ISSN 0976 – 1411

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International eJournals

INTERNATIONAL eJOURNAL OF MATHEMATICS AND ENGINEERING

www.InternationaleJournals.com

International eJournal of Mathematics and Engineering 27 (2010) 262 - 272

ON THE STABILITY OF HARVESTED COMMENSAL – HOST SPECIES PAIR WITH LIMITED RESOURCES

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Abstract

In this paper, we consider a commensal –host model with the commensal harvested at a constant rate. Further both the commensal and host species are with limited resources. The model is characterized by couple of first order non-linear ordinary differential equations. In all six equilibrium points for the model are identified and their stability criteria are discussed.

Keywords: commensal, host, harvesting, Equilibrium points, Normal Steady state, stability, threshold diagrams.

1. Introduction

The ecological symbiosis of living species can be broadly classified as Prey- Predation, Competition, Mutualism, Commensalism, Ammensalism, and Mutation and so on. Meyer [12], Kapoor [6,7] and several others dealt at length in their treatises ,the general concepts of mathematical modeling of ecosystems. The stability of biological communities in nature was discussed by Svirezher and D.O.Logofet [18]. Competition between two and three species with limited and unlimited resources was studied by Srinivas [19] .This was followed by Lakshminarayan and PattabhiRamacharyulu [8,9,10] and with their investigations on Prey-Predator ecological models with partial cover for the prey and alternate food for the predator and also models with harvesting. A Prey-Predator model with a variable cover for the prey and alternate food for the predator was studied by Lakshminarayan andApparao[11].Later PattabhiRamacharyulu et.al [2], Archanareddy[3]and Rama Sarma[4,5] investigated the stability of species in competition. While mutualism was considered by Ravindra Reddy [16].Following this PattabhiRmacharyulu, Phanikumar,et.al investigating the stability of species in commensalism[13,14,15,17] while ammensalism was considered by K.V.L.N Aacharyulu and PattabhiRamacharyulu N.Ch, [1].

This paper deals with an analytical investigation of a two species comensal - host model with constant harvest rate of commensal and both the species with limited resources. The model is characterized by couple of first order non-linear ordinary differential equations. In all six equilibrium points for the model are identified based on model equations and these are spread over two categories: (A) The states in which only commensal survives (B) The states in which both commensal and host survives. It is noticed that all the states are co-existent and criteria for the asymptotic stability of the states have been derived, in which only one equilibrium state is stable. Some threshold result to highlight the region of asymptotic stability.

2. Notation Adopted

N_1 and N_2 are the populations of the commensal and host species with natural growth rates a_1 and a_2 respectively.

a_{11} is rate of decrease of the commensal due to insufficient food.

a_{12} is rate of increase of the commensal due to inhibition by the host.

a_{22} is rate of decrease of the host due to insufficient food.

$h_1 = a_{11} H_1$ is rate of harvest of the commensal

$K_i = a_i / a_{ii}$ are the carrying capacities of N_i, i = 1, 2

$C = a_{12} / a_{11}$ is the coefficient of commensalism.

t^* is the dominance reversal time.

The state variables N_1 and N_2 as well as the model parameters a_1, a_2, a_{11}, a_{22}, K_1, K_2, C, h_1 are assumed to be non-negative constants.

3. Basic Equations

A commensal –host model with limited resources and with constant harvesting rate of commensal is characterized by the following pair of coupled non-linear ordinary differential equations.

(I) Equation for the growth rate of commensal species (N_1)

$$\frac{dN_1}{dt} = a_{11} (K_1 N_1 - N_1^2 + C N_1 N_2 - H_1) \quad (1)$$

(II) Equation for the growth rate of host species (N_2)

$$\frac{dN_2}{dt} = a_{22} (K_2 N_2 - N_2^2) \quad (2)$$

The system under investigation has ***six*** equilibrium states given by $\frac{dN}{dt} = 0$ and these are classified into two categories A and B.

