \rho_1} \quad \dots (4.3.11)$$

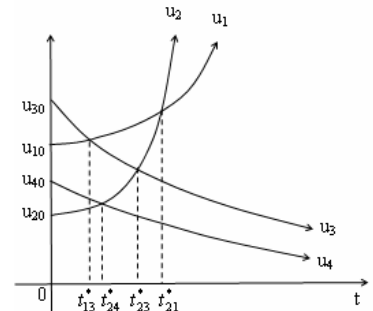
Case (i): If $u_{10} < u_{20} < u_{30} < u_{40}$ and $\rho_1 < \rho_2 < a_3 < a_4$

In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominated over by the Predator (S_2), Prey (S_1) till the time instant t_{23}^*, t_{13}^* respectively and thereafter the dominance is reversed. Also the Host (S_4) of S_2 dominates the Prey (S_1), Predator (S_2) till the time instant t_{14}^*, t_{24}^* , respectively and the dominance gets reversed thereafter.



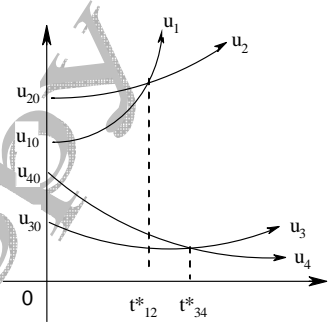
Case (ii): If $u_{20} < u_{40} < u_{10} < u_{30}$ and $\rho_1 < a_4 < a_3 < \rho_2$

In this case the Host (S_4) of S_2 has the least natural birth rate. Initially it is dominated over by the Predator (S_2) till the time instant t_{24}^* and thereafter the dominance is reversed. Also the Host (S_3) of S_1 dominates over the Predator (S_2), Prey (S_1) till the time instant t_{23}^*, t_{13}^* respectively and thereafter the dominance is reversed. Similarly the Prey (S_1) dominates over the Predator (S_2) till the time instant t_{21}^* and the dominance gets reversed thereafter.



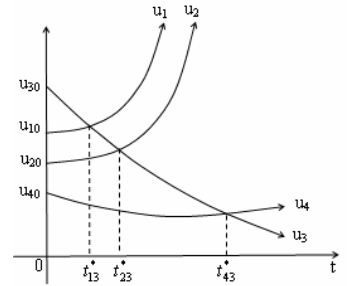
Case (iii): If $u_{30} < u_{40} < u_{10} < u_{20}$ and $a_4 < a_3 < \rho_2 < \rho_1$

In this case the Host (S_4) of S_2 has the least natural birth rate. Initially it is dominated over the Host (S_3) of (S_1) till the time instant t_{34}^* and thereafter the dominance is reversed. Also the Predator (S_2) dominates over the Prey (S_1) till the time instant t_{12}^* and dominance gets reversed thereafter.



Case (iv): If $u_{40} < u_{20} < u_{10} < u_{30}$ and $a_3 < \rho_2 < a_4 < \rho_1$

In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominated over by the Prey (S_1), Predator (S_2), Host (S_4) of S_2 till the time instant t_{13}^* , t_{23}^* , t_{43}^* respectively and thereafter the dominance is reversed.



Trajectories of perturbations:

The trajectories in the $u_1 - u_2, u_1 - u_3, u_1 - u_4, u_2 - u_3, u_2 - u_4, u_3 - u_4$ planes are

$$x = A_2 y_1^{\frac{-\rho_1}{\rho_2}} - B_2 y_1 - C_2 y_1^{\frac{-a_3}{\rho_2}}, \quad x = A_2 y_2^{\frac{\rho_1}{a_3}} - B_2 y_2^{\frac{-\rho_2}{a_3}} - C_2 y_2 \quad \dots (4.3.12)$$

$$x = A_2 y_3^{\frac{\rho_1}{a_4}} - B_2 y_3^{\frac{-\rho_2}{a_4}} - C_2 y_3^{\frac{a_3}{a_4}} \quad \dots (4.3.13)$$

$$\left(\frac{u_2}{u_{20}}\right)^{-a_3} = \left(\frac{u_3}{u_{30}}\right)^{\rho_2}, \quad \left(\frac{u_2}{u_{20}}\right)^{-a_4} = \left(\frac{u_4}{u_{40}}\right)^{\rho_2}, \quad \left(\frac{u_3}{u_{30}}\right)^{a_4} = \left(\frac{u_4}{u_{40}}\right)^{a_3} \quad \dots (4.3.14)$$

respectively

$$\text{where } A_2 = \frac{u_{10} + U + V}{u_{10}}, B_2 = \frac{U}{u_{10}}, C_2 = \frac{V}{u_{10}} \quad \dots (4.3.15)$$

REFERENCES:

- 1) **Archana Reddy .R.**, "On the stability of some Mathematical Models in BioSciences – Interacting Species", Ph.D.Thesis, J.N.T.U., (2009).
- 2) **Bhaskara Rama Sharma .B.**, "Some Mathematical Models in Competitive Eco-Systems", Ph.D. Thesis, Dravidian University, (2009).
- 3) **Hari Prasad.B and Pattabhi Ramacharyulu, N.Ch.**, "On the Stability of a Four Species : A Prey-Predator-Host-Commensal-Syn Eco-System-I" (Fully washed out state) International eJournal of Mathematics and Engineering. 11, (2010), 122-132.

- 4) **Hari Prasad.B and Pattabhi Ramacharyulu, N.Ch.**, “On the Stability of a Four Species : A Prey-Predator-Host-Commensal-Syn Eco-System-II” (Prey and Predator washed out states) International eJournal of Mathematics and Engineering. 5, (2010), 66-75.
- 5) **Hari Prasad.B and Pattabhi Ramacharyulu, N.Ch.**, “On the Stability of a Four Species : A Prey-Predator-Host-Commensal-Syn Eco-System-III” (Co-existent state) International eJournal of Mathematics and Engineering. 16, (2010), 163-173.
- 6) **Kapur J. N.**, Mathematical Modelling in Biology and Medicine, Affiliated East West, (1985).
- 7) **Kapur J. N.**, Mathematical Modelling, Wiley Easter, (1985).
- 8) **Kushing J.M.**, Integro-Differential Equations and Delay Models in Population Dynamics, Lecture Notes in Bio-Mathematics, Springer Verlag, 20, (1977).
- 9) **Lakshmi Narayan K.**, A. Mathematical Study of a Prey-Predator Ecological Model with a Partial Cover for the Prey and Alternate Food for the Predator, Ph. D. Thesis, J. N. T. U., (2005).
- 10) **Lakshmi Narayan K. & Pattabhiramacharyulu. N. Ch.**, “A Prey-Predator Model with Cover for Prey and Alternate Food for the Predator and Time Delay,” International Journal of Scientific Computing. 1, (2007), 7—14.
- 11) **Lotka A. J.**, Elements of Physical Biology, Williams & Wilking, Baltimore, (1925).
- 12) **Meyer W.J.**, Concepts of Mathematical Modeling Mc. Grawhill, (1985).
- 13) **Phani Kumar, Seshagiri Rao and Pattabhi Ramacharyulu N. Ch.**, “On the Stability of a Host – A Flourishing Commensal Species pair with Limited Resources”, International Journal of Logic Based Intelligent Systems, Vol.3, No.1, January-June 2009.
- 14) **Srinivas N. C.**, “Some Mathematical Aspects of Modeling in Bio-medical Sciences” Ph.D Thesis, Kakatiya University, (1991).