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A Finite Difference Method for the Numerical Solution of First Order Nonlinear Differential Equation

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ABSTRACT

In this paper a stable finite difference method for the numerical solution of the first order nonlinear differential equation is developed. The method is both A-stable and L-stable. This method is a modified form of trapezoidal method. Numerical results are given to illustrate the applicability of the method.

Keywords: nonlinear differential equation, finite difference scheme.
AMS (MOS) subject classification: 65F05, 65N30, 65N35, 65Y05.

1. INTRODUCTION

Consider the initial value problem(IVP)

$$y'(x) = f(x, y(x)),\ x \in (a,b), \qquad (1a)$$

$$y(a) = \varphi. \qquad (1b)$$

The trapezoidal method [5] for solving the IVP(1a,b) is

$$y_{n+1} = y_n + (h/2)\,[\,f(x_n, y_n) + f(x_{n+1}, y_{n+1})\,],\ n \geq 0, \qquad (2a)$$

$$y_0 = \varphi \qquad (2b)$$

where $x_n = nh$,$n \geq 0$, and h is the step size. The modified method from trapezoidal method (2a,b) available in the literature are – the method by Gourlay[4]

$$y_{n+1} = y_n + h\ f(x_n + h/2, (y_n + y_{n+1})/2),\ n \geq 0, \qquad (3a)$$

$$y_0 = \varphi\ , \qquad (3b)$$

the method by Evans and Sanugi[1] based on geometric mean,

$$y_{n+1} = y_n + h \sqrt{f(x_n, y_n) f(x_{n+1}, y_{n+1})} \ , \ n \geq 0 , \quad (4a)$$

$$y_0 = \varphi \quad (4b)$$

and the method by Evans and Sanugi[2]

$$y_{n+1} = y_n + h \ f(x_n + h/2 , \sqrt{y_n y_{n+1}} \ ) , n \geq 0 , \quad (5a)$$

$$y_0 = \varphi . \quad (5b)$$

The methods (2a,b)-(5a,b) are of order two. The methods (2a,b) and (3a,b) are A-stable but not L-stable. The methods (4a,b) and (5a,b) are both A-stable and L-stable. And so these methods are applicable for solving stiff differential equations. The presence of square root in the methods (4a,b) and (5a,b) leads to complexity in the evaluation will take more time for the computation. To reduce the computation time and to remove the complexity of the form of the methods, a one step method is presented in this paper in section 2. The stability analysis is given in section 3. In section 4 error analysis is presented. The application of the scheme presented in this paper is given in section 5. Numerical results are given in section 6.

2. ONE STEP METHOD:

The one step method for the numerical solution of the IVP(1a,b) is

$$y_{n+1} = y_n + h \ f(x_n + h/2 , (2 y_n \ y_{n+1}) / (y_n + y_{n+1})) , n \geq 0 , \quad (6a)$$

$$y_0 = \varphi . \quad (6b)$$

The method is based on the harmonic mean of the function values y_n and y_{n+1} at $x = x_n + h/2$. The method is consistent with the IVP(1a,b)as the step size h goes to zero[3].

3. STABILITY ANALYSIS

To study the stability properties of the method(6a,b) one can apply the method to the test equation

$$y'(x) = \lambda y(x) . \quad (7)$$

The equation (7) becomes

$$y_{n+1} = y_n + h \ (2 \lambda \ y_n \ y_{n+1}) / (y_n + y_{n+1}) \ , \quad (8)$$

or

$$y_{n+1} / y_n = \lambda h \pm \sqrt{1 + (\lambda h)^2} \ .$$

Taking only the positive sign, we have

$y_{n+1} / y_n = \lambda h + \sqrt{1+(\lambda h)^2}$.

Writing

$y_{n+1} / y_n = Q(\lambda h)$

the magnification factor , the absolute stability requires that

$$|Q(\lambda h)| < 1.$$

The condition $|Q(\lambda h)| < 1$. implies that

$$|\lambda h + \sqrt{1+(\lambda h)^2}| < 1. \quad (9)$$

To see for what values of λh the inequality (9) is valid. We examine the inequality (9) from two angles

a) **λh is real**

Letting $\lambda h = x$, where x is real, the inequality (9) become

$|x + \sqrt{1+x^2}| < 1.$

By plotting the function $f(z) = x + \sqrt{1+x^2}$ against x ,

we see that $f(x) < 1$ for all $x < 0$ and $f(x) \geq 1$ for all $x \geq 0$.

b) **λh is purely imaginary**

Letting $\lambda h = iy$ where y is real and $i = \sqrt{-1}$, the inequality (9) becomes

$| iy + \sqrt{1+(iy)^2} | < 1.$

Implies $| iy + \sqrt{1-y^2} | < 1.$

That is , $\pm \sqrt{1-y^2+y^2} < 1.$

That is , $\pm 1 < 1.$

This relationship will be true if we include the equality sign. This suggests that the imaginary axis of the complex plane is the boundary for the region for absolute stability of the solution of the numerical method. Thus the one step method (6a,b) is absolute stable (A-stable) for λh lying on the left half of the complex plane.

