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LIAPUNOV'S STABILITY ANALYSIS ON AMMENSAL- ENEMY SPECIES PAIR WITH LIMITED RESOURCES

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ABSTRACT

The paper aims to analyse the global stability of a Mathematical model of Ammensalism between two species surviving with limited resources by Liapnov's stability analysis. It is derived by constructing a suitable Liapanov's function for evaluating the global stability of the model in the case of normal steady state.

AMS Classification: 92 D 25, 92 D 40

Key words: Equilibrium states, Stability, Liapunov's function for global stability

1)INTRODUCTION

K.V.L.N.Acharyulu and and N.ch.Pattabhi Ramacharyulu [1,2,3,] cogitated the Local stability of an Ammensal- enemy co-system on the quasi-linear basic balancing equations. Local stability analysis for an enemy-Ammensal eco-system has been also presented in the author's earlier work [4-10].The present investigation is mainly concentrated on the establishment of the global stability of the co-existent equilibrium state of a mathematical model of Ammensalism between two species surviving with limited resources by employing a property constructed Liapunov's function with expedient Liapunov's criteria for global stability

Discovery and Brief history: Liapunov's stability analysis has been widely used as an effective and efficient tool in theory of control systems, dynamical systems, systems with time lag, power system analysis, multi species ecological systems and so on. A.M. Liapunov initiated an meritorious method in 1892 to examine the global stability of equilibrium points in case of linear and non-linear systems. This method yields stability information directly without solving the differential equations involved in the system. Hence it is also called Liapunov's direct method to detect the criteria for global stability. . His method is based on the chief charactestic of constructing a scalar function called Liapunov's function . The stability behaviour of solutions of linear and weakly non-linear system is done by using the techniques of variation of constants formulae and integral inequalities. So this analysis is confined to a small neighbourhood of operating point i.e., local stability. Further, the techniques used there in require explicit knowledge of solutions of corresponding linear systems. Hence, the stability behaviour of a physical system is curbed by these limitations.in this connection Several autors like Lotka[14], Kapur[12],Pattabhi Ramacharyulu[1-11],Lakiminarayan[13] and Bhaskararama Sarma[11] etc.applied this method in various situations for global stability.

2)Basic properties:

Consider an autonomous system

$$\frac{dx}{dt} = F_1(x, y) \text{ and } \frac{dy}{dt} = F_2(x, y) \qquad (1)$$

Assume that this system has an isolated initial point taken as (0, 0). Consider a function E(x,y) possessing continuous partial derivatives along the path of (1)

This path is represented by C= [(x (t), y (t)] in the parametric form. E(x,y) can be regarded as a function of 't' along C with rate of change

(i)Liapunov's Method for Global stability:

If the total energy of physical system has a local minimum at a certain equilibrium point then the point is said to be stable Liapunov's generalized this principle by constructing a function $E(N_1, N_2)$ whose rate of change is given by

$$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial N_1} \cdot \frac{\partial N_1}{\partial t} + \frac{\partial E}{\partial N_2} \cdot \frac{\partial N_2}{\partial t} = \frac{\partial E}{\partial N_1} F_1 + \frac{\partial E}{\partial N_2} F_2 \qquad (2)$$

Corresponding to the system.

(iv)Definitions

a) E (x,y) is said to be positive definite if E (x,y)>0 $\forall$ (x,y) $\neq$ (0,0)

b) E (x,y) is said to be positive semi-definite if E (x,y) > 0 & E (0,0)=0

c) E (x,y) is said to be negative definite if E (x,y) < 0

d) E (x,y) is said to be negative semi-definite if E (x,y) < 0 & E (0,0)=0

A Positive definite function E (x,y) with the property that (2) is negative semi-definite is called a Liapunov's function for the system (1). The following theorem is the basic discovery.

(ii)Theorem(A): If there exists a Liapunov's function E (x,y) for the system (1), then the critical point (0,0) is stable. Furthermore, if this function has additional property that the function (2) is negative definite, then the critical point (0, 0) is asymptotically stable.

