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Γ-CONGRUENCE and 𝔑-CLASSES in PO-Γ-SEMIGROUPS

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ABSTRACT

The terms Γ-congruence, semilattice, semilattice Γ-congruence and complete are introduced. It is proved that an equivalence relation ρ on a po-Γ-semigroup S is a Γ-congruence if and only if for all $a, b, c, d \in S$, $\alpha \in \Gamma$, $a \,\rho\, b$ and $c \,\rho\, d$ implies $a\alpha c \,\rho\, b\alpha d$. It is proved that if S be a po-Γ- semigroup and ρ_1 and ρ_2 are two left Γ-congruences (resp. right Γ-congruences, Γ-congruences) of S, then ρ_1 o $\rho_2$is a left Γ-congruence (resp. right Γ-congruence, Γ-congruence) of S. Further it is also proved that if S is a po-Γ-semigroup and $\rho_1, \rho_2, \ldots\ldots, \rho_n$ are left Γ-congruences (resp. right congruences, congruences) of S, then $\rho_1 o \rho_2 o \ldots\ldots \rho_n$ is a left Γ-congruence (resp. right Γ-congruence, Γ-congruence) of S. The term, 𝔑-class, 𝔑-simple and 𝔑-subset are introduced. It is proved that if S is a po-Γ-semigroup, $z \in S$ and A is a po-Γ-ideal of an 𝔑-class $(z)_{\mathfrak{N}}$,then A has no proper completely prime po-Γ-ideals. It is proved that if S is a po-Γ-semigroup, A is a completely prime po-Γ-ideal of S,then $A = \cup\{(a)_{\mathfrak{N}} / a \in A\}$. It is proved that every po-Γ-semigroup is a semilattice of 𝔑-simple po-Γ-semigroups. It is proved that every 𝔑-simple po-Γ-subsemigroup T of a po-Γ-semigroup S is contained in an 𝔑-class of S. It is proved that if A is a po-Γ-ideal of a po-Γ-semigroup S, then the conditions (i) A is the intersection of all completely prime po-Γ-ideals of S containing A (ii) A is the intersection of all minimal completely prime po-Γ-ideals of S containing A (iii) A is the union of 𝔑-classes. (iv) A is a completely semiprime po-Γ-ideal of S are equivalent. It is proved that if a nonempty subset $\mathcal{C}$ of a po-Γ-semigroup S is an 𝔑-subset,then $\mathcal{C}$ is a class of semilattice Γ-congruence. It is proved that a po-Γ-semigroup S is separative if and only if S is a semilattice of strongly Γ-cancellative po-Γ-semigroup. If so, the relation σ defined on S by $x \,\sigma\, y$ if for any $a, b \in S$, $x\Gamma a = x\Gamma b$ if and only if $y\Gamma a = y\Gamma b$ is the greatest band Γ-congruence on S all whose classes are strongly Γ-cancellative.

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KEY WORDS: Γ-congruence, semilattice Γ-congruence, 𝔑-class.

1. INTRODUCTION:

Γ-semigroup was introduced by Sen and Saha [15] as a generalization of semigroup. Anjaneyulu. A [1], [2] and [3] initiated the study of ideals and radicals in semigroups. Many classical notions of semigroups have been extended to Γ-semigroups by Madhusudhana Rao, Anjaneyulu and Gangadhara Rao [11]. The concept of po-Γ-semigroup was introduced by Y. I. Kwon and S. K. Lee [10] in 1996, and it has been studied by several authors. In this paper we introduce the notions of Γ-congruence, 𝔑-simple, 𝔑-subset and 𝔑-classes of po-Γ-semigroups and characterize 𝔑-classes.

2. PRELIMINARIES :

DEFINITION 2.1 : Let S and Γ be two non-empty sets. Then S is called a ***Γ-semigroup*** if there exist a mapping from $S \times \Gamma \times S$ to S which maps $(a, \alpha, b) \rightarrow a\alpha b$ satisfying the condition : $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in$ S and $\gamma, \mu \in \Gamma$.

NOTE 2.2 : Let S be a Γ-semigroup. If A and B are two subsets of S, we shall denote the set $\{ a\gamma b : a \in A , b \in B \text{ and } \gamma \in \Gamma \}$ by AΓB.

DEFINITION 2.3: A Γ-semigroup S is said to a partially ordered Γ-semigroup if S is partially ordered set such that $a \le b \Rightarrow a\gamma c \le b\gamma c$ and $c\gamma a \le c\gamma b \ \forall a,b,c \in S$ and $\gamma \in \Gamma$.

NOTE 2.4: A partially ordered Γ-semigroup simply called po-Γ-semigroup or ordered Γ-semigroup.

NOTATION 2.5 : Let S be a po- Γ-semigroup and T is a nonempty subset of S. If H is a nonempty subset of T, we denote the set $\{t \in T : t \le h$ for some $h \in H\}$ by $(H]_T$. The set $\{t \in T : h \le t$ for some $h \in H\}$ by $[H)_T$. $(H]_s$ and $[H)_s$ are simply denoted by (H] and [H) respectively.

DEFINITION 2.6 : Let S be a po- Γ-semigroup. A nonempty subset T of S is said to be a ***po-Γ-subsemigroup*** of S if $a\gamma b \in T$, for all $a, b \in T$ and $\gamma \in \Gamma$ and $t \in T$, $s \in S$, $s \le t \Rightarrow s \in T$.

THEOREM 2.7 : A nonempty subset T of a po-Γ-semigroup S is a po-Γ-subsemigroup of S iff (1) TΓT ⊆ T, (2) (T] ⊆ T.

THEOREM 2.8 : Let S be a po- Γ-semigroup and A is a subset of S. Then for all A, B ⊆ S (i) A ⊆ (A], (ii) ((A]] = (A], (iii) (A]Γ(B] ⊆ (AΓB] and for A ⊆ B (iv) A ⊆ (B], (v) (A] ⊆ (B] for A ⊆ B.

THEOREM 2.9 : The nonempty intersection of two po-Γ-subsemigroups of a po-Γ-semigroup S is a po-Γ-subsemigroup of S.

THEOREM 2.10 : The nonempty intersection of any family of po-Γ-subsemigroups of a po-Γ-semigroup S is a po-Γ-subsemigroup of S.

DEFINITION 2.11 : An element a of po-Γ-semigroup S is said to be ***strongly left Γ-cancellative*** provided $a\Gamma b \le a\Gamma c$ implies $b \le c$.

NOTE 2.12 : An element a of po-Γ-semigroup S is said to be ***strongly left* Γ-*cancellative*** provided $a\alpha b \le a\beta c$, $\alpha, \beta \in \Gamma \Rightarrow b \le c$.

DEFINITION 2.13 : An element a of po-Γ-semigroup S is said to be ***strongly right* Γ-*cancellative*** provided $b\Gamma a \le c\Gamma a$ implies $b \le c$.

NOTE 2.14 : An element a of po-Γ-semigroup S is said to be ***strongly right* Γ-*cancellative*** provided $b\alpha a \le c\beta a$, $\alpha, \beta \in \Gamma \Rightarrow b \le c$.

DEFINITION 2.15 : An element a of po-Γ-semigroup S is said to be ***strongly* Γ-*cancellative*** provided a is a both strongly left **Γ**-cancellative and strongly right **Γ**-cancellative.

NOTE 2.16 : $a\Gamma b \le c\Gamma d$ if and only if $a\Gamma b \subseteq c\Gamma d$.

DEFINITION 2.17 : Po- Γ-semigroup S is said to be ***separative*** if for any $x, y \in S$,

(1) $x\Gamma x \le x\Gamma y$ and $y\Gamma y \le y\Gamma x$ imply $x = y$,
(2) $x\Gamma x \le y\Gamma x$ and $y\Gamma y \le x\Gamma y$ imply $x = y$.

THEOREM 2.18 : In a separative po-Γ-semigroup S, for any $x, y, a, b \in S$, the following statements hold.

- **(i)** $x\Gamma a \le x\Gamma b$ **if and only if** $a\Gamma x \le b\Gamma x$,
- (ii) $x\Gamma x\Gamma a \le x\Gamma x\Gamma b$ **implies** $x\Gamma a \le x\Gamma b$,
- (iii) $x\Gamma y\Gamma a \le x\Gamma y\Gamma b$ **implies** $y\Gamma x\Gamma a \le y\Gamma x\Gamma b$.

DEFINITION 2.19 : A nonempty subset A of a po-Γ-semigroup S is said to be a ***left* po-Γ-*ideal*** of S if

(1) $s \in S, a \in A, \alpha \in \Gamma$ implies $s\alpha a \in A$.

(2) $s \in S, a \in A$, $s \le a \Rightarrow s \in$ A.