A. The states in which only commensal survives:

(A.1) when $H_1 < \frac{K_1^2}{4}$

E₁: $\overline{N_1} = K_1 - \frac{H_1}{K_1}$ **;** $\overline{N_2} = 0$

This would exist only when $K_1^2 > H_1$

E₂: $\overline{N_1} = \frac{H_1}{K_1}$ **;** $\overline{N_2} = 0$

(A.2) when $H_1 = \frac{K_1^2}{4}$

E₃: $\overline{N_1} = \frac{K_1}{2}$ **;** $\overline{N_2} = 0$

B. The states in which both commensal and host survives:

(B.1) when $H_1 < \frac{(K_1 + CK_2)^2}{4}$

E₄: $\overline{N_1} = (K_1 + CK_2) - \frac{H_1}{(K_1 + CK_2)}$ **;** $\overline{N_2} = K_2$

This would exist only when $(K_1 + CK_2)^2 > H_1$

E₅: $\overline{N_1} = \frac{H_1}{K_1 + CK_2}$ **;** $\overline{N_2} = K_2$

(B.2) when $H_1 = \frac{(K_1 + CK_2)^2}{4}$

E₆: $\overline{N_1} = \frac{K_1 + CK_2}{2}$ **;** $\overline{N_2} = K_2$

4. Stability of Equilibrium States

Let N = (N_1, N_2) = $\bar{N}$ + U= $\left(\overline{N_1} + u_1, \overline{N_2} + u_2\right)$ (3)

where U = (u_1, u_2) is a small perturbation over the equilibrium state: $\overline{N} = \left(\overline{N_1}, \overline{N_2}\right)$.Substituting (3) in (1) and (2) and neglecting higher powers of the perturbations u_1, u_2 ,we get

$$\frac{dU}{dt} = AU \tag{4}$$

where A = $\begin{bmatrix} a_{11}(K_1 - 2\bar{N}_1 + C\bar{N}_2) & a_{11}C\bar{N}_1 \\ 0 & a_{22}(K_2 - 2\bar{N}_2) \end{bmatrix}$ (5)

The characteristic equation for the system is

det [A – λ I] = 0 (6)

The equilibrium state is **stable** only when the roots of the equation (6) are negative, in case they are real or have negative real parts, in case they are complex.

4.1 Equilibrium State E₁ :

In this case the characteristic roots of A are $-a_{11}\left(K_1 - \frac{2H_1}{K_1}\right)$ and $a_{22}K_2$. Since one of these roots is positive, the steady state is **unstable**. The equation (4) yields the solution curves:

$$u_1 = P_1 . e^{a_2 t} + (u_{10} - P_1) e^{-a_{11}\left(K_1 - \frac{2H_1}{K_1}\right)t}, u_2 = u_{20}\, e^{a_2 t} \tag{7}$$

where, $P_1 = \dfrac{C a_{11} u_{20}\left(K_1 - \frac{H_1}{K_1}\right)}{a_2 + a_{11}\left(K_1 - \frac{H_1}{K_1}\right)}$

and these are illustrated here under.

when $u_{10} = P_1$ equation (7) be comes

$$u_1 = u_{10} e^{a_2 t} ; u_2 = u_{20} e^{a_2 t}$$

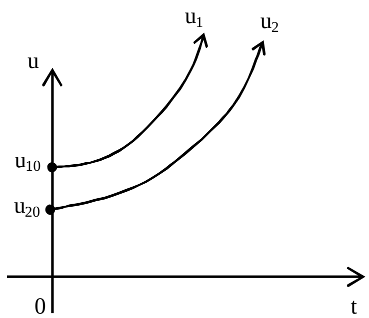

Fig.1

Case (i): $u_{10} > u_{20}$

The commensal dominates the host in natural growth as well as in its initial population strength. In this case the commensal continuously out number the host as shown in Fig.1

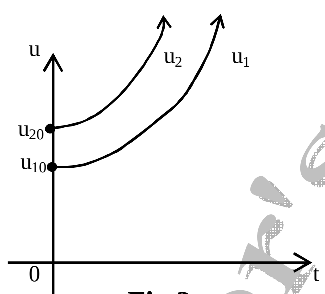

Fig.2

Case (i): $u_{20} > u_{10}$

The host dominates the commensal in natural growth as well as in its initial population strength. In this case the host continuously out number the commensal as shown in Fig.2

Trajectories of perturbed species:

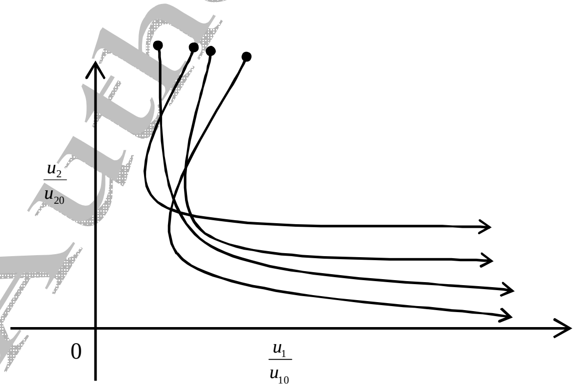

Fig.3

The trajectories obtained by solving (7) in $u_1 - u_2$ plane can be given by

$$\frac{u_1}{u_{10}} = \frac{P_1}{u_{10}}\left(\frac{u_2}{u_{20}}\right) + \left(1 - \frac{P_1}{u_{10}}\right)\left(\frac{u_2}{u_{20}}\right)^{\frac{-a11\left(K_1 - \frac{2H_1}{K_1}\right)}{a_2}}$$

which are sketched in Fig.3.

4.2 Equilibrium State E_2 :

Here the characteristic roots of 'A' are $a_{11}\left(K_1 - \frac{2H_1}{K_1}\right)$ and $a_{22}K_2$ and these are both positive, hence the steady state is **unstable**. The equation (4) yields the solution curves:

$$u_1 = P_2\, e^{a_2 t} + \left(u_{10} - P_2\right) e^{a_{11}\left(K_1 - \frac{2H_1}{K_1}\right)t},\; u_2 = u_{20}e^{a_2 t} \tag{8}$$

where $P_2 = \dfrac{C a_{11} u_{20}\left(\frac{H1}{K_1}\right)}{a_2 - a_{11}\left(K_1 - \frac{2H_1}{K_1}\right)}$

and the solution curves are illustrated as follows

when $u_{10} = P_2$ equation (8) becomes $u_1 = u_{10}e^{a_2 t}$; $u_2 = u_{20}e^{a_2 t}$

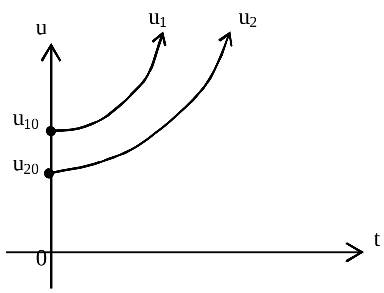

Fig.4

Case (i): $u_{10} > u_{20}$

The commensal dominates the host in its natural growth as well as in initial population strength. In this case the commensal continuously out number the host as shown in Fig.4.

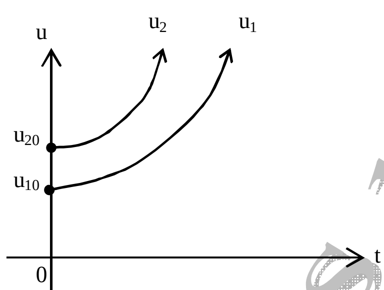

Fig.5

Case (ii): $u_{10} < u_{20}$

The host dominates the commensal in natural growth as well as on its initial population strength. In this case the host continuously out number the commensal as shown in Fig.5.

Trajectories of perturbed species:

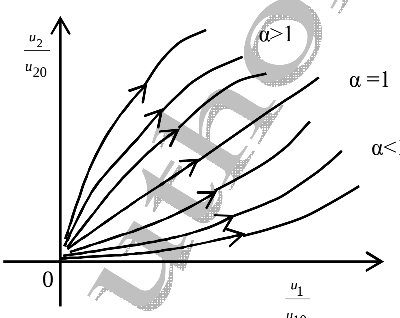

Fig.6

The trajectories obtained by solving (8) in $u_1 - u_2$ plane can be given by

$$\frac{u_1}{u_{10}} = \frac{p_2}{u_{10}}\left(\frac{u_2}{u_{20}}\right) + \left(1 - \frac{p_2}{u_{10}}\right)\cdot\left(\frac{u_2}{u_{20}}\right)^{\alpha}$$

where $\alpha = \dfrac{a_{11}\left(K_1 - \frac{2H_1}{K_1}\right)}{a_2}$

which are illustrated in Fig .6

4.3 Equilibrium State E_3:

The characteristic roots of 'A' in this case are 0, $a_{22}K_2$. Hence the steady state is **unstable**. The equation (4) yields the solution curves.