For the stiff equation , it is desired that $Q(\lambda h) \to 0$ for a very large step size h, that is , as $\lambda h \to -\infty$. This is the L-stability requirement. For the method (6a,b), $Q(\lambda h) \to 0$ as $\lambda h \to -\infty$.We may derive it as follows

As $\lambda h \rightarrow -\infty$., $\lim Q(\lambda h) = \lim [\ \lambda h + \sqrt{1+(\lambda h)^2}\ ]$

$$= \lim [\ x + \sqrt{1+x^2}\ ] \quad \text{as } x = \lambda h \rightarrow -\infty$$

$$= \lim [\ -x + \sqrt{1+x^2}\ ] \quad \text{as } -x = -\lambda h \rightarrow \infty$$

$$= 0.$$

Therefore, the one step method (6a,b) is both A-stable and L-stable.

4. ERROR ANALYSIS

The Taylor's series expansion for $y(x_{n+1})$ about x_n is

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + (h^2/2)y''(x_n) + (h^3/6)y'''(x_n) + O(h^4) \qquad (10)$$

The method (6a,b) can be written as

$$y_{n+1} = y(x_n) + h\ f(x_n + h/2\ , (2\ y(x_n)\ y(x_{n+1})) / (\ y(x_n) + y(x_{n+1}))) .$$

Using this Taylor's series expansion (10) for $y(x_{n+1})$ about x_n

$$y_{n+1} = y(x_n) + hy'(x_n) + (h^2/2)y''(x_n) +$$
$$(h^3/4)\ [\ y'''(x_n) - (y'(x_n))^2\ / y(x_n)\ ]\ \partial f/\partial y\ +$$
$$(h^3/8)\ [\ y'''(x_n) - y''(x_n)\ \partial f/\partial y\ ] + O(h^4) \qquad (11)$$

From (10) and (11) the truncation error τ_{n+1} is of the form

$$\tau_{n+1} = y(x_{n+1}) - y_{n+1}$$
$$= (h^3/24)\ [y'''(x_n)\ ] - (h^3/8)\ [y''(x_n) -2\ (y'(x_n))^2\ / y(x_n)\ ]\ \partial f/\partial y + O(h^4).$$

The truncation error τ_{n+1} is of order $O(h^3)$ and so the one step method (6a,b) is of order two.

Therefore, the one-step method (6a,b) is consistent with the IV P(1a,b), A-stable, L-stable and it is of order two.

5.APPLICATION

Gourlay[4] shows that the application of the method (2a,b) for solving certain types of problems, namely

$$y'(x) = \lambda(x)\ y(x)\ ,\ \lambda(x) \leq 0. \qquad (12)$$

leads to an undesirable property in the results.. In particular, he shows for certain functions $\lambda(x)$, the stability requirement imposes a restriction on the step size h to satisfy

$$h [\lambda(x_n) - \lambda(x_{n+1})] \leq 4 . \qquad (13)$$

Condition(13) is satisfied if $\lambda(x_n) \leq \lambda(x_{n+1})$ but if $\lambda(x_n) > \lambda(x_{n+1})$ then the condition(13) restricts the step size h to lie in the interval

$$0 < h \leq 4 [\lambda(x_n) - \lambda(x_{n+1})]^{-1} . \qquad (14)$$

To remove this restriction on the step size h Gourlay[4] provided the method (3a,b).

It is noted that , the one-step method presented in this paper (6a,b) itself enough for the problem of the type (12) .There is no restriction either on the coefficient $\lambda(x)$ or on the step size h for the stability of the one step method (6a,n).

On applying the method (6a,b) to the YVP(12),we have

$$y_{n+1} = y_n + h \ 2 \lambda (x_n + h/2) y_n \ y_{n+1} / (y_n + y_{n+1}) ,$$

On simplification,

$$Q (\lambda_n)^2 + [1 – 2h \ \lambda (x_n + h/2)] Q (\lambda_n) - 1 = 0 \qquad (15)$$

where $Q (\lambda_n) = y_{n+1} / y_n$. Equation(15) is quadratic in nature and it can be solved to obtain $Q (\lambda_n)$ as

$$Q (\lambda_n) = (-A \pm \sqrt{A + A^2})/2 \qquad (16)$$

where $A = 1 – 2h \lambda (x_n + h/2)$.