The following theorem provides to ascertain definiteness of a Liapunov's function.

(iii)Theorem(B): The function $E(x,y) = ax^2+bxy+cy^2$ is positive definite if a>0 and $b^2 - 4ac<0$ and negative definite if a<0, $b^2 - 4ac<0$.

3) Basic Equations of the model

The equation for the growth rate of Ammensal species (N_1) under limited resources

is given by $$\frac{dN_1}{dt} = N_1 a_{11}[K_1 - N_1 - \alpha N_2] \qquad (3)$$

The equations for the growth rate of enemy species (N_2) under limited resources

is given by $$\frac{dN_2}{dt} = a_{22} N_2 [K_2 - N_2] \qquad (4)$$

Before going to establish the global stability by Liapunov's criteria, we now state the equilibrium states with respective equilibrium points.

4) Equilibrium Points.

The system has only four equilibrium states defined by

$$\frac{dN_1}{dt} = 0, \quad \frac{dN_2}{dt} = 0. \qquad (5)$$

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The system under investigation has four equilibrium points. Hence there exist four corresponding equilibrium states.

I. $E_1 = (0, 0)$: The state in which both the species are washed out.

II. $E_2 = (0, K_2)$: The state in which the enemy survives and the Ammensal is washed out

III. $E_3 = (K_1, 0)$: The state in which, only the Ammensal survives and enemy is washed out.

IV. $E_4 = (K_1 - \alpha K_2, K_2)$: This state would exit only when $\frac{K_1}{\alpha} > K_2$ =carrying capacity of S_2 and this state is known as co-existent state or normal steady state

Where $K_i = \frac{a_i}{a_{ii}}$, i = 1, 2 are the carrying capacities of Ni, $\alpha = \frac{a_{12}}{a_{11}}$, the Ammensal co-efficient.

5)The equilibrium states:

From the author's earlir work of local stability for Ammensal species under limited resources we can state as below

(a) $E_1 = (0, 0)$ is unstable.
(b) $E_2 = (0, K_2)$ is unstable.
(c) $E_3 = (K_1, 0)$ is unstable.
(d) $E_4 = (K_1 - \alpha K_2, K_2)$ is stable. (6)

6)Global Stability of this model by Liapunov's method:

The basic equations of this model are governed as

$$\text{I)}\ \frac{dN_1}{dt} = N_1[a_1 - a_{11}N_1 - a_{12}N_2]; \quad \frac{dN_2}{dt} = N_2[a_2 - a_{22}N_2] \qquad (7)$$

Linearised basic equations are

$$\frac{dU_1}{dt} = -a_{11}U_1\bar{N}_1 - a_{12}\bar{N}_2U_2 \quad and \quad \frac{dU_2}{dt} = -a_{22}\bar{N}_2U_2$$

The characteristic equation is

$$\left(\lambda + a_{11}\bar{N}_1\right)\left(\lambda + a_{22}\bar{N}_2\right) = 0$$

i.e. $\lambda^2 + (a_{11}\bar{N}_1 + a_{22}\bar{N}_2)\lambda + a_{11}a_{22}\bar{N}_1\bar{N}_2 = 0$

$$\Rightarrow \lambda^2 + p\lambda + q = 0$$

$$where\ P = a_{11}\bar{N}_1 + a_{22}\bar{N}_2;\ q = a_{11}a_{22}\bar{N}_1\bar{N}_2 \qquad (8)$$