$$u_1 = P_3 e^{a_2 t} + (u_{10} - P_3)\,; \qquad u_2 = u_{20} e^{a_2 t} \tag{9}$$

where $P_3 = \dfrac{Ca_{11} u_{20} K_1}{2 a_{22} K_2}$

and the solution curves are sketched here under.

when $u_{10} = P_3$ equation (8) becomes

$$u_1 = u_{10} e^{a_2 t}\,; u_2 = u_{20} e^{a_2 t}$$

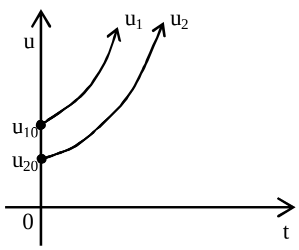

Fig.7

Case (i): u_{10}>u_{20}

The commensal dominates the host in natural growth as well as in its initial population strength. In this case the commensal continuously out number the host as shown in Fig.7.

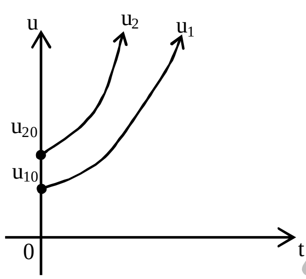

Fig.8

Case (ii): u_{10}<u_{30}

The host dominates the commensal in natural growth as well as on its initial population strength. In this case the host continuously out number the commensal as shown in Fig.8.

Trajectories of perturbed species:

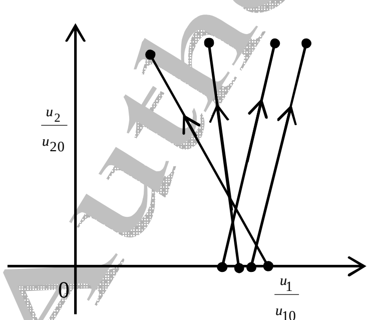

Fig.9

The trajectories obtained by solving (9) in $u_1 - u_2$ plane can be given by

$$\frac{u_1}{u_{10}} - \frac{P_3}{u_{10}}\left(\frac{u_2}{u_{20}}\right) = \left(1 - \frac{P_3}{u_{10}}\right)$$

which are illustrated in Fig .9

4.4 Equilibrium State E_4 :

In this case the characteristic roots of A are $-a_{11}\left[\left(K_1+C\bar{N}_2\right)-\frac{2H_1}{K_1+C\bar{N}_2}\right]$, $-a_2$ and these are both negative. Hence the steady state is **stable**. The equation (4) yields the solution curves:

$$u_1 = P_4 e^{-a_2 t} + \left(u_{10} - P_4\right) e^{-a_{11}\left(\left(K_1+C\bar{N}_2\right)-\frac{2H_1}{K_1+C\bar{N}_2}\right)t} ; \; u_2 = u_{20}\, e^{-a_2 t} \qquad (10)$$

where $P_4 = \dfrac{Ca_{11}\bar{N}_1 u_{20}}{-a_2 + a_{11}\left(\left(K_1+C\bar{N}_2\right)-\frac{2H_1}{K_1+C\bar{N}_2}\right)}$

and these curves are illustrated here under.

when $u_{10} = P_4$,equation (10) be comes $u_1 = u_{10} e^{-a_2 t}$; $u_2 = u_{20} e^{-a_2 t}$

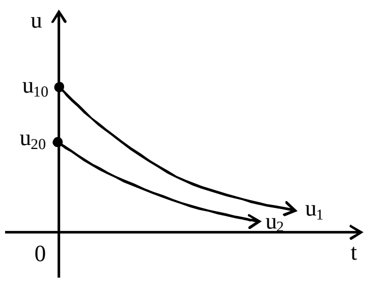

Fig.10

Case (i): $u_{10} > u_{20}$

Initially the commensal dominates the host and it continuous through out its growth. In this case the commensal continuously out number the host as shown in Fig.10. However both converge asymptotically to the equilibrium point.