We require $Q (\lambda_n) \leq 1$ for the relation (16) to be acceptable. The inequality(16) is satisfied if $\lambda(x)$ is a negative function . Therefore, there is no restriction on the step size h and on the coefficient $\lambda(x)$ as for as the solution of $y'(x) = \lambda(x) y(x)$ is concerned by using the one-step method (6a,b) . Unlike the trapezoidal method (2a,b), a modification in the evaluation of the function should not be necessary. This property follows from the fact that the method (6a,b) is L-stable as applied to the trapezoidal for the method (2a,b) which is only A-stable.

6. NUMERIICAL RESULTS

In this section, the method is applied to the test problem[2]

$$y'(x) = \lambda(x) y(x), y(0) = 1 \qquad (17)$$

where

$\lambda(x) = \alpha^2 (x - \beta), 0 \leq x \leq \beta,$

$\lambda(x) = 0, x \geq \beta.$

The solution to this problem is given by

$y(x) = \exp (-\alpha^2 x (\beta - x/2)), 0 \leq x \leq \beta,$

$y(x) = \exp (-\alpha^2 \beta^2 /2), x \geq \beta.$

The stability condition for the trapezoidal method (2a,b) for the IVP(17) is $h \leq 2/\alpha$, but there is no restriction on h if the methods (3a,b), (4a,b) (5a,b) and (6a,n) are to be used. For computation, the choice of constants taken to be

$$\alpha = 1 \text{ and } \beta = 1$$

and for the values of h=0.1 and 0.01 . The numerical results obtained by using the method(6a,b) is shown in Table1. For small values of h the accuracy of the results obtained by all the methods are about the same.

Finally, the method is A-stable, L-stable, second order accurate and computationally cheaper than the methods obtained from arithmetic mean and geometric mean.

CONCLUSION

In this paper a one-step method is presented for the numerical solution of the first order nonlinear differential equation. The classical trapezoidal method is modified using harmonic mean of the function values $f(x_n, y_n)$ and $f(x_{n+1}, y_{n+1})$. The method is A-stable, L-stable, second order accurate and implicit in nature. The method presented in this paper is computationally cheaper than the methods obtained from arithmetic mean and geometric mean.

All computations were performed in Pascal single precision on a Micro Vax II computer at Bharathidasan University, Tiruchirapalli-620 024, Tamil Nadu, India.

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Table1.

h = 0.1 ,

x_i	y_i	$y(x_i)$	$y(x_i)$ - y_i
1.00000E-01	9.09503E-01	9.09373E-01	1.30057E-04
2.00000E-01	8.35475E-01	8.35270E-01	2.04980E-04
3.00000E-01	7.75157E-01	7.74916E-01	2.40743E-04
4.00000E-01	7.26406E-01	7.26149E-01	2..56777E-04
5.00000E-01	6.87550E-01	6.87289E-01	2.61128E-04
6.00000E-01	6.57308E-01	6.57047E-01	2.59280E-04
7.00000E-01	6.34703E-01	6.34448E-01	2.54750E-04
8.00000E-01	6.19033E-01	6.18783E-01	2.50041E-04
9.00000E-01	6.09818E-01	6.09571E-01	2.47180E-04
1.00000E+00	6.03677E-01	6.06531E-01	2,45929E-04

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h = 0.01

x_i	y_i	$y(x_i)$	$y(x_i)$ - y_i
1.00000E-01	9.09376E-01	9.09373E-01	3.33786E-06
2.00000E-01	8.35275E-01	8.35270E-01	5.18560E-06
3.00000E-01	7.74923E-01	7.74917E-01	6.13928E-06
4.00000E-01	7.26156E-01	7.26149E-01	6.61612E-06
5.00000E-01	6.87296E-01	6.87289E-01	6.67572E-06
6.00000E-01	6.57052E-01	6.57047E-01	6.61612E-06
7.00000E-01	6.34454E-01	6.34448E-01	6.49691E-06
8.00000E-01	6.18790E-01	6.18783E-01	6.37770E-06
9.00000E-01	6.09577E-01	6.09571E-01	6.19888E-06
1.00000E+00	6.06537E-01	6.06531E-01	6.13928E-06