THE STABILITY OF A FOUR SPECIES: A PREY-PREDATOR-
HOST-COMMENSAL-SYN ECO-SYSTEM-IV
(Both the Hosts are washed out states)

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ABSTRACT: This paper deals with an investigation on a Four Species Syn-Ecological System
(Both the Hosts are washed out states). The System comprises of a Prey (S₁), a Predator (S₂) that
survives upon S₁, two Hosts S₃ and S₄ for which S₁, S₂ are commensal respectively i.e., S₃ and S₄
benefit S₁ and S₂ respectively, without getting affected either positively or adversely. Further S₃
and S₄ are neutral. The model equations of the system constitute a set of four first order non-
linear ordinary differential coupled equations. In all, there are sixteen equilibrium points.
Criteria for the asymptotic stability of three of the sixteen equilibrium points: Both the Hosts are
washed out states only are established in this paper. The system would be stable if all the
characteristic roots are negative, in case they are real, and have negative real parts, in case they
are complex. The linearized equations for the perturbations over the equilibrium points are
analyzed to establish the criteria for stability and the trajectories illustrated.

1. INTRODUCTION:
Mathematical modeling of Eco-System was initiated in 1925 by Lotka [10] and in 1931 by
Volterra[14]. The general concepts of modeling have been presented in the treatises of
Meyer[11], Kushing[7], Kapur J.N. [5,6] and several others. The ecological interactions can be
broadly classified as Prey-Predator, Commensalism, Competition, Neutralism, Mutualism and so
on. N.C. Srinivas [13] studied competitive eco-systems of two species and three species with
limited and unlimited resources. Later Lakshminarayana [8], Lakshminarayana and Pattabhi
Ramacharyulu [9] studied Prey-Predator ecological models with partial cover for the Prey and
alternate food for the Predator. Recently, Archana Reddy [1] and Bhaskara Rama Sharma [2]
investigated diverse problems related to two species competitive systems with time delay,
employing analytical and numerical techniques. Further Phani Kumar, Seshagiri Rao and
Pattabhi Ramacharyulu [12] studied the stability of a Host-A flourishing commensal species pair
with limited resources. The present authors Hari Prasad, B and Pattabhi Ramacharyulu, N. Ch studied the stability of the fully washed out state [3] and co-existent state [4]. Continuation of this criteria for the stability of the Host (S₃) of S₁ and Host (S₄) of S₂ only washed out states of the system are presented in this paper.

2. BASIC EQUATIONS OF THE MODEL:

Notation Adopted:

- S₁: Prey for S₂ and commensal for S₃.
- S₂: Predator surviving upon S₁ and commensal for S₄.
- S₃: Host for the commensal – Prey (S₁).
- S₄: Host of the commensal – Predator (S₂)
- N₁(t): The Population of the Prey (S₁)
- N₂(t): The Population of the Predator (S₂)
- N₃(t): The Population of the Host (S₃) of the Prey (S₁)
- N₄(t): The Population of the Host (S₄) of the Predator (S₂)
- t: Time instant

a₁, a₂, a₃, a₄: Natural growth rates of S₁, S₂, S₃, S₄

a₁₁, a₂₂, a₃₃, a₄₄: Self inhibition coefficients of S₁, S₂, S₃, S₄

a₁₂, a₂₁: Interaction (Prey-Predator) coefficients of S₁ due to S₂ and S₂ due to S₁

a₁₃: Coefficient for commensal for S₁ due to the Host S₃

a₂₄: Coefficient for commensal for S₂ due to the Host S₄

a₁, a₂, a₃, a₄ : Carrying capacities of S₁, S₂, S₃, S₄

Further the variables N₁, N₂, N₃, N₄ are non-negative and the model parameters a₁, a₂, a₃, a₄; a₁₁, a₂₂, a₃₃, a₄₄; a₁₂, a₂₁, a₁₃, a₂₄ are assumed to be non-negative constants.

The model equations for the growth rates of S₁, S₂, S₃, S₄ are

\[
\frac{dN₁}{dt} = a₁N₁ - a₁₁N₁² - a₁₂N₁N₂ + a₁₃N₁N₃ \quad \ldots \quad (2.1)
\]

\[
\frac{dN₂}{dt} = a₂N₂ - a₂₂N₂² - a₂₁N₁N₂ + a₂₄N₂N₄ \quad \ldots \quad (2.2)
\]

\[
\frac{dN₃}{dt} = a₃N₃ - a₃₃N₃² \quad \ldots \quad (2.3)
\]

\[
\frac{dN₄}{dt} = a₄N₄ - a₄₄N₄² \quad \ldots \quad (2.4)
\]
3 Equilibrium States:
The system under investigation has sixteen equilibrium states defined by
\[ \frac{dN_i}{dt} = 0, \ i = 1, 2, 3, 4 \]
............. (3.1)
are given in the following table.