NOTE 2.20 : A nonempty subset A of a po-Γ-semigroup S is a left po-Γ-ideal of S iff (1) SΓA⊆A, and (2) (A] ⊆ A.

DEFINITION 2.21 : A nonempty subset A of a po-Γ-semigroup S is said to be a ***right po- Γ-ideal*** of S if

(1) $s \in S, a \in A, \alpha \in \Gamma$ implies $a\alpha s \in A$.
(2) $s \in S, a \in A$, $s \le a \Rightarrow s \in$ A.

NOTE 2.22 : A nonempty subset A of a po-Γ-semigroup S is a right po-Γ-ideal of S iff (1) AΓS⊆A and (2) (A] ⊆ A.

DEFINITION 2.23 : A nonempty subset A of a po-Γ-semigroup S is said to be a ***two sided* po-Γ-*ideal*** or simply a ***po*-Γ-*ideal*** of S if
(1) $s \in$ S, $a \in$ A, $\alpha \in \Gamma$ imply $s\alpha a \in$ A, $a\alpha s \in$ A.

(2) $s \in S, a \in A$, $s \le a \Rightarrow s \in$ A.

NOTE 2.24 : A nonempty subset A of a po- Γ-semigroup S is a two sided po-Γ-ideal iff it is both a left po-Γ-ideal and a right po-Γ- ideal of S.

DEFINITION 2.25 : A (left, right) po-Γ-ideal P of a po-Γ-semigroup S is said to be ***completely prime* (*left, right*) *po*-Γ-*ideal*** provided $x, y \in S$ and $x\Gamma y \subseteq P$ implies either $x \in P$ or $y \in P$.

DEFINITION 2.26 : A (left, right) po-Γ-ideal P of a po-Γ-semigroup S is said to be a ***prime* (*left, right*) *po*-Γ-*ideal*** provided A, B are two po- Γ-ideals of S and $A\Gamma B \subseteq P$ $\Rightarrow$ either $A \subseteq P$ or $B \subseteq P$.

THEOREM 2.27 : If P is a prime po-Γ-ideal of a po-Γ-semigroup S, then the following conditions are equivalent.

(1) If A, B are po-Γ- ideals of S and $A\Gamma B \subseteq P$ then either $A \subseteq P$ or $B \subseteq P$.

(2) If $a, b \in S$ such that $a\Gamma S^1\Gamma b \subseteq P$, then either $a \in P$ or $b \in P$.

THEOREM 2.28 : Let P be a po- Γ-ideal of a po- Γ-semigroup S. Then $(a\Gamma S^1\Gamma b] \subseteq P$ if and only if $a\Gamma S^1\Gamma b \subseteq P$.

DEFINITION 2.29 : A po-Γ-ideal A of a po- Γ-semigroup S is said to be a ***completely semiprime po*-Γ- *ideal*** provided $x\Gamma x \subseteq A$; $x \in S$ implies $x \in A$.

DEFINITION 2.30 : A po-Γ- ideal A of a po-Γ-semigroup S is said to be a ***semiprime po*-Γ-*ideal*** provided $x \in S$, $x\Gamma S^1\Gamma x \subseteq A$ implies $x \in A$.

THEOREM 2.31[13] : Every prime ideal P minimal relative to containing a completely semiprime po-Γ-ideal is completely prime.

DEFINITION 2.32 : A Γ-subsemigroup F of a po-Γ-semigroup S is said to be a ***left po*-Γ-*filter*** of S if

(1) $a, b \in S$, $\alpha \in \Gamma$, $a\alpha b \in F$ implies $a \in F$.
(2) $a \in F$ and $a \leq c$ for $c \in S$ implies $c \in F$.

NOTE 2.33 : A Γ-subsemigroup F of a po-Γ-semigroup S is said to be a ***left po*-Γ-*filter*** of S if

(1) for $a, b \in S$, $a\Gamma b \subseteq F$ implies $a \in F$.

(2) $[F) \subseteq F$.

THEOREM 2.34 : A nonempty subset F of a po-Γ-semigroup S is a left po-Γ-filter if and only if S\F is a completely prime right po-Γ-ideal of S or empty.

DEFINITION 2.35 : A Γ-subsemigroup F of a po- Γ-semigroup S is said to be ***right po- Γ-filter*** of S if

(1) $a, b \in S$, $\alpha \in \Gamma$, $a\alpha b \in F$ implies $b \in F$.
(2) $a \in F$ and $a \leq c$ for $c \in S$ implies $c \in F$.

NOTE 2.36: A Γ-subsemigroup F of a po-Γ-semigroup S is said to be a ***right po- Γ-filter*** of S if

(1) a, $b \in$ S, $a\Gamma b \subseteq$ F implies $b \in$ F.
(2) [F) ⊆ F.

THEOREM 2.37 : A nonempty subset F of a po-Γ-semigroup S is a right po-Γ-filter if and only if S\F is a completely prime left po-Γ-ideal of S or empty.

DEFINITION 2.38 : A Γ-subsemigroup F of a po-Γ-semigroup S is said to be ***po-Γ-filter*** of S if

(1) a, $b \in$ S, $\alpha \in \Gamma$, $a\alpha b \in$ F implies a, $b \in$ F.
(2) a ∈ F and $a \leq c$ for $c \in$ S implies $c \in$ F .

NOTE 2.39 : A Γ-subsemigroup F of a po-Γ-semigroup S is said to be ***po-Γ-filter*** of S if

(1) a, $b \in$ S, $a\Gamma b \in$ F implies a, $b \in$ F.
(2) [F) ⊆ F

THEOREM 2.40 : A nonempty subset F of a po-Γ-semigroup S is a po-Γ-filter if and only if S\F is a completely prime po-Γ-ideal of S or empty.

NOTE 2.41 : A Γ-subsemigroup F of a po- Γ-semigroup S is said to be ***po-Γ-filter*** of S iff F is a left po-Γ-filter and a right po-Γ-filter of S.

DEFINITION 2.42 : A po-Γ-filter F of a po-Γ-semigroup S is said to be a ***proper po-Γ-filter*** if F ≠ S.

DEFINITION 2.43 : Let S be a po- Γ-semigroup and A be a nonempty subset of S. The smallest left po-Γ-filter of S containing A is called ***left po-Γ-filter of* S *generated by A*** and it is denoted by L_f(A).

DEFINITION 2.44: Let S be a po- Γ-semigroup and A be a nonempty subset of S. The smallest right po-Γ-filter of S containing A is called ***right po-Γ-ideal of* S *generated by A*** and it is denoted by R_f(A).

DEFINITION 2.45 : Let S be a po- Γ-semigroup and A be a nonempty subset of S. The smallest po-Γ-filter of S containing A is called ***po-Γ-filter of* S *generated by A*** and it is denoted by N(A).

DEFINITION 2.46 : A po-Γ-filter F of a po-Γ-semigroup S is said to be a ***principal po-Γ-filter*** provided F is a po-Γ-filter generated by {a} for some $a \in$ S. It is denoted by N(a).

NOTE 2.47 : For every $a \in$ S, the intersection of all po-Γ-filters containing a is again a po-Γ-filter and thus the least po-Γ-filter containing a.

THEOREM 2.48 : If N(b) ⊆ N(a), then N(a)\N(b), if it is nonempty, is a completely prime po-Γ-ideal of N(a).

LEMMA 2.49 : Let a, $b \in$ S and $b \in$ N(a), then N(b) ⊆ N(a).

COROLLARY 2.50 : Let a, $b \in$ S and $a \leq b$ then N(b) ⊆ N(a).

3. Γ-CONGRUENCES :

DEFINITION 3.1 : A relation ρ on a po-Γ-semigroup S is said to be ***reflexive*** if $x\rho x$ for all $x \in S$.

DEFINITION 3.2 : A relation ρ on a po-Γ-semigroup S is said to be ***symmetric*** if $x, y \in S$ and $x\rho y$ implies $y\rho x$.

DEFINITION 3.3 : A relation ρ on a po-Γ-semigroup S is said to be ***transitive*** if $x, y, z \in S$, $x\rho y$, $y\rho z$ implies $x\rho z$.

DEFINITION 3.4 : A relation ρ on a po-Γ-semigroup S is said to be an ***equivalence relation*** on S if (i) $x\rho x$ for all $x \in S$, (ii) $x, y \in S$, $x\rho y \Rightarrow y\rho x$ (iii) $x, y, z \in S$, $x\rho y$, $y\rho z$ implies $x\rho z$.

NOTE 3.5 : Let S be a po-Γ-semigroup. A relation ρ on S is an equivalence relation on S iff ρ is (i) reflexive (ii) symmetric and (iii) transitive on S.

DEFINITION 3.6 : Let S be a po-Γ-semigroup. If an equivalence relation ρ on S is a ***left Γ-congruence***, then for all $a, b, c \in S$, $a \rho b$ implies $(c\Gamma a)\rho(c\Gamma b)$.