By using $(a_1 + a_2 - K_2a_{12})^2 > 4a_2(a_1 - a_{12}K_2)$, we have p, q >0

The required conditions for Liapunov's function are satisfied

Let $E(U_1, U_2) = \frac{1}{2}(aU_1^2 + 2b\,U_1U_2 + CU_2^2)$ (9)

where $a = \frac{\left(a_{22}\bar{N}_2\right)^2 + a_{11}a_{12}\bar{N}_1\bar{N}_2)}{D}$

$$b = \frac{-a_{12}a_{22}\bar{N}_1\bar{N}_2}{D} \quad and$$

$$c = \frac{\left(a_{11}\bar{N}_1\right)^2 + \left(a_{12}\bar{N}_1\right)^2 + \left(a_{11}a_{22}\bar{N}_1\bar{N}_2\right)}{D}$$

where $D = pq = \left(a_{11}\bar{N}_1 + a_{22}\bar{N}_2\right)\left(a_{11}a_{22}\bar{N}_1\bar{N}_2\right) > 0$ (10)

From (8), it is clear that D>0 and a>0

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$D^2(ac-b^2) = 2a_{11}^2 a_{22}^2\, \overline{N}_1^2\, \overline{N}_2^2 + a_{11}a_{22}\, \overline{N}_1\overline{N}_2\, [a_{22}^2\, \overline{N}_2^2 + a_{11}^2\, \overline{N}_1^2 + a_{12}^2\, \overline{N}_1^2\,] > 0$

$\Rightarrow$ $ac-b^2>0$ (since $D^2>0$)

i.e., $b^2 - ac < 0$

$\therefore$ The function E (U_1,U_2) is positive definite (11)

Further $S = \dfrac{\partial E}{\partial U_1}.\dfrac{\partial U_1}{\partial t} + \dfrac{\partial E}{\partial U_2}.\dfrac{\partial U_2}{\partial t}$

$= (aU_1+bU_2)\,(-a_{11}\,\overline{N}_1 U_1 - a_{12}\,\overline{N}_1 U_2 + (bU_1+CU_2)\,(-a_{22}\overline{N}_2 U_2)$

$= -aa_{11}\,\overline{N}_1 U_1^2 - (aa_{12}\overline{N}_1 + ba_{11}\overline{N}_1 + ba_{22}\overline{N}_2)\, U_1U_2 - (ba_{12}\overline{N}_1 + ca_{22}\overline{N}_2)\, U_2^2$

Substituting the values of a, b and c, we obtain

$$S = \frac{-(a_{22}^2\overline{N}_2^2 + a_{11}a_{22}\overline{N}_1\overline{N}_2)a_{11}\overline{N}_1U_1^2}{D}$$

$$-\left\{\frac{(a_{22}^2\overline{N}_2^2 + a_{11}a_{22}\overline{N}_1\overline{N}_2)a_{12}\overline{N}_1}{D} - \frac{(a_{12}a_{22}\overline{N}_1\overline{N}_2)a_{11}\overline{N}_1}{D}\right.$$

$$\left. - \frac{(a_{12}a_{22}\overline{N}_1\overline{N}_2)a_{22}\overline{N}_2}{D}\right\}U_1U_2$$

$$+\left\{\frac{(a_{12}a_{22}\overline{N}_1\overline{N}_2)a_{12}\overline{N}_1}{D} - \frac{(a_{11}^2\overline{N}_1^2 + a_{12}^2\overline{N}_1^2 + a_{11}a_{22}\overline{N}_1\overline{N}_2)a_{22}\overline{N}_2}{D}\right\}U_2^2$$

$$= -\frac{(a_{11}a_{22}\overline{N}_1\overline{N}_2\,(a_{11}\overline{N}_1 + a_{22}\overline{N}_2))\,U_1^2}{D} - \frac{(a_{11}^2a_{22}\overline{N}_1^2\overline{N}_2 + a_{11}a_{22}^2\overline{N}_1\overline{N}_2)\,U_2^2}{D}$$

$$= -\frac{(a_{11}a_{22}\overline{N}_1\,\overline{N}_2)(a_{11}\overline{N}_1 + a_{22}\overline{N}_2)\,U_1^2}{D} - \frac{(a_{11}a_{22}\overline{N}_1\,\overline{N}_2)(a_{11}\overline{N}_1 + a_{22}\overline{N}_2)\,U_2^2}{D}$$

After simplifying, S can be reduced as

$$S = -\frac{D}{D}U_1^2 - \frac{D}{D}U_2^2 = -\,(U_1^2 + U_2^2)$$

$$\Rightarrow \frac{\partial E}{\partial U_1}.\frac{dU_1}{dt} + \frac{\partial E}{\partial U_2}.\frac{dU_2}{dt} = -(U_1^2 + U_2^2) \qquad (12)$$

which is negative definite. So E (U_1, U_2) is a Liapunov's function for the linear system.