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Equilibrium States</th>
<th>Equilibrium Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Fully Washed out state</td>
<td>( N_1 = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>Only the Host (S₄) of S₂ survives</td>
<td>( N_1 = 0, \overline{N_2} = 0, \overline{N_3} = 0, N_4 = \frac{a_4}{a_{44}} )</td>
</tr>
<tr>
<td>3</td>
<td>Only the Host (S₃) of S₁ survives</td>
<td>( N_1 = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0 )</td>
</tr>
<tr>
<td>4</td>
<td>Only the Predator S₂ survives</td>
<td>( N_1 = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = 0, \overline{N_4} = 0 )</td>
</tr>
<tr>
<td>5</td>
<td>Only the Prey S₁ survives</td>
<td>( N_1 = \frac{a_1}{a_{11}}, N_2 = 0, \overline{N_3} = 0, \overline{N_4} = 0 )</td>
</tr>
<tr>
<td>6</td>
<td>Prey (S₁) and Predator (S₂) washed out</td>
<td>( N_1 = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, N_4 = \frac{a_4}{a_{44}} )</td>
</tr>
<tr>
<td>7</td>
<td>Prey (S₁) and Host (S₃) of S₁ washed out</td>
<td>( N_1 = 0, \overline{N_2} = \frac{a_2a_{44} + a_4a_{24}}{a_{22}a_{44}}, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}} )</td>
</tr>
<tr>
<td>8</td>
<td>Prey (S₁) and Host (S₄) of S₂ washed out</td>
<td>( N_1 = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0 )</td>
</tr>
<tr>
<td>9</td>
<td>Predator (S₂) and Host (S₃) of S₁ washed out</td>
<td>( N_1 = \frac{a_4}{a_{44}}, N_2 = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}} )</td>
</tr>
<tr>
<td>10</td>
<td>Predator (S₂) and Host (S₄) of S₂ washed out</td>
<td>( N_1 = \frac{a_1a_{33} + a_4a_{13}}{a_{11}a_{33}}, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0 )</td>
</tr>
<tr>
<td>11</td>
<td>Prey (S₁) and Predator (S₂) survives</td>
<td>( N_1 = \frac{a_1a_{22} - a_2a_{12}}{a_{11}a_{22} + a_{12}a_{21}}, \overline{N_2} = \frac{a_1a_{21} + a_2a_{11}}{a_{11}a_{22} + a_{12}a_{21}}, \overline{N_3} = 0, \overline{N_4} = 0 )</td>
</tr>
<tr>
<td>12</td>
<td>Only the Prey (S₁) washed out</td>
<td>( N_1 = 0, \overline{N_2} = \frac{a_2a_{44} + a_4a_{24}}{a_{22}a_{44}}, \overline{N_3} = \frac{a_4}{a_{33}}, \overline{N_4} = \frac{a_4}{a_{44}} )</td>
</tr>
<tr>
<td>13</td>
<td>Only the predator (S₂) washed out</td>
<td>( N_1 = \frac{a_1a_{23} + a_4a_{13}}{a_{11}a_{23}}, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = \frac{a_4}{a_{44}} )</td>
</tr>
</tbody>
</table>
The present paper deals with the host (S₃) of S₁ and host (S₄) of S₂ washed out states only. The stability of the other equilibrium states will be presented in the forthcoming communications.

4. Stability of the Host (S₃) of S₁ and Host (S₄) of S₂ washed out equilibrium states (Sl. Nos 4, 5, 11 in the above table)

4.1 Equilibrium point \( \hat{N}_1 = 0, \hat{N}_2 = \frac{a_3}{a_{22}}, \hat{N}_3 = 0, \hat{N}_4 = 0 \):

Let us consider small deviations \( u_1(t), u_2(t), u_3(t), u_4(t) \) from the steady state i.e., \( N_i(t) = \hat{N}_i + u_i(t), i = 1, 2, 3, 4 \) \( \ldots \ldots \ldots \ldots \ldots \ldots (4.1.1) \)

where \( u_i(t) \) is a small perturbation in the species \( S_i \).

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of \( u_1, u_2, u_3, u_4 \)
we get
\[ \frac{du_1}{dt} = r_1 u_1 \quad \text{(4.1.2)}, \quad \frac{du_2}{dt} = \frac{a_2 a_{21}}{a_{22}} u_1 - a_2 u_2 + \frac{a_2 a_{24}}{a_{22}} u_4 \quad \text{.........(4.1.3)} \]
\[ \frac{du_3}{dt} = a_3 u_3 \quad \text{.........(4.1.4)}, \quad \frac{du_4}{dt} = a_4 u_4 \quad \text{.........(4.1.5)} \]
where \( r_i = a_1 - \frac{a_4 a_{12}}{a_{22}} \quad \text{.........(4.1.6)} \)

The characteristic equation of which is
\[ (\lambda + r_i)(\lambda + a_2)(\lambda - a_3)(\lambda - a_4) = 0 \quad \text{.........(4.1.7)} \]

**Case (A):** When \( r_i > 0 \) (i.e., when \( \frac{a_1}{a_2} > \frac{a_{12}}{a_{22}} \))

The roots \( r_i, a_3, a_4 \) are positive and \( -a_2 \) is negative
Hence the steady state is **unstable**.