NOTE 3.7 : An equivalence relation ρ on a po-Γ-semigroup S is said to be a ***left Γ-congruence*** if for all $a, b, c \in S$, $\alpha \in \Gamma$, $a \rho b$ implies $c\alpha a \rho c\alpha b$.

DEFINITION 3.8 : Let S be a po-Γ-semigroup. If an equivalence relation ρ on S is a ***right Γ-congruence***, then for all $a, b, c \in S$, $a \rho b$ implies $(a\Gamma c)\rho(b\Gamma c)$.

NOTE 3.9 : An equivalence relation ρ on a po-Γ-semigroup S is said to be a ***right Γ-congruence*** if for all $a, b, c \in S$, $\alpha \in \Gamma$, $a \rho b$ implies $a\alpha c \rho b\alpha c$.

DEFINITION 3.10 : Let S be a po-Γ-semigroup. If an equivalence relation ρ on S is a ***Γ-congruence***, then for all $a, b, c \in S$, $a \rho b$ implies $(c\Gamma a)\rho(c\Gamma b)$ and $(a\Gamma c)\rho(b\Gamma c)$.

NOTE 3.11 : An equivalence relation ρ on a po-Γ-semigroup S is said to be a ***Γ-congruence*** if for all $a, b, c \in S$, $\alpha \in \Gamma$, $a \rho b$ implies $c\alpha a \rho c\alpha b$, $a\alpha c \rho b\alpha c$.

NOTE 3.12 : An equivalence relation ρ on a po-Γ-semigroup S is said to be a Γ-congruence if it is both a left Γ-congruence and a right Γ-congruence.

THEOREM 3.13 : An equivalence relation ρ on a po-Γ-semigroup S is a Γ-congruence if and only if for all $a, b, c, d \in S$, $\alpha \in \Gamma$, $a \rho b$ and $c \rho d$ implies $a\Gamma c \rho b\Gamma d$.

Proof **:** Let ρ be an equivalence relation on a po-Γ-semigroup S.
Suppose that ρ is Γ-congruence. Let $a, b, c, d \in S$, $\alpha \in \Gamma$, $a \rho b$ and $c \rho d$

$a, b, c \in S$, $a \rho b$ and ρ is right Γ-congruence $\Rightarrow (a\Gamma c)\rho(b\Gamma c)$.
$b, c, d \in S$, $c \rho d$ and ρ is left Γ-congruence $\Rightarrow (b\Gamma c)\rho(b\Gamma d)$.

Now $a\Gamma c \rho b\Gamma c$, $b\Gamma c \rho b\Gamma d$, ρ is transitive $\Rightarrow (a\Gamma c)\rho(b\Gamma d)$.

Conversely suppose that ρ is an equivalence relation on a po-Γ-semigroup S such that $a, b, c, d \in S$, $a \rho b$ and $c \rho d \Rightarrow (a\Gamma c)\rho(b\Gamma d)$.
Now $c \rho c$, $a \rho b \Rightarrow (c\Gamma a)\rho(c\Gamma b) \Rightarrow$ ρ is a left Γ-congruence.

$a\,\rho\, b$, $c\,\rho\, c \Rightarrow (a\Gamma c)\rho(b\Gamma c) \Rightarrow \rho$ is a right Γ-congruence and hence ρ is a Γ-congruence.

NOTATION 3.14 : Let ρ be a Γ-congruence relation on a po-Γ-semigroup S. We denote the set $a_\rho = \{b \in S \,/\, a\rho b\}$ and is called ρ-class containing a. The set of all ρ-classes is denoted by the set S/ρ.

THEOREM 3.15 : The set of all ρ-classes S/ρ is a Γ-semigroup.

Proof **:** If a_ρ, $b_\rho \in S/\rho$, then we define the multiplication on S/ρ, given by $(a_\rho)\alpha(b_\rho) = (a\alpha b)_\rho$ for all $a, b \in S$, $\alpha \in \Gamma$. This is well defined, since for all $a, b, c, d \in S$ and $\alpha \in \Gamma$, $a_\rho = b_\rho$ and $c_\rho = d_\rho \Rightarrow (a, b), (c, d) \in \rho \Rightarrow (a\alpha c, b\alpha c), (b\alpha c, b\alpha d) \in \rho \Rightarrow (a\alpha c, b\alpha d) \in \rho \Rightarrow (a\alpha c)_\rho = (b\alpha d)_\rho$. Let $(a)_\rho$, $(b)_\rho$, $(c)_\rho \in S/\rho$, α , $\beta \in \Gamma$. Then $[(a)_\rho\alpha(b)_\rho]\beta(c)_\rho = (a\alpha b)_\rho\beta(c)_\rho = [(a\alpha b)\beta c]_\rho = [a\alpha(b\beta c)]_\rho = (a)_\rho\alpha(b\beta c)_\rho = (a)_\rho\alpha[(b)_\rho\beta(c)_\rho]$. Then S/ρ is a Γ-semigroup.

DEFINITION 3.16 : Let ρ be a Γ-congruence relation on a po-Γ-semigroup S. Then the set S/ρ of all ρ-classes with the multiplication $(a_\rho)\alpha(b_\rho) = (a\alpha b)_\rho$ for all $a, b \in S$, $\alpha \in \Gamma$ is called the ***quotient* Γ*-semigroup*** relative to the Γ-congruence ρ.

NOTE 3.17 : If S is a po-Γ-semigroup and ρ is a Γ-congruence on S, then is the set S/ρ a po-Γ-semigroup? A probable order on S/ρ could be the relation $\preccurlyeq$ on S/ρ defined by means of the order ≤ on S, that is, $(a)_\rho \preccurlyeq (b)_\rho \Leftrightarrow$ there exist $x \in (a)_\rho$ and $y \in (b)_\rho$ such that $x \leq y$. But the relation is not an order, in general. We show it in the following example.

EXAMPLE 3.18 : We consider the po-Γ-semigroup S = {a, b, c, d, e} and Γ = {α, β } defined by the multiplication and the order ≤ below:

α	a	b	c	d	e
a	a	e	c	d	e
b	a	e	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

β	a	b	c	d	e
a	a	e	c	d	e
b	a	b	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

and ≤ = { $(a, a), (a, d), (b, b), (c, c), (c, e), (d, d), (e, e)$ }.

For $x, y, z \in S$ and $\gamma, \mu \in \Gamma$, we have

$$(x\square y)\square a = a = x\square(y\square a),\ (x\square y)\square c = c = x\square(y\square c)$$

$$(x\square y)\square d = d = x\square(y\square d),\ (x\square y)\square e = e = x\square(y\square e)$$

$$(x\square y)\square b = e = x\square(y\square b),$$

$$(x\square y)\square b = e = x\square(y\square b) \text{ if } y \neq b$$

$$(x\square b)\square b = e = x\square(b\square b) \text{ if } x \neq b$$

$$(b\square b)\square b = e = b\square(b\square b),\ (b\square b)\square b = b = b\square(b\square b).$$

Then S is a Γ-semigroup. Since $x\square a \leq x\square d$, $a\square x = d\square x$, $x\square c \leq x\square e$, $c\square x = e\square x$ for all $x \in S$ and $\square \in \Gamma$, S is a po-Γ-semigroup.

Let ρ be the Γ-congruence on S defined as follows:

□ = { (a, a), (b, b), (c, c), (d, d), (e, e), (a, e), (e, a), (c, d), (d, c)}.

Let ≼ be an order on S/□ defined by means of the order ≤ on S, that is,

$(a)_\rho \preccurlyeq (b)_\rho \Leftrightarrow$ there exist $x \in (a)_\rho$ and $y \in (b)_\rho$ such that $x \le y$.

We have $a_□$ = {a, e}, $b_□$ = {b} and $c_□$ = {c, d}. Also we have $a_□ \preccurlyeq c_□$ and $c_□ \preccurlyeq a_□$ but $a_□ \neq c_□$. Thus ≼ is not an order relation on S/□ .

THEOREM 3.19 : Let S be a po-Γ- semigroup. If ρ_1 and ρ_2 are two left Γ-congruences (resp. right Γ-congruences, Γ-congruences) of S, then (ρ_1 o ρ_2) is a left Γ-congruence (resp. right Γ-congruence, Γ-congruence) of S.

Proof : Let ρ_1 and ρ_2 be two left Γ-congruences of S.
Suppose $a(\rho_1 \text{ o } \rho_2)b$ holds for $a, b \in$ S.

Then there exists c ∈ S such that $a\ \rho_1 c$ and $c\ \rho_2\ b$ hold.

