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# PRIMARY IDEALS IN TERNARY SEMIGROUPS

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## ABSTRACT

In this paper, the terms left primary ideal, lateral primary ideal, right primary ideal, primary ideal, left primary ternary semigroup, lateral primary ternary semigroup , right primary ternary semigroup, primary ternary semigroup are introduced. It is proved that A be an ideal in a ternary semigroup T and if X, Y, Z are three ideals of T such that 1) XYZ$\subseteq$A and Y $\nsubseteq$ A, Z $\nsubseteq$ A, implies X $\subseteq\sqrt{A}$ $\sqrt{A}$ iff $x, y, z \in$T, $< x > < y > < z > \subseteq$A and $y \notin$A, $z \notin$A implies x $\in \sqrt{A}$. 2) XYZ$\subseteq$A and X $\nsubseteq$A, Z $\nsubseteq$A, implies Y$\subseteq\sqrt{A}$ if and only if $x, y, z \in$T, $< x >< y >< z > \subseteq$A and $x \notin$A, $z \notin$A implies y $\in\sqrt{A}$. 3) XYZ$\subseteq$A and X $\nsubseteq$ A, Y $\nsubseteq$ A, implies Z $\subseteq\sqrt{A}$ iff x, y, z $\in$T, $< x >< y >< z > \subseteq$A and x$\notin$A, y$\notin$A implies z$\in\sqrt{A}$. Further it is proved that if T be a commutative ternary semigroup and A be an ideal of T, then the conditions, 1) A is left primary ideal, 2) X, Y, Z are three ideals of T such that XYZ$\subseteq$A and Y $\nsubseteq$ A, Z $\nsubseteq$ A, implies X $\subseteq\sqrt{A}$, 3) x, y, z $\in$T, $< x >< y >< z > \subseteq$A and y $\notin$A, z $\notin$A implies x $\in\sqrt{A}$ are equivalent. It is also proved that if T be a commutative ternary semigroup and A be an ideal of T, then the conditions, 1) A is lateral primary ideal, 2) X, Y, Z are three ideals of T such that XYZ$\subseteq$A and X $\nsubseteq$ A, Z $\nsubseteq$A, implies Y$\subseteq\sqrt{A}$, 3) $x, y, z \in$T, $< x >< y >< z > \subseteq$A and $x \notin$A, $z \notin$A implies $y \in \sqrt{A}$. Further the conditions for an ideal in a commutative ternary semigroup T, 1) A is right primary ideal, 2) X, Y, Z are three ideals of T such that XYZ$\subseteq$A and X $\nsubseteq$A, Y $\nsubseteq$A, implies Z $\subseteq\sqrt{A}$, 3) $x, y, z \in$T, $x\ y\ z \subseteq$A and $x \notin$A, $y \notin$A implies $z \in \sqrt{A}$ are equivalent. It is proved that every ideal A in a ternary semigroup T, 1) T is a left primary if and only if every ideal A satisfies X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and Y $\nsubseteq$A, Z $\nsubseteq$ A, implies X $\subseteq \sqrt{A}$, 2) T is a lateral primary if and only if every ideal A satisfies X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and X $\nsubseteq$A, Z $\nsubseteq$A,

implies Y $\subseteq \sqrt{A}$, 3) T is a right primary if and only if every ideal A satisfies X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and X $\nsubseteq$A, Y $\nsubseteq$A, implies Z $\subseteq \sqrt{A}$. It is proved that T be a ternary semigroup with identity and M be the unique maximal ideal in T. If $\sqrt{A}$ = M for some ideal A in T, then A is a primary ideal. Further it proved that if T is a ternary semigroup with identity and M is the unique maximal ideal of T , then for any odd natural number n, $M^n$ is a primary ideal of T. It is proved that if A is an ideal of quasi commutative ternary semigroup T, then 1) A is primary, 2) A is left primary, 3) A is lateral primary and 4) A is right primary are equivalent.

**SUBJECT CLASSIFICATION (2010): 20M07, 20M11, 20M12.**

**KEY WORDS :** Left primary ideal, Lateral primary ideal, right primary ideal, primary ideal, left primary ternary semigroup, lateral primary ternary semigroup , right primary ternary semigroup, primary ternary semigroup.

## 1. INTRODUCTION :

The algebraic theory of semigroups was widely studied by CLIFFORD and PRESTON [4], [5]; PETRICH [11]. The ideal theory in commutative semigroups was developed by BOURNE [3], HARBANS LAL [6], SATYANARAYANA [15], [16], MANNEPALLI and NAGORE [10]. The ideal theory in general semigroups was developed by ANJANEYULU [1], [2], HOEHNKE [7] and KAR.S and MAITY. B. K[8], [9]. SANTIAGO [14] developed the theory of ternary semigroups. SARALA. Y, ANJANEYULU. A AND MADHUSUDHANA RAO.D [12], [13] introduced the ideal theory in ternary semigroups. They also introduced the notion of prime ideals in ternary semigroups and characterized by using the properties of quasi-ideals. In this paper we introduce the notions of primary ideals in ternary semigroups and characterize primary ideals in ternary semigroups.