The solutions of the equations (4.1.1) (4.1.2), (4.1.3), (4.1.4) are
\[ u_1 = u_{10} e^{r_1 t} \quad \text{......... (4.1.8)} \]
\[ u_2 = (u_{20} - m - n) e^{-a_2 t} + n e^{a_2 t} + m e^{a_1 t} \quad \text{......... (4.1.9)} \]
\[ u_3 = u_{30} e^{a_3 t} \quad \text{.........(4.1.10)}, \quad u_4 = u_{40} e^{a_4 t} \quad \text{......... (4.1.11)} \]
Where \( m = \frac{a_2 a_{21} u_{10}}{a_{22}(r_i + a_2)}, \quad n = \frac{a_2 a_{23} u_{40}}{a_{22}(a_4 + a_2)} \quad \text{......... (4.1.12)} \)

There would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates \( a_1, a_2, a_3, a_4 \) and the initial values of the perturbations \( u_{10}(t), \ u_{20}(t), \ u_{30}(t), \ u_{40}(t) \) of the species \( S_1, S_2, S_3, S_4 \). Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations.

**Case (i):** If \( u_{10} < u_{20} < u_{30} < u_{40} \) and \( r_i < a_2 < a_4 < a_3 \)

In this case the Prey \( (S_1) \) has the least natural birth rate. Initially the Host \( (S_4) \) of \( S_2 \) dominates over the Host \( (S_3) \) of \( S_1 \) till the time instant \( t_{34}^* \) and there after the dominance is reversed. The time \( t_{34}^* \) may be called the dominance time of the Host \( (S_4) \) of \( S_2 \) over the host \( (S_3) \) of \( S_1 \).

Here \( t_{34}^* = \frac{1}{a_3 - a_4} \log \left( \frac{u_{40}}{u_{30}} \right) \quad \text{.........(4.1.13)} \)
**Case (ii):** If \( u_{20} < u_{40} < u_{30} < u_{10} \) and \( a_2 < a_3 < r_1 < a_4 \).

In this case the Predator (\( S_2 \)) has the least natural birth rate. Initially the Prey (\( S_1 \)) dominates over the Host (\( S_4 \)) of \( S_2 \) till the time instant \( t_{41}^* \) and there after the dominance is reversed. Also the Host (\( S_3 \)) of \( S_1 \) dominates over the Host (\( S_4 \)) of \( S_2 \) till the time instant and the dominance gets reversed there after.

Here \( t_{41}^* = \frac{1}{r_1 - a_4} \log \left( \frac{u_{40}}{u_{10}} \right) \) ....(4.1.14)

**Case (iii):** If \( u_{30} < u_{10} < u_{40} < u_{20} \) and \( a_3 < r_1 < a_2 < a_4 \)

In this case the Host (\( S_4 \)) of \( S_2 \) has the least natural birth rate. Initially it is dominated over by the Prey (\( S_1 \)), Host (\( S_3 \)) of \( S_1 \) till the time instant \( t_{41}^* \), \( t_{34}^* \) respectively and there after the dominance is reversed. Also the Prey (\( S_1 \)) dominates over its Host (\( S_3 \)) till the time instant \( t_{31}^* \) and the dominance gets reversed there after.

Here \( t_{31}^* = \frac{1}{r_1 - a_3} \log \left( \frac{u_{30}}{u_{10}} \right) \) ....(4.1.15)

**Case (iv):** If \( u_{40} < u_{20} < u_{10} < u_{30} \) and \( r_1 < a_3 < a_4 < a_2 \)

In this case the Prey (\( S_1 \)) has the least natural birth rate. Initially it is dominated over by the Predator (\( S_2 \)), Host (\( S_3 \)) of \( S_2 \) till the time instant \( t_{21}^* \), \( t_{41}^* \) respectively and there after the dominance is reversed. Also the Host (\( S_3 \)) of \( S_1 \) dominates over the Prey (\( S_2 \)), Host (\( S_4 \)) of \( S_2 \) till the time instant \( t_{23}^* \), \( t_{43}^* \) respectively and the dominance is gets reversed there after.