Since ρ_1, ρ_2 are left Γ-congruences of S, it follows that $(s□a)\rho_1(s\alpha c)$ and $(s\alpha c)\rho_2(s\alpha b)$ for all $s \in$ S, □ ∈ Γ . This implies that $(s\alpha a)(\rho_1 o \rho_2)(s\alpha b)$ hold for all $s \in$ S, □ ∈ Γ and hence $(\rho_1 o \rho_2)$ is a left Γ-congruence of S. Similarly, we can prove the remaining cases.

From theorem 319, it can be easily prove the following result by induction:

COROLLARY 3.20 : Let S be a po-Γ-semigroup. If $\rho_1, \rho_2, ..., \rho_n$ are left □-congruences (resp. right congruences, congruences) of S, then $\rho_1 o \rho_2 o.....\rho_n$ is a left □-congruence (resp. right congruence, congruence) of S.

THEOREM 3.21 : The intersection of family of □-congruences on a po-□-semigroup S is again a □-congruence on S.

Proof **:** Let $\{\rho_i \,/\, i \in \Delta\}$ be the set of all Γ-congruences on S.

Let $\psi = \bigcap\limits_{i\in\Delta} \rho_i$. Let $a, b, c, d \in$ S, □ ∈ Γ. Suppose that $a\ \psi\ b$, $c\ \psi\ d$.

$a\ \psi\ b,\ c\ \psi\ d \Rightarrow a \bigcap\limits_{i\in\Delta} \rho_i\ b,\ c \bigcap\limits_{i\in\Delta} \rho_i\ d \Rightarrow a\ \rho_i\ b,\ c\ \rho_i\ d$ for all ρ_i

$\Rightarrow a□c\ \rho_i\ b□d$ for all $\rho_i \Rightarrow a□c \bigcap\limits_{i\in\Delta} \rho_i\ b□d \Rightarrow a□c\ \psi\ b□d.$

Hence the intersection of any family of Γ-congruences on a po-Γ-semigroup S is again a Γ-congruence on S.

THEOREM 3.22 : The union of a non-empty family of Γ-congruences on a po-Γ-semigroup S is a □-congruence on S.

Proof **:** Let $\{\rho_i \,/\, i \in \Delta\}$ be the set of all Γ-congruencies on set S.

Let $\psi = \bigcup\limits_{i\in\Delta} \rho_i$ where ρ_i is a Γ-congruence on po-Γ-semigroup S.

Let $a, b, c \in$ S, □ ∈ Γ. Suppose that $a\ \psi\ b$.
$a\ \psi\ b \Rightarrow a \bigcup\limits_{i\in\Delta} \rho_i\ b \Rightarrow a\ \rho_i\ b$ for some ρ_i on S

$\Rightarrow a\ \rho_i\ b$ for some ρ_i on S, ρ_i is a Γ-congruence $\Rightarrow c\square a\ \rho_i\ c\square b \Rightarrow c\square a \bigcup\limits_{i\in\Delta} \rho_i\ c\square b$

$\Rightarrow c\square a\ \psi\ c\square b \Rightarrow \psi$ is left Γ-congruence.

Now $a\ \rho_i\ b$ for some ρ_i on S, ρ_i is a Γ-congruence $\Rightarrow a\square c\ \rho_i\ b\square c \Rightarrow a\square c \bigcup\limits_{i\in\Delta} \rho_i\ b\square c$

$\Rightarrow a\square c\ \psi\ b\square c \Rightarrow \psi$ is right Γ-congruence.

Ψ is a Γ-congruence on po-Γ-semigroup S. Therefore the union of a non-empty family of Γ-congruences on a po-Γ-semigroup S is a Γ-congruence on S.

NOTE 3.23 : The set of Γ-congruences on a po-Γ-semigroup *S* is denoted by *C*(*S*).

DEFINITION 3.24 : The intersection of all Γ-congruences on a po-Γ-semigroup S containing a binary relation ρ on S is called the ***□-congruence generated by □.***

DEFINITION 3.25 : A po-Γ-semigroup S is said to be a **Γ-*band*** if every element of S is a Γ-idempotent.

DEFINITION 3.26 : A po-Γ-semigroup S is said to be a ***□-semilattice*** if S is a commutative Γ-band.

DEFINITION 3.27 : A Γ-congruence ρ on a po-Γ-semigroup S is said to be ***□-semilattice* Γ*-congruence*** if for all $a, b \in$ S, $\square \in \Gamma \Rightarrow a\square a\ \rho\ a$ and $a\square b\ \rho\ b\square a$.

DEFINITION 3.28 : A semilattice Γ-congruence ρ on a po-Γ-semigroup S is said to be ***complete*** if for any $a, b \in$ S, $\square \in \Gamma$, $a \le b$ implies $a\ \rho\ a\square b$.

NOTE 3.29 : A semilattice Γ-congruence ρ on a po-Γ-semigroup S is said to be ***complete*** iff $a, b \in$ S, $a \le b$ implies $a\rho(a\Gamma b)$.

4. <u>□-CLASSES IN PO-□-SEMIGROUPS</u> :

NOTATION 4.1 : We denote □ the Γ-congruence relation on po-Γ-semigroup S defined by □ = {(a, b) ∈ S × S / N(a) = N(b)} and let $a \in$ S, $(a)_\square$ denotes the □-class of S containing a. Let S/□ = {$(a)_\square$ / $a \in$ S}. Let $(a)_\square$, $(b)_\square \in$ S/□, □ ∈ Γ, then there is a well-defined multiplication on the quotient set S/□, given by $(a)_\square\ \alpha(b)_\square = (a\square b)_\square$.

The following theorem is due to K.Hila and E. Pisha [11]

THEOREM 4.2 : Let S/□ = {$(a)_\square$ / $a \in$ S} and $(a)_\square$, $(b)_\square \in$ S/□. If we define $(a)_\square \preccurlyeq (b)_\square$ if and only if $(a)_\square = (a\gamma b)_\square$ for all □ ∈ Γ, then the set S/□ is a po-Γ-semigroup induced by the complete semilattice Γ-congruence □ on S.

The following theorem is due to K.Hila and E. Pisha [11]

THEOREM 4.3 : The □-class $(a)_\square$ is a po-□-subsemigroup of S.

THEOREM 4.4 : If $a \le b$ for a, b in a po-□-semigroup S, then (a, $a\square b$) ∈ □ for every □ ∈ □.

Proof **:** Let *a*, *b* ∈ S and *a* ≤ *b*. Since N(*a*) is a po-Γ-filter generated by *a*. We have *b* ∈ N(*a*). Since N(*a*) is a po-Γ-subsemigroup and *a*, *b* ∈ N(*a*), we have *a*□*b* ∈ N(*a*) for all □ ∈ Γ. Hence N(*a*□*b*) ⊆ N(*a*). Now *a*□*b* ∈ N(*a*□*b*) and N(*a*□*b*) is a Γ-filter and hence *a* ∈ N(*a*□*b*). Thus N(*a*) ⊆ N(*a*□*b*). Therefore N(*a*□*b*) = N(*a*) and hence (*a*, *a*□*b*) ∈ □.

THEOREM 4.5 : In a po-□-semigroup S, $(a)_□ = (b)_□$ if and only if $(a)_□ = (a□b)_□$ for all □ ∈ □.

Proof **:** Let $(a)_□ = (b)_□$ ⇒ N(*a*) = N(*b*) ⇒ *a*, *b* ∈ N(*a*) ⇒ *a*□*b* ∈ N(*a*) for all □ ∈ Γ. Hence N(*a*□*b*) ⊆ N(*a*). Now *a*□*b* ∈ N(*a*□*b*) implies *a* ∈ N(*a*□*b*). Thus N(*a*) ⊆ N(*a*□*b*). Therefore N(*a*□*b*) = N(*a*) and hence $(a)_□ = (a□b)_□$ for all □ ∈ Γ.
Conversely suppose that $(a)_□ = (a□b)_□$ for all □ ∈ Γ. Then $(a)_□ = (a□b)_□$ ⇒ N(*a*□*b*) = N(*a*) ⇒ *a*□*b* ∈ N(*a*) ⇒ *b* ∈ N(*a*) ⇒ N(*a*) = N(*b*). Therefore $(a)_□ = (b)_□$.

THEOREM 4.6 : Let S be a po-□-semigroup, F is a po-□-filter of S, *a* ∈ F ∩ $(z)_□$ for *z* ∈ S. Then $(z)_□$ ⊆ F.

Proof **:** Let *y* ∈ $(z)_□$. Then $(y)_□ = (z)_□ = (a)_□$. Therefore (*y*, *a*) ∈ □ and N(*y*) = N(*a*).
Since F is a po-Γ-filter of S, *a* ∈ F, we have N(*a*) ⊆ F. Thus *y* ∈ N(*y*) = N(*a*) ⊆ F and hence $(z)_□$ ⊆ F.