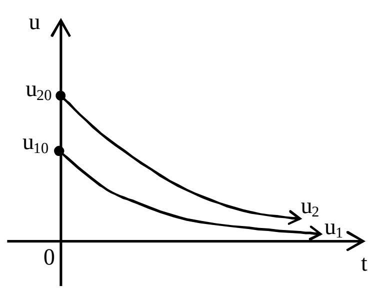

Fig.11

Case (ii): $u_{10} < u_{20}$

Initially the host dominates the commensal and it continuous to be so throughout its growth. In this case the host continuously out number the commensal as shown in Fig.11.However both converges asymptotically to the equilibrium point.

Trajectories of perturbed species:

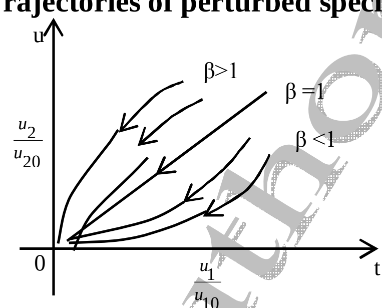

Fig.13

The trajectories obtained by solving (10) in $u_1 - u_2$ plane can be given by

$$\frac{u_1}{u_{10}} = \frac{P_4}{u_{10}}\left(\frac{u_2}{u_{20}}\right) + \left(1 - \frac{P_4}{u_{10}}\right)\left(\frac{u_2}{u_{20}}\right)^{\beta}$$

where $p = \dfrac{a_{11}\left(\left(K_1 + C\overline{N_2}\right) - \dfrac{2H_1}{\left(K_1 + C\overline{N_2}\right)}\right)}{a_2}$

which are sketched in Fig.13.

4.5 Equilibrium state E_5 :

Here the characteristic roots of A are $a_{11}\left(\left(K_1+C\overline{N_2}\right)-2\overline{N_1}\right)$ and $-a_2$ since one of these roots is positive, the steady state is **unstable**. The equation (4) yields the solution curves,

$$u_1=-P_5e^{-a_2t}+\left(u_{10}+P_5\right)e^{-a_{11}\left(\left(K_1+C\overline{N_2}\right)-2\overline{N_1}\right)t};\ u_2=e^{-a_{22}\left(K_2-\frac{2H_2}{K_2}\right)t} \qquad (11)$$

where $P_5=\dfrac{Ca_{11}\overline{N_1}u_{20}}{a_2+a_{11}\left[\left(K_1+C\overline{N}_2\right)-2\overline{N}_1\right]}$

and the solution curves are illustrated here under.

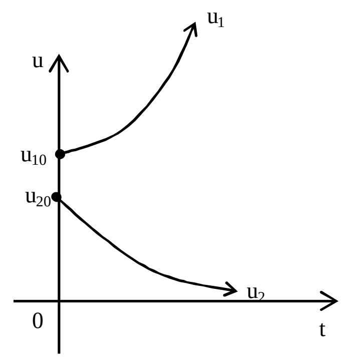

Fig.14

Case (i): $u_{10}>u_{20}$

In this case the commensal dominates the host in natural growth as well as in its population strength. The commensal species is noted to be going away from the equilibrium point as shown in Fig.14 while the host species asymptotic to the equilibrium point.

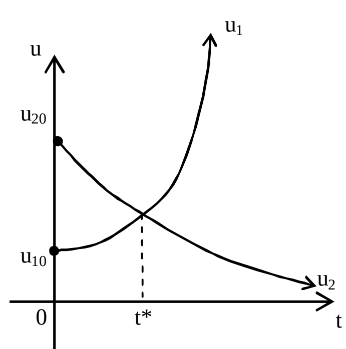

Fig.15

Case (ii): $u_{10}<u_{20}$

The commensal dominates the host in natural growth but its initial strength is less than at host. In the case the host out number the commensal till the time instant $t^*=\dfrac{1}{a_2+\left(\left(K_1+C\overline{N_2}\right)-2\overline{N_1}\right)}\log\left[\dfrac{u_{10}+P_5}{1+P_5}\right]$ after which the commensal to be going away from the equilibrium point while the host species asymptotic to the equilibrium point. Hence the equilibrium point is unstable, as shown in Fig.15

Trajectories of perturbed species:

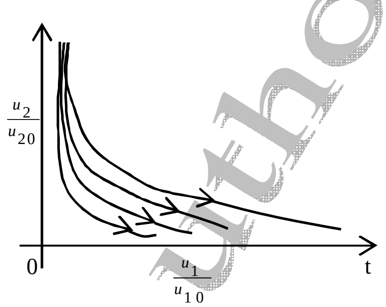

Fig.16

The trajectories obtained by solving (11) in u_1-u_2 plane given by

$$\frac{u_1}{u_{10}}=\left(1+\frac{P_5}{u_{10}}\right).\left(\frac{u_2}{u_{20}}\right)^{\frac{-a_{11}\left(\left(K_1+C\overline{N}_2\right)-2\overline{N}_1\right)}{a_2}}-\frac{P_5}{u_{10}}\left(\frac{u_2}{u_{20}}\right)$$

which are illustrated in Fig. 16

4.6 Equilibrium state E_6 :

In this case the characteristic roots of A are $0, -a_2$. The equation (4) yields the solution curves.

$$u_1 = u_{10} + P_6\left(1 - e^{-a_2 t}\right), \qquad u_2 = u_{20}\, e^{-a_2 t} \qquad (12)$$

where $P_6 = \dfrac{C a_{11} u_{20}\left(k_1 + C K_2\right)}{2 a_2}$

Since one of the characteristic roots is zero, the equilibrium state $\left(\overline{N_1}, \overline{N_2}\right)$ is **unstable** in the sense

the perturbations u_1, u_2 do not diminish as $t \to \infty$ .However it is noted that u_1, u_2 approach

$\left(u_{10} + P_6, 0\right)$ i.e., u_1-t curve is asymptotic to the line $u_1 = u_{10} + P_6$

and which curves are illustrated here under.

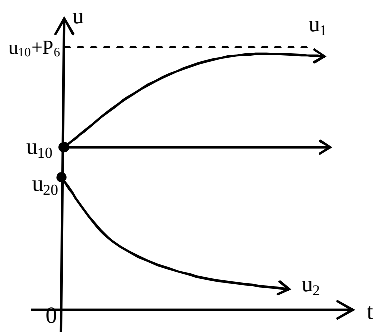

Fig.17

Case (i) $\boldsymbol{u_{10} > u_{20}}$

In this case the commensal species dominates the host in natural growth as well as in its initial population strength. Further the commensal species is noted to be going away from equilibrium point while the host species is asymptotic to the equilibrium point. as shown in Fig.17

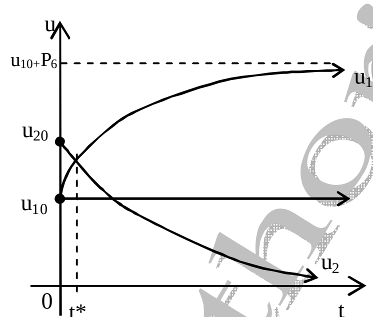

Fig.18

Case (ii) $\boldsymbol{u_{10} < u_{20}}$

The commensal species dominates the host in natural growth rate but is initial population strength is less than the host. In this case the host out numbers the commensal till the time instant $t^* = \dfrac{1}{a_2}\log\left(\dfrac{u_{10} + P_6}{u_{20} + P_6}\right)$ after which the commensal out number the host as shown in Fig.18. Further the commensal species going away from the equilibrium point while the host species is asymptotic to the equilibrium point then the steady state is unstable. shown in fig 18.

Trajectories of perturbed species:

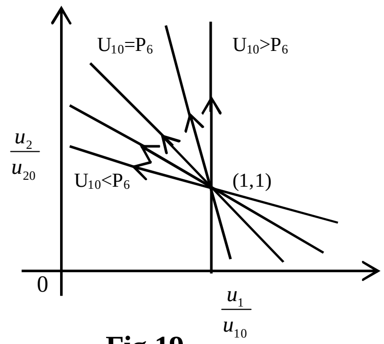

Fig.19

The trajectories obtained by solving (12) in $u_1 - u_2$ plane can be given by $\frac{u_1}{u_{10}} = \left(1+\frac{P_6}{u_{10}}\right) - \left(\frac{P_6}{u_{10}}\right)\frac{u_2}{u_{20}}$ which are illustrated in Fig. 19

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