**4.1.A. Trajectories of perturbations:**

The trajectories in the \( u_1 - u_3 \) plane given by

\[
\left( \frac{u_1}{u_{10}} \right)^{a_3} = \left( \frac{u_1}{u_{30}} \right)^{a_3} \quad \ldots(4.1.16)
\]

and are showing in fig. 1
Also the trajectories in the $u_1 - u_4$, $u_3 - u_4$ planes are
\[
\left( \frac{u_1}{u_{10}} \right)^{a_1} = \left( \frac{u_4}{u_{40}} \right)^{a_4}, \quad \left( \frac{u_3}{u_{30}} \right)^{a_3} = \left( \frac{u_4}{u_{40}} \right)^{a_4}
\]
respectively
\[
\ldots \ldots (4.1.17)
\]
and the trajectories in the $u_1 - u_2$, $u_2 - u_3$, $u_2 - u_4$ planes are
\[
y = Px^{\frac{-a_1}{\eta}} + Qx + Rx^{\frac{a_4}{\eta}}, \quad y = Px^{\frac{-a_1}{\eta}} + Qx^{\frac{a_4}{\eta}} + Rx^{\frac{a_4}{\eta}}
\]
\[
\ldots \ldots (4.1.18)
\]
\[
y = Px_{2}^{\frac{-a_2}{\eta}} + Qx_{2}^{\frac{a_4}{\eta}} + Rx_{2}
\]
respectively
\[
\ldots \ldots (4.1.19)
\]
Where $x = \frac{u_1}{u_{10}}$, $y = \frac{u_2}{u_{20}}$, $x_1 = \frac{u_3}{u_{30}}$, $x_2 = \frac{u_4}{u_{40}}$
\[
\ldots \ldots (4.1.20)
\]
and $P = \frac{u_{20} - m - n}{u_{20}}$, $Q = \frac{m}{u_{20}}$, $R = \frac{n}{u_{20}}$
\[
\ldots \ldots (4.1.21)
\]

**Case (B):** When $r_i < 0$ (ie, when $\frac{a_i}{a_2} < \frac{a_{12}}{a_{22}}$)

The roots $a_1$, $a_4$ are positive and $r_i$, $-a_i$ are negative.

Hence the study state in **unstable**.

In this case the solutions are same as in case (A)

**Case (i):** If $u_{10} < u_{20} < u_{30} < u_{40}$ and $a_1 < r_i < a_2 < a_4$

In this case the Prey ($S_1$) has the least natural birth rate. Initially the Host ($S_3$) of $S_1$ dominates over the Predator ($S_2$) till the time instant $t_{23}$ and there after the dominance is reversed.

**Case (ii):** If $u_{20} < u_{10} < u_{30} < u_{40}$ and $r_i < a_4 < a_3 < a_2$

In this case the Prey ($S_1$) has the least natural birth rate. Initially the Prey ($S_1$), Host ($S_4$) of $S_2$, Host ($S_3$) of $S_1$ dominates over the Predator ($S_2$) till the time instant $t_{21}^{*}$, $t_{24}^{*}$, $t_{23}^{*}$, respectively and there after the dominance is reversed.
Case (iii): If $u_{30} < u_{10} < u_{20} < u_{40}$ and $a_4 < r_1 < a_2 < a_3$

In this case the Prey ($S_1$) has the least natural birth rate. Initially the Prey ($S_1$), Predator ($S_2$), Host ($S_4$) of $S_2$ dominates over the Host ($S_3$) of $S_1$ till the time instant $t^*_{31}$, $t^*_{32}$, $t^*_{34}$ respectively and thereafter the dominance is reversed. Also the Host ($S_4$) of $S_2$ dominates over the Predator $S_2$ till the time instant $t^*_{34}$ and the dominance gets reversed there after.

Case (iv): If $u_{40} < u_{20} < u_{10} < u_{30}$ and $a_4 < r_1 < a_3 < a_2$

In this case the Prey ($S_1$) has the least natural birth rate. Initially it is dominated over by the Predator ($S_2$), Host ($S_4$) of $S_2$ till the time instant $t^*_{21}$, $t^*_{41}$ respectively and thereafter the dominance is reversed. Also the Host ($S_4$) of $S_1$ dominates over the Predator ($S_2$) till the time instant $t^*_{23}$ and the dominance gets reversed there after.

4.1.B: Trajectories of perturbations:

The trajectories in the $u_1 - u_3$ plane given by

$$\left( \frac{u_1}{u_{10}} \right)^{a_3} = \left( \frac{u_3}{u_{30}} \right)^{r_1} \quad \text{.... (4.1.22)}$$

and are shown in Fig. 2

Also the trajectories in the $u_1 - u_4$, $u_3 - u_4$ planes are

$$\left( \frac{u_1}{u_{10}} \right)^{a_4} = \left( \frac{u_4}{u_{40}} \right)^{r_1}, \quad \left( \frac{u_3}{u_{30}} \right)^{a_4} = \left( \frac{u_4}{u_{40}} \right)^{a_4} \quad \text{respectively} \quad \text{....(4.1.23)}$$

and the trajectories in the $u_1 - u_2$, $u_2 - u_3$, $u_2 - u_4$ planes are

$$y = Px_1^{a_1} + Qx_1^{a_1} + Rx_1^{a_1}, \quad y = Px_1^{a_1} + Qx_1^{a_1} + Rx_1^{a_1} \quad \text{....... (4.1.24)}$$

$$y = Px_2^{a_2} + Qx_2^{a_2} + Rx_2^{a_2} \quad \text{respectively} \quad \text{.... (4.1.25)}$$
4.2. Equilibrium point $\bar{N}_1 = \frac{a_1}{a_{11}}, \bar{N}_2 = 0, \bar{N}_3 = 0, \bar{N}_4 = 0$:

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of $u_1, u_2, u_3, u_4$ we get

\[
\frac{du_1}{dt} = -a_1u_1 - \frac{a_1a_{12}}{a_{11}}u_2 + \frac{a_1a_{11}}{a_{11}}u_3 \quad \ldots (4.2.1)
\]

\[
\frac{du_2}{dt} = l_2u_2 \quad \ldots (4.2.2), \quad \frac{du_3}{dt} = a_3u_3 \quad \ldots (4.2.3)
\]

\[
\frac{du_4}{dt} = a_4u_4 \quad \ldots (4.2.4)
\]

Where $l_2 = \left( a_2 + \frac{a_1a_{21}}{a_{11}} \right) > 0 \quad \ldots (4.2.5)$

The characteristic equation of which is

\[
(\lambda + a_1)(\lambda - l_2)(\lambda - a_3)(\lambda - a_4) = 0 \quad \ldots (4.2.6)
\]

The roots $l_2, a_3, a_4$ are positive and $-a_1$ is negative.

Hence the steady state is unstable.

The solutions of the equations (4.2.1), (4.2.2), (4.2.3) (4.2.4) are

\[
u_1 = (u_{10} + m_1 - n_1)e^{-a_1t} - m_1e^{-l_2t} + n_1e^{a_4t} \quad \ldots (4.2.7)
\]

\[
u_2 = u_{20}e^{l_1t} \quad \ldots (4.2.8), \quad u_3 = u_{30}e^{a_3t} \quad \ldots (4.2.9)
\]

\[
u_4 = u_{40}e^{a_4t} \quad \ldots (4.2.10)
\]

Here $m_1 = \frac{a_1a_{12}u_{20}}{a_{11}(l_2 + a_1)} > 0, \quad n_1 = \frac{a_1a_{13}u_{30}}{a_{11}(a_3 + a_1)} > 0 \quad \ldots (4.2.11)$

and $l_2 = \frac{a_2 + a_{21}}{a_1 + a_{11}} > 0 \quad \ldots (4.2.12)$

Case (i): If $u_{10} < u_{20} < u_{30} < u_{40}$ and $a_1 < l_2 < a_3 < a_4$

In this case the Prey ($S_1$) has the least natural birth rate and the Host ($S_4$) of $S_2$ dominates the prey ($S_1$) Predator ($S_2$), Host ($S_3$) of $S_1$ in natural growth rate as well as in its population strength.
Case (ii): If \( u_{20} < u_{30} < u_{40} < u_{10} \) and \( a_3 < a_1 < l_2 < a_4 \)

In this case the Host (\( S_1 \)) of \( S_2 \) has the least natural birth rate. Initially it is dominated over by the Predator (\( S_2 \)) till the time instant \( t_{23}^* \) and thereafter the dominance is reversed. Also the Prey (\( S_1 \)) dominates over the Predator (\( S_2 \)), Host (\( S_4 \)) of \( S_2 \) till the time instant \( t_{21}^* , t_{41}^* \) respectively and the dominance gets reversed there after.

Case (iii): If \( u_{10} < u_{20} < u_{40} < u_{30} \) and \( a_4 < a_3 < l_2 < a_1 \)

In this case the Host (\( S_4 \)) of \( S_2 \) has the least natural birth rate. Initially it is dominated over by the Prey (\( S_1 \)), Predator (\( S_2 \)), Host (\( S_3 \)) of \( S_1 \) till the time instant \( t_{14}^* , t_{34}^* , t_{34}^* \) respectively and there after the dominance is reversed. Also the Predator (\( S_2 \)), Prey (\( S_1 \)) dominates over the Host (\( S_4 \)) of \( S_2 \) till the time instant \( t_{42}^* , t_{41}^* \) respectively and the dominance gets reversed there after.

Case (iv): If \( u_{40} < u_{10} < u_{20} < u_{30} \) and \( a_3 < a_1 < l_2 < a_4 \)

In this case the Host (\( S_3 \)) of \( S_1 \) has the least natural birth rate. Initially it is dominated over by the Host (\( S_4 \)) of \( S_2 \), Predator (\( S_2 \)), Prey (\( S_1 \)) till the time instant \( t_{43}^* , t_{23}^* , t_{13}^* \) respectively and there after the dominance is reversed. Also the Predator (\( S_2 \)), Prey (\( S_1 \)) dominates over the Host (\( S_4 \)) of \( S_2 \) till the time instant \( t_{42}^* , t_{41}^* \) respectively and the dominance gets reversed there after.