Next we will prove that E $(U_1, U2)$ is also a Liapunov's function for the non-Linear system

Define $F_1(N_1,N_2) = N_1\,(a_1 - a_{11}\,N_1 - a_{12}\,N_2)$ and

$F_2(N_1,N_2) = N_2\,(a_2 - a_{22}\,N_2)$

By putting $N_1 = \overline{N_1} + U_1$ and $N_2 = \overline{N_2} + U_2$ in (7)

$$\frac{dU_1}{dt} = (\overline{N_1} + U_1)\,(a_1 - a_{11}\overline{N_1} - a_{11}U_1 - a_{12}\overline{N_2} - a_{12}U_2)$$

$$= a_1\overline{N_1} - a_{11}\overline{N_1}^2 - a_{11}\overline{N_1}U_1 - a_{12}\overline{N_1}\,\overline{N_2} - a_{12}\overline{N_1}U_2 + a_1U_1 - a_{11}\overline{N_1}U_1 - a_{11}U_1^2$$

$$- a_{12}\overline{N_2}U_1 - a_{12}U_1U_2$$

$$= -a_{11}\overline{N_1}U_1 - a_{12}\overline{N_1}U_2 - U_2\,(a_1 - a_{11}\overline{N_1} - a_{12}\overline{N_2}) - a_{11}U_1^2 - a_{12}U_1U_2$$

$$\Rightarrow F_1(U_1,U_2) = \frac{dU_1}{dt} = -a_{11}\overline{N}_1\,U_1 - a_{12}\overline{N}_1\,U_2 + f_1(U_1, U_2) \qquad (13)$$

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where $f_1(U_1, U_2) = -a_{11}U_1^2 - a_{12}U_1U_2$

similarly $F_2(U_1,U_2) = \dfrac{dU_1}{dt} = -a_{22}\overline{N_2}\,U_2 + f_2(U_1, U_2)$ where $f_2(U_1, U_2) = -a_{22}U_2^2$ (14)

we have

$$\frac{\partial E}{\partial U_1}F_1 + \frac{\partial E}{\partial U_2}F_2 = -(U_1^2 + U_2^2) + (aU_1 + bU_2)\, f_1(U_1, U_2) + (bU_1 + cU_2)\, f_2(U_1, U_2)$$

By introducing polar coordinates, we get

$$\frac{\partial E}{\partial U_1}F_1 + \frac{\partial E}{\partial U_2}F_2 = -r^2 + r[(a\cos\theta + b\sin\theta)\, f_1(U_1, U_2) + (b\cos\theta + c\sin\theta)\, f_2(U_1, U_2)]$$

Denote largest of the numbers |a|, |b|, |c| by M

Then $|f_1(U_1, U_2)| < \dfrac{r}{6k}$ and $|f_2(U_1, U_2)| < \dfrac{r}{6M}$ for all satisfying small r > 0

so $$\frac{\partial E}{\partial U_1}F_1 + \frac{\partial E}{\partial U_2}F_2 < -r^2 + \frac{4kr^2}{6k} = \frac{-r^2}{3} < 0 \qquad (15)$$

Thus $E(U_1, U_2)$ is a positive definite in with the property that

$\dfrac{\partial E}{\partial U_1}F_1 + \dfrac{\partial E}{\partial U_2}F_2$ is negative definite.

$\therefore$ The equilibrium point is asymptotically stable.

Conclusion: The global stability of an Ammensal-Enemy species with limited resources in the case of the co-existent state is established by using Liapunov's stability analysis.

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