## 2. PRILIMINARIES :

**DEFINITION 2.1** : Let T be a non-empty set. Then T is said to be a ***ternary semigroup*** if there exist a mapping from T×T×T to T which maps $(x_1, x_2, x_3) \rightarrow [x_1x_2x_3]$ satisfying the condition : $[(x_1x_2x_3)x_4x_5] = [x_1(x_2x_3x_4)x_5] = [x_1x_2(x_3x_4x_5)]$ $\forall$ $x_i \in$ T, $1 \leq i \leq 5$.

**NOTE 2.2** : For the convenience we write $x_1x_2x_3$ instead of $[x_1x_2x_3]$

**NOTE 2.3** : Let T be a ternary semigroup. If A,B and C are three subsets of T , we shall denote the set ABC = $\{abc : a \in A, b \in B, c \in C\}$.

**DEFINITION 2.4** : A ternary semigroup T is said to be ***commutative*** provided *abc* = *bca* = *cab* = *bac* = *cba* = *acb* for all *a*,*b*,*c* $\in$ T.

**DEFINITION 2.5** : A ternary semigroup T is said to be ***quasi commutative*** provided for each $a, b, c \in$ T, there exists a natural number $n$ such that $abc = b^n ac = bca = c^n ba = cab = a^n cb$.

**DEFINITION 2.6** : A nonempty subset A of a ternary semigroup T is said to be ***left ternary ideal*** or ***left ideal*** of T if $b, c \in$ T, $a \in$ A implies $bca \in$ A.

**NOTE 2.7** : A nonempty subset A of a ternary semigroup T is a left ideal of T if and only if TTA $\subseteq$ A.

**DEFINITION 2.8** : A nonempty subset of a ternary semigroup T is said to be a ***lateral ternary ideal*** or simply ***lateral ideal*** of T if $b, c \in$ T , $a \in$ A implies $bac \in$ A.

**NOTE 2.9** : A nonempty subset of A of a ternary semigroup T is a lateral ideal of T if and only if TAT $\subseteq$ A.

**DEFINITION 2.10** : A nonempty subset A of a ternary semigroup T is a ***right ternary ideal*** or simply ***right ideal*** of T if $b, c \in$ T , $a \in$ A implies $abc \in$ A

**NOTE 2.11** : A nonempty subset A of a ternary semigroup T is a right ideal of T if and only if ATT $\subseteq$ A.

**DEFINITION 2.12** : A nonempty subset A of a ternary semigroup T is a ***two sided ternary ideal*** or simply ***two sided ideal*** of T if $b, c \in$ T , $a \in$ A implies $bca \in$ A, $abc \in$A.

**NOTE 2.13** : A nonempty subset A of a ternary semigroup T is a two sided ideal of T if and only if it is both a left ideal and a right ideal of T .

**DEFINITION 2.14** : A nonempty subset A of a ternary semigroup T is said to be ***ternary ideal*** or simply an ***ideal*** of T if $b, c \in$ T , $a \in$ A implies $bca \in$ A, $bac \in$A, $abc \in$A.

**NOTE 2.15** : A nonempty subset A of a ternary semigroup T is an ideal of T if and only if it is left ideal, lateral ideal and right ideal of T .

**DEFINITION 2.16** : An ideal A of a ternary semigroup T is said to be a ***proper ideal*** of T if A is different from T.

**DEFINITION 2.17** : An ideal A of a ternary semigroup T is said to be a ***trivial ideal*** provided T\ A is singleton.

**DEFINITION 2.18** : An ideal A of a ternary semigroup T is said to be a ***maximal ideal*** provided A is a proper ideal of T and is not properly contained in any proper ideal of T.

**THEOREM 2.19 : If T is a ternary semigroup with unity 1 then the union of all proper ideals of T is the unique maximal ideal of T.**

**DEFINITION 2.20** : An ideal A of a ternary semigroup T is said to be ***a principal ideal*** provided A is an ideal generated by $\{a\}$ for some $a \in$T. It is denoted by J ($a$) (or) < $a$ >.

**NOTATION 2.21** : Let T be a ternary semigroup. If T has an identity, let $T^1$ = T and if T does not have an identity , let $T^1$ be the ternary semigroup T with an identity adjoined usually denoted by the symbol 1.

**NOTATION 2.22 :** Let T be a ternary semigroup. if T has a zero, let $T^0$ = T and if T does not have a zero , let $T^0$ be the ternary semigroup T with zero adjoined usually denoted by the symbol 0.

**DEFINITION 2.23** : An ideal A of a ternary semigroup T is said to be a ***completely prime ideal*** of T provided $x, y, z \in$ T and $xyz \in$A implies either $x \in$ A or $y \in$A or $z \in$A.