Trajectories of perturbations:

The trajectories in the \( u_2 - u_3 \) plane given by

\[
\left( \frac{u_2}{u_{20}} \right)^{a_2} = \left( \frac{u_3}{u_{40}} \right)^{l_2} \quad \text{……(4.2.13)}
\]

and are shown in fig. 3

Also the trajectories in the \( u_2 - u_4 \), \( u_3 - u_4 \), \( u_1 - u_2 \), \( u_1 - u_3 \), \( u_1 - u_4 \) planes are

\[
\left( \frac{u_2}{u_{20}} \right)^{a_4} = \left( \frac{u_4}{u_{40}} \right)^{l_2}, \quad \left( \frac{u_3}{u_{30}} \right)^{a_4} = \left( \frac{u_4}{u_{40}} \right)^{a_3}
\]

\[\text{……(4.2.14)}\]
\[ x = P_{1} x_{1} - Q_{1} y + R_{1} y^{2}, \quad x = P_{2} x_{2}^{2} - Q_{1} x_{1}^{2} + R_{1} x_{1}^{2} \]  
\[ \ldots (4.2.15) \]

\[ x = P_{2} x_{2}^{2} - Q_{1} x_{1}^{2} + R_{1} x_{2}^{2} \quad \text{respectively} \quad \ldots (4.2.16) \]

where

\[ P_{1} = \frac{u_{10} + m_{1} - n_{1}}{u_{10}}, \quad Q_{1} = \frac{m_{1}}{u_{10}}, \quad R_{1} = \frac{u_{1}}{u_{10}} \quad \ldots (4.2.17) \]

4.3. Equilibrium point: 

\[ \bar{N}_{1} = \frac{a_{i} a_{22} - a_{2} a_{12}}{a_{11} a_{22} + a_{12} a_{21}}, \quad \bar{N}_{2} = \frac{a_{i} a_{21} + a_{2} a_{11}}{a_{11} a_{22} + a_{12} a_{21}}, \quad \bar{N}_{3} = 0, \quad \bar{N}_{4} = 0: \]

Substituting (4.1.4) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of \( u_{1}, u_{2}, u_{3}, u_{4} \) we get

\[ \frac{du_{1}}{dt} = \mu_{1} u_{1} - \frac{a_{12} u_{1}}{\mu} u_{2} + \frac{a_{13} u_{1}}{\mu} u_{3} \quad \ldots (4.3.1) \]

\[ \frac{du_{2}}{dt} = \frac{a_{12} \gamma_{1}}{\mu} u_{1} + \gamma_{2} u_{2} + \frac{a_{24} \gamma_{1}}{\mu} u_{4} \quad \ldots (4.3.2) \]

\[ \frac{du_{3}}{dt} = a_{3} u_{3} \quad \ldots (4.3.3), \quad \frac{du_{4}}{dt} = a_{4} u_{4} \quad \ldots (4.3.4) \]

where \( \mu_{1} = \mu_{1} - \frac{2 a_{11} u_{1} - a_{12} \gamma_{1}}{\mu} \quad \ldots (4.3.5) \)

\[ \mu_{2} = a_{1} a_{22} - a_{2} a_{12} ; \quad \gamma_{1} = a_{1} a_{21} + a_{2} a_{11} > 0 \quad \ldots (4.3.6) \]

\[ \gamma_{2} = a_{2} - \frac{2 a_{21} \gamma_{1}}{\mu} + \frac{a_{1} a_{11}}{\mu} \quad \mu = a_{11} a_{22} + a_{12} a_{21} > 0 \quad \ldots (4.3.7) \]

The characteristic equation of which is

\[ \left[ \lambda^{2} - (\mu_{1} + \gamma_{2}) \lambda - \frac{a_{12} a_{11} \mu_{1}}{\mu} \right] \left[ (\mu_{1} - a_{4}) (\mu_{1} - a_{4}) = 0 \quad \ldots (4.3.8) \right] \]

Two of the four roots \( a_{3} \) and \( a_{4} \) are positive.

Hence the steady state is \textbf{unstable}.

Let \( \lambda_{1}, \lambda_{2} \) be the zeros of the quadratic polynomial on the L.H.S. of the equation (4.3.8)

Case (A): When the roots \( \lambda_{1} \) and \( \lambda_{2} \) have opposite signs
The solutions of the equations (4.3.1), (4.3.2), (4.3.3), (4.3.4) are

\[
    u_1 = \left[ \frac{a_{12} \mu_2 (\beta - u_{20}) - (\alpha - u_{10}) (\mu_1 - \lambda_2) \mu}{\mu (\lambda_1 - \lambda_2)} \right] e^{\lambda_1 t} \\
    + \left[ \frac{a_{12} \mu_2 (\beta - u_{20}) - (\alpha - u_{10}) (\mu_1 - \lambda_1) \mu}{a_{12} \mu_2 (\lambda_1 - \lambda_2)} \right] e^{\lambda_2 t} + A e^{\alpha t} - B e^{\beta t} \quad \ldots \ldots (4.3.9)
\]