THEOREM 4.7 : Let S be a po-□-semigroup and *z* ∈ S. If A is a po-□-ideal of an □-class $(z)_□$, then A has no proper completely prime po-□-ideals.

Proof **:** Let S be a po-Γ-semigroup, *z* ∈ S and A be a po-Γ-ideal of $(z)_□$. It suffices to show that A, itself, is the only po-Γ-filter of A.

Let F be a nonempty po-Γ-filter of A. Let *a* ∈ F □ ∈ Γ. Let T = { *x* ∈ S / $(a□)^2 x$ ∈ F}.
a ∈ F ⇒ $(a\alpha)^2 a$ ∈ F ⇒ *a* ∈ T. Therefore T ≠ ∅.

Let *x*, *y* ∈ T. Now *y* ∈ T ⇒ $(a\alpha)^2 y$ ∈ F.

Since F ⊆ A ⊆ $(z)_□$, $(a\alpha)^2 y$ ∈ F ⇒ $(a\alpha)^2 y \in (z)_□$. Therefore $((a\alpha)^2 y)_□ = (z)_□$ and $(a\alpha)^2 y$ ∈ F ⇒ *a*, *y* ∈ F ⇒ *y*□*a* ∈ F ⇒ $(y□a)_□ = ((a\alpha)^2 y)_□ = (z)_□$. Hence *y*□*a* ∈ $(z)_□$ and thus *y*□*a* ∈ A. Further, $(a\alpha)^2 y(□a)^2 = ((a\alpha)^2 y)(□a)^2$ ∈ F so that $y(□a)^2$ ∈ F.
Similarly $(a\alpha)^2 x$ ∈ F ⇒ $a\alpha x \in (z)_□$, which in turn yields $a\alpha x\beta y \in (z)_□$ for □ ∈ Γ

⇒ $(a\alpha)^2 x\beta y$ ∈ A. Since $(a\alpha)^2 x$, $y(□a)^2$ ∈ F ⇒ $(a\alpha)^2 x\beta y(□a)^2$ ∈ F ⇒ $(a\alpha)^2 x\beta y$ ∈ F.

Therefore $x\beta y$ ∈ T.

Conversely, let $x\beta y$ ∈ T for *x*, *y* ∈ S, □ ∈ Γ. Then $(a\alpha)^2 x\beta y$ ∈ F.

Therefore $(a\alpha)^2 x\beta y\,(\Gamma a)^2 = [(a\alpha)^2 x\beta y]\,(□a)^2$ ∈ F ⇒ $(a\alpha)^2 x\beta y \in (z)_□$ ⇒ $((a\alpha)^2 x\beta y)_□ = (z)_□$.

$(a\alpha x)_□\beta(a\alpha y)_□ = (z)_□$ ⇒ $(a\alpha x)_□\beta\,(y\alpha a)_□ = (z)_□$ ⇒ $(a\alpha x)\beta\,(y\alpha a) \in (z)_□$ ⇒ $(a\alpha x\beta y\alpha a)_□ = (z)_□$

⇒ $a\alpha x\beta y\alpha a \in (z)_□$ ⇒ $a\alpha x$, $y\alpha a \in (z)_□$ and hence $(a\alpha)^2 x$, $y(\Gamma a)^2$ ∈ A. But since $(a\alpha)^2 x\beta y(□a)^2$ ∈ F, it follows that $(a\alpha)^2 x$, $y(\Gamma a)^2$ ∈ F. As before, we infer that $(a\alpha)^2 y$ ∈ F. Since $(a\alpha)^2 y(□a)^2 = (a\alpha)^2\,(y(□a)^2)$ ∈ F and $(a\alpha)^2 y$ ∈ F. Thus *x*, *y* ∈ T.

Let *x* ∈ T, *y* ∈ S, *x* ≤ *y*. Now *x* ≤ *y* ⇒ $(a\alpha)^2 x \le (a\alpha)^2 y$ for all □ ∈ Γ.

x ∈ T ⇒ $(a\alpha)^2 x$ ∈ F ⇒ *a*, *x* ∈ F ⇒ *a* ∈ A, A is a po-Γ-ideal of $(z)_□$ ⇒ $(a\alpha)^2 y$ ∈ A.

Then, since F is a po-Γ-filter of A, we have $(a\alpha)^2 y \in$ F, $y \in$ T. Therefore T is a po-Γ-filter.

Now $(a\alpha)^2 y = a\alpha(a\alpha y)$; a, $a\alpha y \in$ F ⊆ A ⊆ $(z)_{\square} \Rightarrow a\alpha y \in (z)_{\square}$. Then, since A is po-Γ-ideal of $(z)_{\square}$, we have $a\alpha(a\alpha y) \in$ A Γ $(z)_{\square}$ ⊆ A and hence $(a\alpha)^2 y \in$ A.

$(a\alpha)^2 x \in$ F ⊆ A ⊆ $(z)_{\square} \Rightarrow (z)_{\square} = ((a\alpha)^2 x)_{\square} = (a\alpha a)_{\square}\, \Gamma\, (x)_{\square} = (a)_{\square}\, \Gamma\, (x)_{\square} = (a\alpha x)_{\square}$.

Now $x \leq y \Rightarrow a\alpha x \leq a\alpha y$ for $\square \in \Gamma \Rightarrow (a\alpha x,\ a\alpha x \square a\alpha y) \in \square \Rightarrow (a\alpha x)_{\square} = (a\alpha x \square a\alpha y)_{\square} = (a)_{\square}\alpha(x)_{\square}\beta(a)_{\square}\alpha(y)_{\square} = (a\alpha a)_{\square}\beta(x)_{\square}\alpha(y)_{\square} = ((a\alpha a\beta x)_{\square}\alpha(y)_{\square} = (z)_{\square}\Gamma(y)_{\square}$ where $z = a\alpha a\beta x \Rightarrow (a)_{\square}\alpha(y)_{\square} = (a\alpha y)_{\square}$.

Hence $a\alpha y \in (a\Gamma y)_{\square} = (a\alpha x)_{\square} = (z)_{\square}$. Since T is a po-Γ-filter of S, $a \in$ T, $a \in (z)_{\square}$.

By theorem 4.6, $(z)_{\square}$ ⊆ T.

Let $x \in$ F. Since $a\alpha a \in$ F, we have $(a\alpha)^2 y \in$ F, i.e., $y \in$ T.

Therefore F ⊆ T and F ⊆ A ⇒ F ⊆ T ∩ A. Let $y \in$ T ∩ A. Since $y \in$ T, $(a\alpha)^2 y \in$ F. Then, since $a\alpha a \in$ F ⊆ A, $y \in$ A, F is a po-Γ-filter of A, we have $y \in$ F and hence F = T ∩ A. Thus we get F ⊆ A ⊆ $(z)_{\square}$ ⊆ T ⇒ A ⊆ T ⇒ A ∩ T = A ⇒ A ∩ T = F = A.

COROLLARY 4.8 : Let S is a po-□-semigroup, A is a completely prime po-□-ideal of S. Then A = ∪{$(a)_{\square}$ / $a \in$ A}.

Proof **:** Let $t \in (a)_{\square}$ for some $a \in$ A. Since $(a)_{\square}$ is a po-Γ-ideal of $(a)_{\square}$, by theorem 4.7, $(a)_{\square}$ does not contain proper completely prime po-Γ-ideals. We prove that $(a)_{\square}$ ∩ A is a completely prime po-Γ-ideal of $(a)_{\square}$.

$a \in (a)_{\square}$, $a \in$ A ⇒ $a \in (a)_{\square}$ ∩ A. Therefore ∅ ≠ $(a)_{\square}$ ∩ A ⊆ $(a)_{\square}$. For all $\alpha \in \Gamma$, $(a)_{\square}\alpha((a)_{\square} \cap A) \subseteq ((a)_{\square}\alpha(a)_{\square}) \cap ((a)_{\square}\alpha A) = (a\alpha a)_{\square} \cap ((a)_{\square}\alpha A) \subseteq (a)_{\square} \cap (S\Gamma A) \subseteq (a)_{\square} \cap A$.
Similarly For all $\alpha \in \Gamma$, $((a)_{\square} \cap A)\alpha(a)_{\square} \subseteq ((a)_{\square}\alpha(a)_{\square}) \cap (A\alpha(a)_{\square}) \subseteq (a)_{\square} \cap (A\Gamma S) \subseteq (a)_{\square} \cap A$.

Let $a \in (a)_{\square} \cap$A, $b \in (a)_{\square}$ such that $b \leq a$. Since $b \leq a$, A is a po-Γ-ideal of S, we have $b \in$ A. Thus $b \in (a)_{\square} \cap$A.

Let $b, c \in (a)_{\square}$, $\square \in \Gamma$, $b\alpha c \in (a)_{\square} \cap$A. Since $b\alpha c \in$ A, A is a completely prime po-Γ-ideal of S, we have $b \in$ A or $c \in$ A. Hence $b \in (a)_{\square} \cap$A or $c \in (a)_{\square} \cap$A.

The following theorem is due to K.Hila [9]

THEOREM 4.9 : Let $a \in$ S. Then $(a)_{\square}$ is a semiprime po-□-ideal of N(a).

DEFINITION 4.10 : A po-Γ-semigroup without proper completely prime po-Γ-ideals is ***□-simple.***

NOTE 4.11 : In view of the theorem 4.7, every po-Γ-ideal of an □-class is □-simple.

COROLLARY 4.12 : Every po-□-semigroup is a semilattice of □-simple po-□-semigroups.

Proof **:** Let S be a po-Γ-semigroup. Clearly S can be partitioned into □-classes and those □-classes form a □-class which is □-simple. Therefore po-Γ-semigroup is semilattice of □-simple po-Γ-semigroups.

THEOREM 4.13 : If A is a completely prime po-□-ideal of a po-□-semiproup S, then J = { $(x)_{□}$ ∈ S/□ : x ∈ A} is a completely prime po-□-ideal of S/□. Conversely, if J is a completely prime po-□-ideal of S/□, then A = {x ∈ S : $(x)_{□}$ ∈ J} is a completely prime po-□-ideal of S. This establishes a one-to-one, order preserving(relative to inclusion) correspondence between the partially ordered set of all completely prime po-□-ideals of S and the partially ordered set of all completely prime po-□-ideals of S/□.

Proof **:** Suppose that $x, y \in$ S and $x\Gamma y \subseteq$ A ⇒ either $x \in$ A or $y \in$ A.

Let $(x)_{□}$, $(y)_{□} \in$ S/□, □ ∈ Γ and $(x)_{□}\alpha(y)_{□} \in$ J.

$(x)_{□}\alpha(y)_{□} \in$ J ⇒ $(x\alpha y)_{□} \in$ J ⇒ $x\alpha y \in$ A ⇒ either $x \in$ A or $y \in$ A ⇒ $(x)_{□} \in$ J or $(y)_{□} \in$ J.

Therefore J is a completely prime po-Γ-ideal of S/□.

Conversely suppose that J is a completely po-Γ-ideal of S/□.

Let $x, y \in$ S, □ ∈ Γ and $x\alpha y \in$ A.

Then $x\alpha y \in$ A ⇒ $(x\alpha y)_{□} \in$ J ⇒ $(x)_{□}\alpha(y)_{□} \in$ J ⇒ either $(x)_{□}$ or $(y)_{□} \in$ J

⇒ either $x \in$ A or $y \in$ A. Therefore A is a completely prime po-Γ-ideal of S.

Define the function φ from the set of all completely prime po-Γ-ideal of S to the set of all completely prime po-Γ-ideal of S/□ as φ(A) = J_A.

Suppose that $J_{A_1} = J_{A_2}$. $x \in A_1 \Leftrightarrow (x)_{□} \in J_{A_1} = J_{A_2} \Leftrightarrow x \in A_2$. Hence $A_1 = A_2$.

Therefore φ is one-to-one, order preserving correspondence.

THEOREM 4.14 : Every □-simple po-□-subsemigroup of a po-□-semigroup S is contained in an □-class of S.

Proof **:** Let T be a po-Γ-subsemigroup. If T ⊈ $(a)_{□}$ for any $a \in$ S. Then there exists $b \in$ T such that $b \notin (a)_{□}$. $b \notin (a)_{□}$ ⇒$(a)_{□} \neq (b)_{□}$.
Suppose that $(a)_{□} \supseteq (a\alpha b)_{□}$ for □ ∈ Γ, then F = T ∩ $(a)_{□}$ is a proper po-Γ-filter of T,

since $a \in$ F, $b \notin$ F. It is a contradiction for the fact that T is □-simple.

Therefore T ⊆ $(a)_{□}$ for some a. Hence T is contained in an □-class of S.

COROLLARY 4.15 : A po-Γ-semigroup S is □-simple if and only if S has a single □-class. The □-classes of S are precisely all maximal □-simple po-Γ-subsemigroups of S. The set $(a)_{□}$ can be characterized either as union of all □-simple po-Γ-subsemigroups of S containing *a* or as the greatest such po-Γ-subsemigroup.

Proof **:** Suppose that S is □-simple. Therefore S is an □-simple po-Γ-semigroup of S.
By the theorem 4.14, S is contained in an □-class of S. Therefore S has a single □-class.

Conversely suppose that S has a single □-class $(a)_{□}$. Therefore S is an po-Γ-ideal of $(a)_{□}$.
Hence S has no proper completely prime po-Γ-ideals. Therefore S is □-simple.

Let $(a)_{□}$ ⊆ T ⊆ S and T is □-simple.

Suppose T ≠ S. Then T is a □-simple po-Γ-subsemigroup.

Therefore T is contained in an □-class and hence T = $(a)_{□}$. ∴ $(a)_{□}$ is maximal. Hence the set $(a)_{□}$ can be characterized either as the union of all □-simple po-Γ-subsemigroup of S containing a or as the greatest such po-Γ-subsemigroup.

THEOREM 4.16 : The following condition on a po-□-ideal A of a po-□-semigroup S are equivalent

(i) A is the intersection of all completely prime po-□-ideals of S containing A.
(ii) A is the intersection of all minimal completely prime po-□-ideals of S containing A.
(iii) A is the union of □-classes.
(iv) A is a completely semiprime po-□-ideal of S.

***Proof* :** (i) ⇒ (ii)

Let $A=\bigcap_{\alpha\in\Delta} A_\alpha$ where each A_α is a completely prime po-Γ-ideals. Fix $\alpha \in \Delta$ and let □ be the partially ordered set of all completely prime po-Γ-ideals J of S for which $A \subseteq J \subseteq A_\alpha$. Then $A_\alpha \in$ □ and hence □ ≠ ∅. Let □ be a chain in □ and let $B=\bigcap_{D\in C} D$. Then $A \subseteq B \subseteq A_\alpha$ and B is a po-Γ-ideal of S. Now the partially ordered set { S\D : D ∈ □} form a chain. Let $x, y \in \bigcup_{D\in C} S\backslash D \Rightarrow x \in S\backslash D_1, y \in S\backslash D_2 \Rightarrow x \notin D_1, y \notin D_2$

Assume $D_1 \subseteq D_2$. So $x, y \notin D_1$. If possible for all $\gamma \in \Gamma$, $x□y \notin \bigcup_{D\in C} S\backslash D \Rightarrow x□y \notin S\backslash D_1$,
$\Rightarrow x□y \in D_1 \Rightarrow$ either $x \in D_1$ or $y \in D_1$. It is a contradiction.
Therefore $x□y \in S\backslash B=\bigcup_{D\in C} S\backslash D$ is a po-Γ-subsemigroup of S or empty, which show that B is a completely prime po-Γ-ideal. Therefore B ∈ □. Hence there is a minimal completely prime po-Γ-ideal $J_{□}$ (say) with the property such that $A \subseteq J_{□} \subseteq A_\alpha$. Therefore $A\subseteq \bigcap_{\alpha\in\Delta} J_\alpha \subseteq \bigcap_{\alpha\in\Delta} A_\alpha$ = A and hence A = $\bigcap_{\alpha\in\Delta} J_\alpha$.

(ii) ⇒ (iii) : Suppose that $A=\bigcap_{\alpha\in\Delta} J_\alpha$, the intersection of minimal completely prime po-Γ-ideal containing it. By theorem 4.8, $J_\alpha=\bigcup (x)_{□}$, for all $\alpha \in \Delta$. A = $\bigcap_{\alpha\in\Delta}\bigcup_{x\in J_x}(x)_{□}$.

Hence A is the union of □-classes.

(iii) ⇒ (iv) : If for all $\alpha \in \Gamma$, $x\alpha x \in A$, then $(x\alpha x)_{□} \subseteq A$. Now $(x\alpha x)_{□} = (x)_{□} \Rightarrow (x)_{□} \subseteq A \Rightarrow x \in A$. Therefore A is completely semiprime.

(iv) ⇒ (i) : We know that every semiprime po-Γ-ideal is the intersection of minimal prime po-Γ-ideals containing it. For every $d \notin A$, there exists a minimal prime po-Γ-ideal J containing A and not containing d. By theorem 2.31., J is completely prime. Thus A is the intersection of all completely prime po-Γ-ideals containing it. Therefore $A=\bigcap_{\alpha\in\Delta} J_\alpha$.

COROLLARY 4.17 : A po-□-semigroup S □-simple if and only if it contains no proper completely semiprime po-□-ideals.

***Proof* :** Suppose that po-Γ-semigroup S is □-simple, then S has no proper completely prime po-Γ-ideals. Let A be a proper completely prime po-Γ-ideal of S. Therefore A is the intersection of completely prime po-Γ-ideals. Therefore A = S. It is a contradiction. Hence S has no proper completely semiprime po-Γ-ideals.

Conversely suppose that S has no proper completely semiprime po-Γ-ideals. If A is a proper completely prime po-Γ-ideal of S, then A is a proper completely semiprime po-Γ-ideal. It is a contradiction. Therefore S has no proper completely prime po-Γ-ideals and hence S is □-simple.

COROLLARY 4.18 : If A is a completely semiprime po-□-ideal of a po-□-semiproup S, then J = { $(x)_{□}$ ∈ S/□ : x ∈ A} is a po-□-ideal of S/□. Conversely, if J is a po-□-ideal of S/□, then A = {x ∈ S : $(x)_{□}$ ∈ J} is a completely semiprime po-□-ideal of S. This establish a one-to-one, order preserving(relative to inclusion) correspondence between the partially ordered set of all completely semiprime po-□-ideal of S and the partially ordered set of all po-□-ideal of S/□.

***Proof* :** Suppose that A is a completely semiprime po-Γ-ideal of a po-Γ-semigroup S.

J ={$(x)_{□}$ ∈ S/□ : x ∈ A}. Let $(x)_{□}$ ∈ S/□ and $(a)_{□}$ ∈ J

⇒ for all $\alpha \in \Gamma$, $(x)_{□}\alpha(a)_{□} = (x\alpha a)_{□}$ ∈ J.

Now $(x)_{□} \subseteq (a)_{□}$ ∈ J ⇒ $(x)_{□}$ ∈ J. Therefore J is a po-Γ-ideal of S/□.

Conversely suppose that J is a po-Γ-ideal of S/□. Let $x \in$ S, $\alpha \in \Gamma$ $x\alpha x \in$ A.

$x\alpha x \in$ A ⇒ $(x)_{□}\alpha(x)_{□} = (x\alpha x)_{□}$ ∈ J ⇒ $(x)_{□}$ ∈ J ⇒ x ∈ A.

Therefore A is a completely semiprime po-Γ-ideal of S.

Define the function φ from the set of all completely semiprime po-Γ-ideal of S to the set of all po-Γ-ideal of S/□ as φ(A) = J_A. Suppose that $J_{A_1} = J_{A_2}$. $x \in A_1$

⇔$(x)_{□} \in J_{A_1} = J_{A_2}$ ⇔ $x \in A_2$. Hence $A_1 = A_2$.

Therefore φ is one-to-one, order preserving correspondence.

DEFINITION 4.19 : A nonempty subset □ of a po-Γ-semigroup S is an □-***subset*** if □ is completely semiprime and satisfies the condition: for any $y \in$ S, $z \in S^1$, $x \in$ □, $\alpha \in \Gamma$ and $y\alpha z \in$ □ implies $x\alpha y \in$ □ and $z\alpha x \in$ □.

NOTE 4.20 : A nonempty subset □ of a po-Γ-semigroup S is an □-***subset*** if □ is completely semiprime and satisfies the condition: for any x, $y \in$ S and $z \in S^1$, $x \in$ □ and $y\Gamma z \subseteq$ □ implies $x\Gamma y \subseteq$ □ and $z\Gamma x \subseteq$ □.

THEOREM 4.21 : Let □ be an □-subset of a po-□-semigroup S. Then for any a, b ∈ S and $x \in S^1$, x□a□b ⊆ □ implies x□b□a ⊆ □.

***Proof* :** Suppose c, $d \in$ S, $c\Gamma d \subseteq$ □, then $d\Gamma(c\Gamma d) \subseteq$ □, $c\Gamma d \subseteq$ □.

Hence $(d\Gamma c)\Gamma(d\Gamma c) = d\Gamma(c\Gamma d)\Gamma c \subseteq$ □ ⇒ $d\Gamma c \subseteq$ □. Thus $c\Gamma d \subseteq$ □ ⇒ $d\Gamma c \subseteq$ □. We will use this several times.

Let $x, a, b \in$ S with $x\Gamma a\Gamma b \subseteq \square$. Then $(a\Gamma b)\Gamma x \subseteq \square$, $b\Gamma(x\Gamma a) \subseteq \square$.
So that $a\Gamma(b\Gamma x\Gamma b) = (a\Gamma b)\Gamma x\Gamma b \subseteq \square$.
Hence $(b\Gamma x\Gamma b)\Gamma a \subseteq \square$, $x\Gamma(a\Gamma b) \subseteq \square \Rightarrow b\Gamma(x\Gamma b\Gamma a\Gamma x) = (b\Gamma x\Gamma b)\, \Gamma a\Gamma x \subseteq \square$.
Consequently $(x\Gamma b\Gamma a\Gamma x)\Gamma b \subseteq \square$, $a\Gamma(b\Gamma x) \subseteq \square$.
So that $(x\Gamma b\Gamma a)\Gamma(x\Gamma b\Gamma a) = (x\Gamma b\Gamma a\Gamma x)\Gamma b\Gamma a \subseteq \square$ and hence $x\Gamma b\Gamma a \subseteq \square$.

THEOREM 4.22 : A nonempty subset □ of a po-□-semigroup S is an □-subset. Then □ is a class of semilattice □-congruences.

Proof **:** Suppose that □ is an □-subset. Define a relation ρ on S by $a\rho b$ if for every $x \in S^1$, $x\Gamma a \subseteq \square$ if and only if $x\Gamma b \subseteq \square$. It is clear that ρ is reflexive and symmetric. Suppose $a\rho b$, $b\rho c$. If $a\rho b \Rightarrow x\Gamma a \subseteq \square$ iff $x\Gamma b \subseteq \square$ and $b\rho c \Rightarrow x\rho b \subseteq \square$ iff $x\rho c \subseteq \square$. Therefore $x\Gamma a \subseteq \square$ iff $x\Gamma b \subseteq \square$ iff $x\rho c \subseteq \square \Rightarrow x\Gamma a \subseteq \square$ iff $x\rho c \subseteq \square \Rightarrow a\rho c$. Therefore ρ is transitive and hence ρ is an equivalence relation.
Let $a\rho b$ and $c \in S \Rightarrow x\Gamma a \subseteq \square$ iff $x\Gamma b \subseteq \square \Rightarrow (x\Gamma c)\Gamma a \subseteq \square$ iff $(x\Gamma c)\Gamma b \subseteq \square$

$\Rightarrow$ $x\Gamma(c\Gamma a) \subseteq \square$ iff $x\Gamma(c\Gamma b) \subseteq \square \Rightarrow (c\Gamma a)\, \rho (c\Gamma b)$. Therefore ρ is left Γ-congruence.

Let $a\rho b$ and $c \in S \Rightarrow x\Gamma a \subseteq \square$ iff $x\Gamma b \subseteq \square$. Let $x \in S^1$, $x\Gamma(a\Gamma c) \subseteq \square$ iff $x\Gamma(c\Gamma a) \subseteq \square$ iff $(x\Gamma c)\Gamma a \subseteq \square$ iff $(x\Gamma c)\Gamma b \subseteq \square$ iff $x\Gamma(c\Gamma b) \subseteq \square$ iff $x\Gamma(b\Gamma c) \subseteq \square$.

Therefore $x\Gamma(a\Gamma c) \subseteq \square$ iff $x\Gamma(b\Gamma c) \subseteq \square$ $\Rightarrow$ for all $\alpha \in \Gamma$, $(a\alpha c)\rho(b\alpha c)$ and hence ρ is right Γ-congruence. Therefore ρ is Γ-congruence. Let $x\Gamma(a\Gamma b) \subseteq \square$ iff $x\Gamma(b\Gamma a) \subseteq \square$

$\Rightarrow$ for all $\alpha \in \Gamma$, $(a\alpha b)\rho(b\alpha a)$. Therefore S/ρ is a commutative po-Γ-semigroup.

THEOREM 4.23 : A po-□-semigroup S is separative if and only if S is a semilattice of strongly □-cancellative po-□-semigroup. If so, the relation □ defined on S by *x* □ *y* if for any *a, b* ∈ S, *x*□*a* = *x*□*b* if and only if *y*□*a* = *y*□*b* is the greatest band □-congruence on S all whose classes are strongly □-cancellative.

Proof **:** Let σ defined on S by $x\ \sigma\ y$ if for any $a, b \in S$, $x\Gamma a = x\Gamma b$ if and only if $y\Gamma a = y\Gamma b$.

For any $a, b \in S$, $x\Gamma a = x\Gamma b$ if and only if $x\Gamma a = x\Gamma b \Rightarrow x\ \sigma\ x$ and hence σ is reflexive.

Now let $x\ \sigma\ y$. $x\sigma y \Rightarrow$ for any $a, b \in S$, $x\Gamma a = x\Gamma b$ if and only if $y\Gamma a = y\Gamma b \Rightarrow y\ \sigma\ x$ and hence σ is symmetric.

Let $x\ \sigma\ y$ and $y\ \sigma\ z$. $x\ \sigma\ y \Rightarrow$ for any $a, b \in S$, $x\Gamma a = x\Gamma b$ if and only if $y\Gamma a = y\Gamma b$ and

$y\ \sigma\ z \Rightarrow$ for any $a, b \in S$, $y\Gamma a = y\Gamma b$ if and only if $z\Gamma a = z\Gamma b$.

Therefore $x\Gamma a = x\Gamma b$ if and only if $z\Gamma a = z\Gamma b$

$\Rightarrow x\ \sigma\ z$

$\Rightarrow$ σ is transitive and hence σ is an equivalence relation.

Let $x\ \sigma\ y$ and $z \in S$. Suppose that $(x\Gamma z)\Gamma a = (x\Gamma z)\Gamma b \Leftrightarrow x\Gamma(z\Gamma a) = x\Gamma(z\Gamma b)$

$\Leftrightarrow y\Gamma(z\Gamma a) = y\Gamma(z\Gamma b) \Leftrightarrow (y\Gamma z)\Gamma a = (y\Gamma z)\Gamma b$.

Therefore $(x\Gamma z)\ \sigma\ (y\Gamma z)$ and hence σ is right Γ-congruence.

If $(z\Gamma x)\Gamma a = (z\Gamma x)\Gamma b \Leftrightarrow z\Gamma x\Gamma a = z\Gamma x\Gamma b \Leftrightarrow x\Gamma a\Gamma z = x\Gamma b\Gamma z$ (by theorem 2.18 (i))

$\Leftrightarrow x\Gamma(a\Gamma z) = x\Gamma(b\Gamma z)$ (by definition of σ) $\Leftrightarrow y\Gamma(a\Gamma z) = y\Gamma(b\Gamma z)$ (by theorem 2.18 (i)) $\Leftrightarrow (a\Gamma z)\Gamma y = (b\Gamma z)\,\Gamma y \Leftrightarrow a\Gamma(z\Gamma y) = b\Gamma(z\Gamma y)$ (by theorem 2.18 (i)) $\Leftrightarrow (z\Gamma y)\Gamma a = (z\Gamma y)\Gamma b \Leftrightarrow (z\Gamma x)\ \sigma\ (z\Gamma y)$. Therefore σ is left Γ-congruence. Therefore σ is a Γ-congruence.

By theorem 2.18 (ii), for any $x, y, a, b \in S$, $(x\Gamma)^2a = (x\Gamma)^2b \Rightarrow x\Gamma a = x\Gamma b$

and if $x\Gamma a = x\Gamma b \Rightarrow x\Gamma(x\Gamma a) = x\Gamma(x\Gamma b) \Rightarrow (x\Gamma)^2a = (x\Gamma)^2b$.

Therefore $(x\Gamma)^2a = (x\Gamma)^2b$ if and only if $x\Gamma a = x\Gamma b$ and hence $(x\Gamma x)\ \sigma\ x$.

Therefore S/σ is a band. Therefore σ is a band Γ-congruence.

Again by theorem 2.18 (iii) for any $x, y, a, b \in S$, $x\Gamma y\Gamma a = x\Gamma y\Gamma b$ implies $y\Gamma x\Gamma a = y\Gamma x\Gamma b$

and $y\Gamma x\Gamma a = y\Gamma x\Gamma b$ implies $x\Gamma y\Gamma a = x\Gamma y\Gamma b \Rightarrow x\Gamma y\Gamma a = x\Gamma y\Gamma b$ if and only if $y\Gamma x\Gamma a = y\Gamma x\Gamma b$.

Therefore $(x\Gamma y)\ \sigma\ (y\Gamma x)$ and hence S/σ is commutative.

Therefore σ is a semilattice Γ-congruence.

Suppose that $z\Gamma x = z\Gamma y$ and $x\ \sigma\ z$, $y\ \sigma\ z$.

By definition of σ, for any $x, y \in S$, $x\ \sigma\ z \Rightarrow x\Gamma x = x\Gamma y$ if and only if $z\Gamma x = z\Gamma y$ and $y\ \sigma\ z \Rightarrow z\Gamma x = z\Gamma y \Leftrightarrow y\Gamma x = y\Gamma y$. Therefore $x\Gamma x = x\Gamma y$, $y\Gamma x = y\Gamma y$

and S is separative po-Γ-semigroup $\Rightarrow x = y$.

Therefore $z\Gamma x = z\Gamma y \Rightarrow x = y$ and hence σ is strongly left Γ-cancellative.

Again suppose that $x\Gamma z = y\Gamma z$ with $x\ \sigma\ z$, $y\ \sigma\ z$.

By theorem 2.18 (i), $x\Gamma z = y\Gamma z$ if and only if $z\Gamma x = z\Gamma y$. But $z\Gamma x = z\Gamma y \Rightarrow x = y$.

Therefore $x\Gamma z = y\Gamma z \Rightarrow x = y$ and hence each σ is strongly right Γ-cancellative.

Therefore each σ-class is strongly Γ-cancellative.

Conversely suppose that τ is a semilattice Γ-congruence on S all of whose classes are strongly Γ-cancellative. Let $x, y \in S$.

If $x\Gamma x = x\Gamma y$ and $y\Gamma y = y\Gamma x$, then $x\ \tau\ (x\Gamma y)$ and $y\ \tau\ (y\Gamma x)$.

$y\ \tau\ (y\Gamma x) \Rightarrow (y\Gamma x)\ \tau\ y \Rightarrow (x\Gamma y)\ \tau\ y$. Therefore $x\ \tau\ (x\Gamma y)$, $(x\Gamma y)\ \tau\ y \Rightarrow x\ \tau\ y$.

Thus the equation $x\Gamma x = x\Gamma y$ implies $x = y \Rightarrow x\Gamma x \le x\Gamma y$ implies $x = y$.

Similarly $x\Gamma x = y\Gamma x$ and $y\Gamma y = x\Gamma y$ implies $x = y \Rightarrow$ for all $\alpha \in \Gamma$, $x\alpha x \le y\alpha x$ and $y\alpha y \le x\alpha y$ implies $x = y$. Therefore S is separative.

Let ξ is a band Γ-congruence of S all of whose classes are strongly Γ-cancellative.

If $x\ \xi\ y$ and $x\Gamma a = x\Gamma b$. Then $x\ \xi\ y \Rightarrow (x\Gamma a)\ \xi\ (y\Gamma a)$ and $(x\Gamma b)\ \xi\ (y\Gamma b) \Rightarrow (x\Gamma a)\ \xi\ (y\Gamma b)$. Consequently the three sets $x\Gamma a$, $y\Gamma a$, $y\Gamma b$ are all contained in the same ξ-class.

By theorem 2.18 (iii), to $y\Gamma x\Gamma a = y\Gamma x\Gamma b \Rightarrow x\Gamma y\Gamma a = x\Gamma y\Gamma b$

and again to $(a\Gamma x)\Gamma(y\Gamma a) = (a\Gamma x)\Gamma(y\Gamma b) \Rightarrow (x\Gamma a)\Gamma(y\Gamma a) = (x\Gamma a)\Gamma(y\Gamma b)$. Hence the strongly Γ-cancellation law in the ξ-class containing $x\Gamma a$, $y\Gamma a$, $y\Gamma b$ implies $y\Gamma a = y\Gamma b$.

By symmetry, we conclude that $y\Gamma a = y\Gamma b$ implies $x\Gamma a = x\Gamma b$, which shows that $x\ \sigma\ y$. Therefore $\xi \subseteq \sigma$.

COROLLARY 4.24 : A po-□-semigroup S is separative if and only if every □-class of S is strongly □-cancellative.

THEOREM 4.25 : If S is a commutative and separative po-□-semigroup, then S is a semilattice of commutative strongly □-cancellative po-□-semigroup.

Proof **:** Define $x\ \sigma\ y$ if for all $a, b \in S$ $x\Gamma a = x\Gamma b$ if and only if $y\Gamma a = y\Gamma b$. By theorem 4.24, σ is Γ-congruence. Therefore σ is semilattice Γ-congruence. Each σ-class is a subset of S and S is commutative implies each σ-class is commutative. Therefore S is a semilattice of commutative strongly Γ-cancellative po-Γ-semigroup.

COROLLARY 4.26 : If a po-□-semigroup S is commutative and separative, then every □-class of S is commutative and strongly □-cancellative.

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