**DEFINITION 2.24** : An ideal A of a ternary semigroup T is said to be a ***prime ideal*** of T provided X,Y,Z are ideals of T and XYZ $\subseteq$ A $\Rightarrow$ X $\subseteq$A or Y $\subseteq$A or Z $\subseteq$A.

**DEFINITION 2.25** : If A is an ideal of a ternary semigroup T , then the intersection of all prime ideals of T containing A is called ***prime radical*** or simply ***radical*** of A and it is denoted by $\sqrt{A}$ or *rad* A.

**DEFINITION 2.26 :** If A is an ideal of a ternary semigroup T , then the intersection of all completely prime ideals of T containing A is called ***completely prime radical*** or simply ***complete radical*** of A and it is denoted by *c.rad* A.

**COROLLARY 2.27 : If $a \in \sqrt{A}$, then there exist a positive integer *n* such that $a^n \in$A for some odd natural number *n* ∈ N.**

**COROLLARY 2.28 : If A is an ideal of a commutative ternary semigroup T, then *rad* A = *c.rad* A.**

**DEFINITION 2.29** : An element $a$ of ternary semigroup T is said to be ***left identity*** of T provided $aat = t$ for all $t \in T$.

**NOTE 2.30** : Left identity element $a$ of a ternary semigroup T is also called as ***left unital element.***

**DEFINITION 2.31** : An element $a$ of a ternary semigroup T is said to be a ***lateral identity*** of T provided $ata = t$ for all $t \in T$.

**NOTE 2.32** : Lateral identity element $a$ of a ternary semigroup T is also called as ***lateral unital element.***

**DEFINITION 2.33**: An element $a$ of a ternary semigroup T is said to be a ***right identity*** of T provided $taa = t \ \forall \ t \in T$.

**NOTE 2.34** : Right identity element $a$ of a ternary semigroup T is also called as ***right unital element.***

**DEFINITION 2.35** : An element $a$ of a ternary semigroup T is said to be a ***two sided identity*** of T provided $aat = taa = t \ \forall \ t \in T$.

**NOTE 2.36** : Two-sided identity element of a ternary semigroup T is also called as ***bi-unital element***.

**DEFINITION 2.37** : An element $a$ of a ternary semigroup T is said to be an ***identity*** provided $aat = taa = ata = t \ \forall \ t \in T$.

**NOTE 2.38 :** An identity element of a ternary semigroup T is also called as ***unital element.***

**NOTE 2.39** : An element $a$ of a ternary semigroup T is an ***identity*** of T iff $a$ is left identity , lateral identity and right identity of T.

**DEFINITION 2.40** : An ideal A of a ternary semigroup T is said to be a ***proper ideal*** of T if A is different from T.

**DEFINITION 2.41** : An ideal A of a ternary semigroup T is said to be a ***trivial ideal*** provided T\ A is singleton.

**DEFINITION 2.42** : An ideal A of a ternary semigroup T is said to be a ***maximal ideal*** provided A is a proper ideal of T and is not properly contained in any proper ideal of T.

**THEOREM 2.43 : If A is an ideal of a ternary semigroup T with unity 1 and 1 $\in$ A then A = T.**

**THEOREM 2.44 : If T is a ternary semigroup with unity 1 then the union of all proper ideals of T is the unique maximal ideal of T.**

**DEFINITION 2.45** : An ideal A of a ternary semigroup T is said to be a ***completely semiprime ideal*** provided $x \in$ T, $x^n \in$ A for some odd natural number $n$ >1 implies $x \in$ A.

**DEFINITION 2.46** : An ideal A of a ternary semigroup T is said to be ***semiprime ideal*** provided X is an ideal of T and $X^n \subseteq$ A for some odd natural number $n$ implies X ⊆ A.

**THEOREM 2.47 :** An ideal Q of ternary semigroup T is a semiprime ideal of T if and only if $\sqrt{Q}$ =Q.

**THEOREM 2.48 : If A, B and C are any three ideals of a ternary semigroup T , then**

**i)** $A \subseteq B \Rightarrow \sqrt{A} \subseteq \sqrt{B}$

**ii) if** $A \cap B \cap C \neq \emptyset$ **then** $\sqrt{ABC} = \sqrt{A \cap B \cap C} = \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$

**iii)** $\sqrt{\sqrt{A}} = \sqrt{A}$.

**THEOREM 2.49 : If P is a prime ideal of a ternary semigroup T, then** $\sqrt{(P)^n} = P$ **for all odd natural numbers $n \in$ N.**

**THEOREM 2.50 : A commutative ternary semigroup T is regular if and only if every ideal of T is semiprime.**

## 3. PRIMARY IDEALS :

**DEFINITION 3.1 :** An ideal A of a ternary semi group T is said to be a ***left primary ideal*** if

i) X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and Y $\not\subseteq$ A, Z $\not\subseteq$ A, implies X $\subseteq \sqrt{A}$.
ii) $\sqrt{A}$ is a prime ideal.

**DEFINITION 3.2 :** An ideal A of a ternary semi group T is said to be a ***lateral primary ideal*** if
i) X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and X $\not\subseteq$ A, Z $\not\subseteq$ A, implies Y $\subseteq \sqrt{A}$.
ii) $\sqrt{A}$ is a prime ideal.

**DEFINITION 3.3 :** An ideal A of a ternary semi group T is said to be a ***right primary ideal*** if
i) X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and X $\not\subseteq$ A, Y $\not\subseteq$ A, implies Z $\subseteq \sqrt{A}$.
ii) $\sqrt{A}$ is a prime ideal.

**DEFINITION 3.4 :** An ideal A of a ternary semigroup T is said to be a ***primary ideal*** if A is left primary, lateral primary and right primary ideal of T.

**THEOREM 3.5: Let A be an ideal A in a ternary semigroup T. X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and Y $\not\subseteq$ A, Z $\not\subseteq$ A, implies X $\subseteq \sqrt{A}$ if and only if *x, y, z* $\in$ T, $< x > < y > < z > \subseteq$ A and *y* $\notin$ A, *z* $\notin$ A implies $x \in \sqrt{A}$.**

***Proof*** **:** Let A be an ideal of a ternary semigroup T.

Suppose that X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and Y $\not\subseteq$ A, Z $\not\subseteq$ A, implies X $\subseteq \sqrt{A}$. Let $x, y, z \in$ T, $< x > < y > < z > \subseteq$ A and $y \notin$ A, $z \notin$ A.

Since $y \notin$ A, $z \notin$ A then $< y > \not\subseteq$ A, $< z > \not\subseteq$ A.

Then by assumption $< x > < y > < z > \subseteq$ A and $<y> \not\subseteq$ A, $< z > \not\subseteq$ A, implies $<x> \subseteq \sqrt{A}$. Therefore $x \in \sqrt{A}$.

Conversely suppose that $x, y, z \in$ T, $< x > < y > < z > \subseteq$ A and $y \notin$ A, $z \notin$ A implies $x \in \sqrt{A}$. Let X, Y, Z be three ideals of T such that XYZ $\subseteq$ A and Y $\not\subseteq$ A, Z $\not\subseteq$ A.

Suppose if possible X $\not\subseteq \sqrt{A}$. Then there exists $x \in$ X such that $x \notin \sqrt{A}$.

Since Y $\not\subseteq$ A, Z $\not\subseteq$ A, there exists $y \in$ Y, $z \in$ Z such that $y \notin$ A, $z \notin$ A.

Now $x \in$ X, $y \in$ Y and $z \in$ Z, implies that $< x > < y > < z > \subseteq$ XYZ $\subseteq$ A and $y \notin$ A, $z \notin$ A, implies that $x \in \sqrt{A}$. It is a contradiction. Therefore X $\subseteq \sqrt{A}$.

**THEOREM 3.6 : Let A be an ideal A in a ternary semigroup T. X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and X $\nsubseteq$ A, Z $\nsubseteq$ A, implies Y $\subseteq \sqrt{A}$ if and only if *x, y, z* $\in$ T, $<x><y><z> \subseteq$ A and *x* $\notin$ A, *z* $\notin$ A implies $y \in \sqrt{A}$.**

***Proof* :** Let A be an ideal of a ternary semigroup T.

Suppose that X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and X $\nsubseteq$ A, Z $\nsubseteq$ A, implies Y $\subseteq \sqrt{A}$. Let x, y, z $\in$ T, $<x><y><z> \subseteq$ A and $x \notin$ A, $z \notin$ A.

Since $x \notin$ A, $z \notin$ A then $<x> \nsubseteq$ A, $<z> \nsubseteq$ A.

Then by assumption $<x><y><z> \subseteq$ A and $<x> \nsubseteq$ A, $<z> \nsubseteq$ A, implies $<y> \nsubseteq \sqrt{A}$. Therefore $y \in \sqrt{A}$.

Conversely, suppose that *x, y, z* $\in$ T, $<x><y><z> \subseteq$ A and $x \notin$ A, $z \notin$ A implies $y \in \sqrt{A}$. Let X, Y, Z be three ideals of T such that XYZ $\subseteq$ A and X $\nsubseteq$ A, Z $\nsubseteq$ A.

Suppose if possible Y $\nsubseteq \sqrt{A}$. Then there exists $y \in$ Y such that $y \notin \sqrt{A}$.

Since X $\nsubseteq$ A, Z $\nsubseteq$ A, there exists $x \in$ X, $z \in$ Z such that $x \notin$ A, $z \notin$ A.

Now $x \in$ X, $y \in$ Y and $z \in$ Z, implies that $<x><y><z> \subseteq$ XYZ $\subseteq$ A and $x \notin$ A, $z \notin$ A, implies that $y \in \sqrt{A}$. It is a contradiction. Therefore Y $\subseteq \sqrt{A}$.

**THEOREM 3.7 : Let A be an ideal A in a ternary semigroup T. X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and X $\nsubseteq$ A, Y $\nsubseteq$ A, implies Z $\subseteq \sqrt{A}$ iff *x, y, z* $\in$ T, $<x><y><z> \subseteq$ A and *x* $\notin$ A, *y* $\notin$ A implies $z \in \sqrt{A}$.**

***Proof* :** Let A be an ideal of a ternary semigroup T.

Suppose that X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and X $\nsubseteq$ A, Y $\nsubseteq$ A, implies Z $\subseteq \sqrt{A}$. Let *x, y, z* $\in$ T, $<x><y><z> \subseteq$ A and $x \notin$ A, $y \notin$ A.

Since $x \notin$ A, $y \notin$ A then $<x> \nsubseteq$ A, $<y> \nsubseteq$ A.

Then by assumption $<x><y><z> \subseteq$ A and $<x> \nsubseteq$ A, $<y> \nsubseteq$ A, implies $<z> \subseteq \sqrt{A}$. Therefore $z \in \sqrt{A}$.

Conversely suppose that *x, y, z* $\in$ T, $<x><y><z> \subseteq$ A and $x \notin$ A, $y \notin$ A implies $z \in \sqrt{A}$. Let X, Y, Z be three ideals of T such that XYZ $\subseteq$ A and X $\nsubseteq$ A, Y $\nsubseteq$ A.

Suppose if possible Z $\nsubseteq \sqrt{A}$. Then there exists $z \in$ Z such that $z \notin \sqrt{A}$.

Since X $\nsubseteq$ A, Y $\nsubseteq$ A, there exists $x \in$ X, $y \in$ Y such that $x \notin$ A, $y \notin$ A.

Now $x \in$ X, $y \in$ Y and $z \in$ Z, implies that $< x > < y > < z > \subseteq$ XYZ $\subseteq$ A and

$x \notin$ A, $y \notin$ A, implies that $z \in \sqrt{A}$. It is a contradiction. Therefore Z $\subseteq \sqrt{A}$.

**THEOREM 3.8 : Let T be a commutative ternary semigroup and A be an ideal of T. Then the following conditions are equivalent.**

**1.A is a left primary ideal.**

**2. X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and Y $\nsubseteq$ A, Z $\nsubseteq$ A, implies X $\subseteq \sqrt{A}$.**

**3. $x, y, z \in$ T, $< x > < y > < z > \subseteq$ A and $y \notin$ A, $z \notin$ A implies $x \in \sqrt{A}$.**

***Proof* : (1) $\Rightarrow$(2)** : Suppose that A is a left primary ideal.

By definition 1, we get X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and

Y $\nsubseteq$ A, Z $\nsubseteq$ A, implies X $\subseteq \sqrt{A}$.

(2) $\Rightarrow$(3) : Suppose that X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and

Y $\nsubseteq$ A, Z $\nsubseteq$ A, implies X $\subseteq \sqrt{A}$.

Let $x, y, z \in$ T, $xyz \in$ A and $y \notin$ A, $z \notin$ A.

Now, $xyz \in$ A, implies that $<xyz> \subseteq$ A $\Rightarrow <x> <y> <z> \subseteq$ A.

Now $y \notin$ A, $z \notin$ A, implies that $< y > \nsubseteq$ A, $< z > \nsubseteq$ A. Since $< x > < y > < z > \subseteq$ A and

$< y > \nsubseteq$ A, $< z > \nsubseteq$ A. Therefore by our assumption, $< x > \subseteq \sqrt{A}$. Thus $x \in \sqrt{A}$.

(3) $\Rightarrow$(1) : Suppose that $x, y, z \in$ T, $< \mathbf{x} > < \mathbf{y} >< \mathbf{z} > \subseteq$ A and $y \notin$ A, $z \notin$ A implies

$x \in \sqrt{A}$. Let X, Y, Z be three ideals of T such that XYZ $\subseteq$ A and Y $\nsubseteq$ A, Z $\nsubseteq$ A.

Now Y $\nsubseteq$ A, Z $\nsubseteq$ A implies that $y \in$ Y such that $y \notin$ A and $z \in$ Z such that $z \notin$ A.

Suppose, if possible X $\nsubseteq \sqrt{A}$. Then there exists $x \in$ X such that $x \notin \sqrt{A}$.

Now $xyz \in$ XYZ $\subseteq$ A. Therefore $xyz \in$ A and $y \notin$ A $z \notin$ A, $x \notin \sqrt{A}$.

It is a contradiction. Therefore X $\subseteq \sqrt{A}$.

Let $x, y, z \in$ T and $xyz \in \sqrt{A}$. Suppose that $y \notin \sqrt{A}$ and $z \notin \sqrt{A}$.

Now $xyz \in \sqrt{A}$, implies $(xyz)^m \in$ A, for some odd positive integer m $\Rightarrow x^m y^m z^m \in$ A.

$y \notin \sqrt{A}$, $z \notin \sqrt{A} \Rightarrow y^m \notin$ A and $z^m \notin$ A.

Now $x^m y^m z^m \in$ A and $y^m \notin$ A and $z^m \notin$ A $\Rightarrow$ $x^m \in \sqrt{A} \Rightarrow x \in \sqrt{\sqrt{A}} = \sqrt{A} \Rightarrow x \in \sqrt{A}$.

Therefore $\sqrt{A}$ is prime ideal. Therefore A is left primary ideal.

**THEOREM 3.9 : Let T be a commutative ternary semigroup and A be an ideal of T. Then the following conditions are equivalent.**

**1. A is a lateral primary ideal.**
**2. X, Y, Z are three ideals of T such that XYZ$\subseteq$A and X $\not\subseteq$A, Z $\not\subseteq$A, implies Y $\subseteq \sqrt{A}$.**
**3. $x, y, z \in$T, $< x > < y > < z > \subseteq$A and $x \notin$A, $z \notin$A implies $y \in \sqrt{A}$.**

***Proof* : (1) $\Rightarrow$(2) :** Suppose that A is a lateral primary ideal.

By definition 1, we get X, Y, Z are three ideals of T such that XYZ$\subseteq$A and X $\not\subseteq$A, Z $\not\subseteq$ A, implies Y $\subseteq \sqrt{A}$.

(2) $\Rightarrow$(3) : Suppose that X, Y, Z are three ideals of T such that XYZ$\subseteq$A and X $\not\subseteq$A, Z $\not\subseteq$A, implies Y $\subseteq \sqrt{A}$. Let $x, y, z \in$T, $xyz \in$A and $x \notin$ A, $z \notin$A.

Now, $xyz \in$A, implies that $\langle xyz\rangle \subseteq$A $\Rightarrow \langle x\rangle \langle y\rangle \langle z\rangle \subseteq$A.

Now $y \notin$ A, $z \notin$A, implies that $< x > \not\subseteq$ A, $< z > \not\subseteq$ A .

Since $< x > < y > < z > \subseteq$A and $< x > \not\subseteq$ A, $< z > \not\subseteq$ A.

Therefore by our assumption, $< y > \subseteq \sqrt{A}$. Thus $y \in \sqrt{A}$.

(3) $\Rightarrow$(1) : Suppose that $x, y, z \in$T, $< x > < y > < z > \subseteq$A and $x \notin$A, $z \notin$A implies $y \in \sqrt{A}$. Let X, Y, Z be three ideals of T such that XYZ$\subseteq$A and X $\not\subseteq$A, Z $\not\subseteq$A.

Now X $\not\subseteq$A, Z $\not\subseteq$A implies that $x \in$ Y such that $x \notin$ A and $z \in$ Z such that $z \notin$ A.

Suppose, if possible X $\not\subseteq \sqrt{A}$. Then there exists $y \in$X such that $y \notin \sqrt{A}$.

Now $xyz \in$ XYZ $\subseteq$ A. Therefore $xyz \in$ A and $x \notin$ A $z \notin$ A, $y \notin \sqrt{A}$.

It is a contradiction. Therefore Y $\subseteq \sqrt{A}$.

Let $x, y, z \in$T and $xyz \in \sqrt{A}$. Suppose that $x \notin \sqrt{A}$ and $z \notin \sqrt{A}$.

Now $xyz \in \sqrt{A}$, implies $(xyz)^{m} \in$A, for some odd positive integer m $\Rightarrow x^{m}y^{m}z^{m} \in$A.

$x \notin \sqrt{A}$, $z \notin \sqrt{A} \Rightarrow x^{m} \notin$A and $z^{m} \notin$A .

Now $x^{m}y^{m}z^{m} \in$A and $x^{m} \notin$A and $z^{m} \notin$A $\Rightarrow y^{m} \in \sqrt{A} \Rightarrow y \in \sqrt{\sqrt{A}} = \sqrt{A} \Rightarrow y \in \sqrt{A}$.

Therefore $\sqrt{A}$ is prime ideal. Therefore A is lateral primary ideal.

**THEOREM 3.10 : Let T be a commutative ternary semigroup and A be an ideal of T. Then the following conditions are equivalent.**

**1.A is a right primary ideal.**

**2. X, Y, Z are three ideals of T such that XYZ$\subseteq$A and X $\not\subseteq$A, Y $\not\subseteq$A, implies Z $\subseteq \sqrt{A}$.**

**3. $x, y, z \in$T, $xyz \subseteq$A and $x \notin$A, $y \notin$A implies $z \in \sqrt{A}$.**

***Proof* : (1) $\Rightarrow$(2) :** Suppose that A is a right primary ideal.

By definition 1, we get X, Y, Z are three ideals of T such that XYZ$\subseteq$A and X $\not\subseteq$A, Y $\not\subseteq$A, implies Z $\subseteq \sqrt{A}$.

**(**2) $\Rightarrow$(3) : Suppose that X, Y, Z are three ideals of T such that XYZ$\subseteq$A and X $\not\subseteq$A, Y $\not\subseteq$A, implies Z $\subseteq \sqrt{A}$. Let $x, y, z \in$T, $xyz \in$A and $x \notin$ A, $y \notin$A.

Now, $xyz \in$A, implies that $\langle xyz\rangle \subseteq$A $\Rightarrow \langle x\rangle\langle y\rangle\langle z\rangle \subseteq$A.

Now $x \notin$ A, $y \notin$A, implies that $< x > \not\subseteq$ A, $< y > \not\subseteq$ A .

Since $\langle x\rangle\langle y\rangle\langle z\rangle \subseteq$A and $< x > \not\subseteq$ A, $< y > \not\subseteq$ A.

Therefore by our assumption, $< z > \subseteq \sqrt{A}$. Thus $z \in \sqrt{A}$.

**(**3) $\Rightarrow$(1) : Suppose that $x, y, z \in$T, $xyz \subseteq$A and $x \notin$A, $y \notin$A implies $z \in \sqrt{A}$.

Let X, Y, Z be three ideals of T such that XYZ$\subseteq$A and X $\not\subseteq$A, Y $\not\subseteq$A.

Now X $\not\subseteq$A, Y $\not\subseteq$A implies that $x \in$ X such that $x \notin$ A and $y \in$ Y such that $y \notin$ A.

Suppose, if possible Z $\not\subset \sqrt{A}$. Then there exists $z \in$Z such that $z \notin \sqrt{A}$.

Now $xyz \in$XYZ$\subseteq$A. Therefore xyz $\in$ A and $x \notin$ A, $y \notin$ A, $z \notin \sqrt{A}$.

It is a contradiction. Therefore Z $\subseteq \sqrt{A}$. Let $x, y, z \in$T and $xyz \in \sqrt{A}$.

Suppose that $x \notin \sqrt{A}$ and $y \notin \sqrt{A}$.

Now $xyz \in \sqrt{A}$, implies $(xyz)^m \in$A, for some odd positive integer m $\Rightarrow x^m y^m z^m \in$A.

$x \notin \sqrt{A}$, $y \notin \sqrt{A}$, $x^m \notin$A and $y^m \notin$A .

Now $x^m y^m z^m \in$A and $x^m \notin$A and $y^m \notin$A $\Rightarrow z^m \in \sqrt{A} \Rightarrow z \in \sqrt{\sqrt{A}} = \sqrt{A} \Rightarrow z \in \sqrt{A}$.

Therefore $\sqrt{A}$ is prime ideal. Therefore A is right primary ideal.

**NOTE 3.11 :** In an arbitrary ternary semigroup a left primary ideal is not necessarily a right primary ideal.

**EXAMPLE 3.12 :** Let T = { 0, *a*, *b*, *c*, *1* } and the ternary operation [ ] as [*xyz*] = *x*(*yz*) = (*xy*)*z*. Then (T, [ ] ) is a ternary semigroup.

| **.** | *0* | *a* | *b* | *c* | *1* |
|---|---|---|---|---|---|
| **0** | *0* | *0* | *0* | *0* | *0* |
| *a* | *0* | *0* | *0* | *a* | *A* |
| *b* | *0* | *0* | *b* | *b* | *B* |
| *c* | *0* | *0* | *b* | *c* | *C* |
| *1* | *0* | *a* | *b* | *c* | *1* |

Let A = {*0*}, B = {0, *a*}, C = {0, *b* }, D = {0, *a*, *b*}, E = {*0*, *b*, *c*} and F = {0, *a*, *b*, *c* }. Then A, B, C, D, E and F are all ideals of T. Now FED ⊆ C and E ⊈ C, F ⊈ C and D ⊆ √C and √C is a prime ideal of T. Therefore C is a left primary ideal of T. But FED ⊆ C and D ⊈ C, E ⊈ C and also F ⊈ √C. Therefore C is a left primary ideal of T but not right primary..

**THEOREM 3.13 : Every ideal A in a ternary semigroup T is a left primary if and only if every ideal A satisfies X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and Y ⊈A, Z ⊈ A, implies X $\subseteq \sqrt{A}$ .**

***Proof*** **:** If every ideal A in T is left primary, then clearly every ideal satisfies If X, Y, Z are three ideals of T such that XYZ$\subseteq$A and Y ⊈ A, Z ⊈A, implies X $\subseteq \sqrt{A}$.

Conversely suppose that for every ideal A of T satisfies that X, Y, Z are three ideals of T such that XYZ$\subseteq$A and Y $\not\subset$A, Z $\not\subset$A, implies X $\subseteq \sqrt{A}$.

Let A be any ideal in T. Suppose that $< x> <y> <z> \subseteq \sqrt{A}$.

If $< y > \not\subseteq \sqrt{A}$, $< z > \not\subseteq \sqrt{A}$, then by our assumption $x \in \sqrt{\sqrt{A}} = \sqrt{A}$.

Therefore $\sqrt{A}$ is a prime ideal. Hence A is left primary.

**THEOREM 3.14 : Every ideal A in a ternary semigroup T is a lateral primary if and only if every ideal A satisfies X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and X ⊈A, Z ⊈A, implies Y $\subseteq \sqrt{A}$ .**

***Proof*** **:** If every ideal A in T is laterly primary, then clearly every ideal satisfies If X, Y, Z are three ideals of T such that XYZ$\subseteq$A and X $\not\subseteq$A, Z $\not\subseteq$A, implies Y $\subseteq \sqrt{A}$.

Conversely suppose that for every ideal A of T satisfies that X, Y, Z are three ideals of T such that XYZ$\subseteq$A and X $\not\subseteq$A, Z $\not\subseteq$A, implies Y $\subseteq \sqrt{A}$.

Let A be any ideal in T. Suppose that $<x><y><z> \subseteq \sqrt{A}$.

If $<x> \not\subseteq \sqrt{A}$, $<z> \not\subseteq \sqrt{A}$, then by our assumption $y \in \sqrt{\sqrt{A}} = \sqrt{A}$.

Therefore $\sqrt{A}$ is a prime ideal. Hence A is lateral primary.

**THEOREM 3.15 : Every ideal A in a ternary semigroup T is a right primary if and only if every ideal A satisfies X, Y, Z are three ideals of T such that XYZ $\subseteq$ A and X $\not\subseteq$A, Y $\not\subseteq$A, implies Z $\subseteq \sqrt{A}$.**

***Proof*** **:** If every ideal A in T is right primary, then clearly every ideal satisfies If X, Y, Z are three ideals of T such that XYZ$\subseteq$A and X $\not\subseteq$A, Y $\not\subseteq$A, implies Z $\subseteq \sqrt{A}$.

Conversely suppose that for every ideal A of T satisfies that X, Y, Z are three ideals of T such that XYZ$\subseteq$A and X $\not\subseteq$A, Y $\not\subseteq$A, implies Z $\subseteq \sqrt{A}$.

Let A be any ideal in T. Suppose that $<x> <y> <z> \subseteq \sqrt{A}$.

If $<x> \not\subseteq \sqrt{A}$, $<y> \not\subseteq \sqrt{A}$, then by our assumption $z \in \sqrt{\sqrt{A}} = \sqrt{A}$.

Therefore $\sqrt{A}$ is a prime ideal. Hence A is right primary.

**DEFINITION 3.16 :** A ternary semigroup T is said to be a ***left primary*** provided every ideal in T is a left primary ideal.

**DEFINITION 3.17 :** A ternary semigroup T is said to be a ***lateral primary*** provided every ideal in T is a laterly primary ideal.

**DEFINITION 3.18 :** A ternary semigroup T is said to be a ***right primary*** provided every ideal in T is a right primary ideal.

**DEFINITION 3.19 :** A ternary semigroup T is said to be a ***primary*** provided every ideal in T is a primary ideal.

**THEOREM 3.20 : If T is a ternary semigroup with identity, then for any odd natural number n, $M^n$ is a primary ideal of T where M is the unique maximal ideal of T.**

***Proof*** **:** Since M is the only prime ideal containing $M^n$ , we have $\sqrt{M^n}$ = M and hence by theorem 2.49, $M^n$ is a primary ideal.

**NOTE 3.21 :** If T has no identity then theorem 3.19 is not true. In example 3.11, M = { *a, b*} is the unique maximal ideal, but $M^2$= {a} is a primary ideal.

**THEOREM 3.22 : In a quasi commutative ternary semigroup T, an ideal A of T is left primary iff right primary**

***Proof*** **:** suppose that A is a left primary ideal. Let $xyz \in A$ and $x \notin A$, $y \notin A$.
Since T is a quasi commutative ternary semigroup, we have $xyz = yzx = z^n yx = zxy = x^n yz$ for some odd natural number n. So $z^n yx \in A$ and $x \notin A$, $y \notin A$.
Since A is left primary, we have $z^n \in \sqrt{A}$ and since $\sqrt{A}$ is a prime ideal, $z \in \sqrt{A}$ .
Therefore Ais a right primary ideal.
Similarly we can prove that if A is a right primary ideal then A is a left primary ideal.

**THEOREM 3.23 : In a quasi commutative ternary semigroup T, an idealg A of T is left primary if and only if A is lateral primary.**

***Proof*** **:** suppose that A is a left primary ideal. Let $xyz \in A$ and $x \notin A$, $z \notin A$.
Since T is a quasi commutative ternary semigroup,
we have $xyz = yzx = z^n yx = zxy = x^n yz$ for some odd natural number *n*.
So $yzx \in A$ and $x \notin A$, $z \notin A$. Since A is left primary, we have $y \in \sqrt{A}$ and since $\sqrt{A}$ is a prime ideal, $y \in \sqrt{A}$ . Therefore A is a lateral primary ideal.
Similarly we can prove that if A is a lateral primary ideal then A is a left primary ideal.

**COROLLARY 3.24 : If A is an ideal of quasi commutative ternary semigroup T, then the following are equivalent**

1) **A is primary**
2) **A is left primary**
3) **A is lateral primary**
4) **A is right primary**

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