\[
    u_2 = \left[ \frac{a_{12} \mu_2 (\beta - u_{20}) - (\alpha - u_{10}) (\mu_1 - \lambda_2) \mu}{a_{12} \mu_2 (\lambda_1 - \lambda_2)} \right] e^{\lambda_1 t} \\
    + \left[ \frac{a_{12} \mu_2 (\beta - u_{20}) - (\alpha - u_{10}) (\mu_1 - \lambda_1) \mu}{a_{12} \mu_2 (\lambda_1 - \lambda_2)} \right] e^{\lambda_2 t} \\
    + \left[ \frac{\mu A (\mu_1 - a_3) + a_{13} u_{30}}{a_{12} \mu_2} \right] e^{\alpha t} + \frac{\mu B}{a_{12} \mu_2} (a_4 - \mu_1) e^{\beta t} \quad \ldots \ldots (4.3.10)
\]

\[
    u_3 = u_{30} e^{\alpha t} \quad \ldots \ldots (4.3.11), \quad u_4 = u_{40} e^{\beta t} \quad \ldots \ldots (4.3.12)
\]

where

\[
    A = \frac{\bar{A}}{a_3^{\gamma_2} (\mu_1 + \gamma_2) a_3 + C}, \quad B = \frac{\bar{B}}{a_3^{\gamma_2} (\mu_1 + \gamma_2) a_3 + C} \quad \ldots \ldots (4.3.13)
\]

\[
    \bar{A} = \frac{a_{13} \mu_2 u_{30}}{\mu} (a_3 - \gamma_2), \quad \bar{B} = \frac{a_{12} a_{24} \gamma_2 \mu_2}{\mu} u_{40} \quad \ldots \ldots (4.3.14)
\]

\[
    C = m_1 \gamma_2 + \frac{a_{12} a_{21} \gamma_2 \mu_2}{\mu^2}, \quad \alpha = A - B \quad \ldots \ldots (4.3.15)
\]

\[
    \beta = \frac{a_{13} u_{30}}{a_{12}} + \frac{\mu}{a_{12} \mu_2} [A (\mu_1 - a_3) + B (a_4 - \mu_1)] \quad \ldots \ldots (4.3.16)
\]

**Case (i):** If \( u_{10} < u_{20} < u_{30} < u_{40} \) and \( a_4 < a_3 < a_1 < a_2 \)

In this case the Prey (\( S_1 \)) has the least natural birth rate. Initially the Host (\( S_3 \)) of \( S_1 \) dominates over the Predator (\( S_2 \)) till the time instant \( t_{23}^* \) and thereafter the dominance is reversed.
Case (ii): If $u_{20} < u_{30} < u_{40} < u_{10}$ and $u_4 < u_3 < u_1 < u_2$

In this case the Host ($S_1$) of $S_2$ has the least natural birth rate. Initially it is dominated over by the Predator ($S_2$), Host ($S_3$) of $S_1$ till the time instant $t_{24}^*$, $t_{34}^*$ respectively and thereafter the dominance is reversed. Also the Host ($S_3$) of $S_1$ dominates over the Predator ($S_2$) till the time instant $t_{23}^*$ and thereafter the dominance is reversed. Similarly the Prey ($S_1$) dominates over the Predator ($S_2$) till the time instant $t_{21}^*$ and the dominance gets reversed there after.

Case (iii): If $u_{30} < u_{20} < u_{10} < u_{40}$ and $a_2 < a_4 < a_3 < a_1$

In this case the Predator ($S_2$) has the least natural birth rate. Initially it is dominated over by the Host ($S_3$) of $S_1$ till the time instant $t_{32}^*$ and thereafter the dominance is reversed. Also the Host ($S_3$) of $S_2$ dominates over the Prey ($S_1$) till the time instant $t_{i4}^*$ and dominance gets reversed thereafter.

Case (iv): If $u_{40} < u_{30} < u_{10} < u_{20}$ and $a_4 < a_3 < a_2 < a_1$

In this case the Host ($S_3$) has the least natural birth rate. Initially it is dominated over by the Host ($S_4$) of $S_2$ till the time instant $t_{43}^*$ and thereafter the dominance is reversed. Also the Prey ($S_1$), Predator ($S_2$) dominates over the Host ($S_4$) of $S_2$ till the time instant $t_{s1}^*$, $t_{s2}^*$ respectively and the dominance gets reversed thereafter.

Case (B): When the roots $\lambda_1$ and $\lambda_2$ have same signs

In this case the solutions are same as in case (A).

Case (i): If $u_{10} < u_{20} < u_{30} < u_{40}$ and $a_2 < a_1 < a_3 < a_4$

In this case the Predator ($S_2$) has the least natural birth rate. Initially it is dominated over by the Prey ($S_1$) till the time instant $t_{12}^*$ and thereafter the dominance is reversed.
**Case (ii):** If $u_{20} < u_{30} < u_{40}$ and $a_3 < a_1 < a_2 < a_4$

In this case the Host ($S_3$) of $S_1$ has the least natural birth rate. Initially it is dominated over by the Predator ($S_2$) till the time instant $t_{23}$ and thereafter the dominance is reversed. Also the Prey ($S_1$) dominates over the Predator ($S_2$) till the time instant $t_{21}$ and the dominance gets reversed thereafter.

**Case (iii):** If $u_{30} < u_{40} < u_{20} < u_{10}$ and $a_1 < a_4 < a_3 < a_2$

In this case the Prey ($S_1$) has least natural birth rate. Initially it is dominated over by the Predator ($S_2$), Host ($S_3$) of $S_1$, Host ($S_4$) of $S_2$ till the time instant $t_{21}$, $t_{31}$, $t_{41}$ respectively and thereafter the dominance is reversed. Also the Host ($S_i$) of $S_2$ dominates over the Host ($S_3$) of $S_1$ till the time instant $t_{34}$ and the dominance gets reversed thereafter.

**Case (iv):** If $u_{40} < u_{20} < u_{30} < u_{40}$ and $a_1 < a_4 < a_3 < a_2$

In this case the Prey ($S_1$) has the least natural birth rate. Initially it is dominated over by the Predator ($S_2$), Host ($S_4$) of $S_2$, Host ($S_3$) of $S_1$ till the time instant $t_{21}$, $t_{41}$, $t_{31}$ respectively and thereafter the dominance is reversed. Also the Host ($S_3$) of $S_1$ dominates over the Predator ($S_2$), Host ($S_4$) of $S_2$ till the time instant $t_{23}$, $t_{43}$ respectively and the dominance gets reversed thereafter.

**Trajectories of Perturbations:**

The trajectories in the $u_3 - u_4$ plane given by

$$\left(\frac{u_3}{u_{30}}\right)^{a_4} = \left(\frac{a_4}{a_{40}}\right)^{a_3}$$

... $(4.3.17)$

and are shown in Fig. 4
And the trajectories in the \( u_1 - u_3, u_1 - u_4, u_2 - u_1, u_2 - u_4 \) planes are

\[
x = A_1 x_1^{a_1} + B_1 x_1^{a_2} + C_1 x_1^{a_3} - D_1 x_1^{a_4} \quad \ldots \ldots \ldots \ldots (4.3.18)
\]

\[
x = A_1 x_2^{a_1} + B_1 x_2^{a_2} + C_1 x_2^{a_3} - D_1 x_2^{a_4} \quad \ldots \ldots \ldots \ldots (4.3.19)
\]

\[
y = A_2 x_1^{a_1} + B_2 x_1^{a_2} + C_2 x_1^{a_3} - D_2 x_1^{a_4} \quad \ldots \ldots \ldots \ldots (4.3.20)
\]

\[
y = A_2 x_2^{a_1} + B_2 x_2^{a_2} + C_2 x_2^{a_3} - D_2 x_2^{a_4} \quad \ldots \ldots \ldots \ldots (4.3.21)
\]

Where

\[
A_1 = \frac{a_{12} \mu_2 (\beta - u_{20}) - (\alpha - u_{10}) (\mu_1 - \lambda_2) \mu}{u_{10} \mu (\lambda_1 - \lambda_2)} \quad \ldots \ldots \ldots \ldots (4.3.22)
\]

\[
B_1 = \frac{a_{12} \mu_2 (\beta - u_{20}) - (\alpha - u_{10}) (\mu_1 - \lambda_1) \mu}{u_{10} \mu (\lambda_2 - \lambda_1)} \quad \ldots \ldots \ldots \ldots (4.3.23)
\]

\[
C_1 = \frac{A}{u_{10}}, \quad D_1 = \frac{B}{u_{10}} \quad \ldots \ldots \ldots \ldots (4.3.24)
\]

\[
A_2 = \frac{a_{12} \mu_2 (\beta - u_{20}) - (\alpha - u_{10}) (\mu_1 - \lambda_2) \mu}{u_{20} \mu_2 a_{12} (\lambda_1 - \lambda_2)} (\mu_1 - \lambda_2) \quad \ldots \ldots \ldots \ldots (4.3.25)
\]

\[
B_2 = \frac{a_{12} \mu_2 (\beta - u_{20}) - (\alpha - u_{10}) (\mu_1 - \lambda_1) \mu}{u_{20} \mu_2 a_{12} (\lambda_2 - \lambda_1)} (\mu_1 - \lambda_2) \quad \ldots \ldots \ldots \ldots (4.3.26)
\]

\[
C_2 = \frac{1}{u_{20}} \left[ \frac{\mu A}{a_{12} \mu_2} (\mu_1 - a_3) + \frac{a_{13} u}{a_{12}} \right] \quad \ldots \ldots \ldots \ldots (4.3.27)
\]

\[
D_2 = \frac{\mu B}{u_{20} a_{12} \mu_2} (a_4 - \mu_1) \quad \ldots \ldots \ldots \ldots (4.3.28)
\]
REFERENCES:


