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β-Homeomorphisms in Topological Ordered Spaces

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ABSTRACT

In this paper, we introduce I-β-homeomorphisms, D-β-homeomorphisms and B-β-homeomorphisms for topological ordered spaces after introducing I-β-continuous maps, D-β-continuous maps, B-β-continuous maps, I-β-open maps, D-β-open maps, B-β-open maps, I-β-closed maps, D-β-closed maps and B-β-closed maps for topological ordered spaces together with their characterizations.

KEYWORDS AND PHRASES. Topological ordered spaces, semi-open sets, semi-closed sets, pre-open sets, pre-closed sets, α-open sets, α-closed sets, β-open sets, β-closed sets increasing sets, decreasing sets, semi-continuous map, pre-continuous map, α-continuous map, β-continuous map, semi-open map, pre-open map, α-open map, β-open map, semi-closed map. pre-closed map, α-closed map and β-closed map.

1. **INTRODUCTION**. Leopoldo Nachbin [11] initiated the study of topological ordered spaces. A topological ordered space is a triple (X, τ, ≤), where τ is a topology on X and ≤ is a partial order on X. Let (X, τ, ≤) be a topological ordered space. For any $x \in X$, $[x, \rightarrow] = \{y \in X / x \leq y\}$ and $[\leftarrow, x] = \{y \in X / y \leq x\}$. A subset A of a topological ordered space (X, τ, ≤) is said to be increasing if A = i(A) and decreasing if A =d(A), where $i(A) = \bigcup_{a \in A} [a, \rightarrow]$ and $d(A) = \bigcup_{a \in A} [\leftarrow, a]$. Observe that the complement of an increasing set is a decreasing set and the complement of a decreasing set is an increasing set. A subset of a topological ordered space (X, τ, ≤) is said to be balanced if it is both increasing and decreasing. M.K.R.S. Veera Kumar [13] studied different types of maps between topological ordered-spaces. Abd-El-Monsef et al [1] introduced β-open sets and β-closed sets. A subset A of a topological space (X, τ) is called an α-open set [12] if $A \subseteq int(cl(int(A)))$ and α-closed [8] set if $cl(int(cl(A))) \subseteq A$. A subset A of a topological space (X, τ) is called β-open set [1] (semi-pre-open set [2])if $A \subseteq cl(int(cl(A)))$ and β-closed set [1](semi-pre-closedset) if $int(cl(int(A))) \subseteq A$. A subset A of a topological space (X, τ) is called a pre-open set [7] if $A \subseteq int(cl(A))$ and a pre-closed set if $cl(int(A)) \subseteq A$, a semi-open set [6] if $A \subseteq cl(int(A)$ and a semi-closed set if $int(cl(A)) \subseteq A$.

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LEMMA 1.1. Every semi-closed set is a β-closed set.
LEMMA 1.2. Every pre-closed set is a β-closed set.
LEMMA 1.3. Every α-closed set is a β-closed set.

Note that the complement of β-open set is β-closed and vice versa. We denote the complement of A by C (A).
For a subset A of a topological ordered space (X, τ, ≤), we define
iβcl(A) = ∩{F/F is an increasing β-closed subset of X containing A },
dβcl(A) = ∩{F/F is decreasing β-closed subset of X containing A},
bβcl(A) = ∩{F/F is a balanced β-closed subset of X containing A},
$A^{i\beta o}$ = ∪{G/G is an increasing β-open subset of X contained in A},
$A^{d\beta o}$ = ∪{ G/G is an decreasing β-open subset of X contained in A} and
$A^{b\beta o}$ = ∪{G/G is a balanced β-open subset of X contained in A}.

Clearly iβcl(A) (resp. dβcl (A), bβcl (A)) is the smallest increasing (resp. decreasing, balanced) β-closed set containing A.

IβO(X) (resp. DβO(X), BβO(X)) denotes the collection of all increasing (resp. decreasing, balanced) β-open subsets of a topological ordered space (X, τ, ≤). IβC(X) (resp. DβC(X), BβC(X)) denotes the collection of all increasing (resp. decreasing, balanced) β-closed subsets of a topological ordered space (X, τ, ≤).

A function f : (X, τ)→ (Y, σ) is called β-continuous [1] if f^{-1}(V) is a β-closed set of (X, τ) for every closed set V of (Y, σ). A function f : (X, τ) → (Y, σ) is called β-open [1] if f(G) is β-open set in Y for every open set G of X. A function f : (X, τ) → (Y, σ) is called β-closed [1] if f(F) is β-closed set in Y for each closed set F of X. A function f : (X, τ) → (Y, σ) is called semi-continuous [6] if f^{-1}(V) is a semi-open set of (X, τ) for every open set V of (Y, σ). A function f : (X, τ) → (Y, σ) is called pre-continuous [7] if f^{-1}(V) is a pre-closed set of (X, τ) for every closed set V of (Y, σ). A function f : (X, τ) → (Y,σ) is called α-continuous [8] if f^{-1}(V) is an α-closed set of (X, τ) for every closed set V of (Y,σ). A function f : (X, τ) → (Y,σ) is called pre-open map if f(G) is a pre-open set in (Y, σ) for every open set G of (X, τ). A function f : (X, τ) → (Y,σ) is called α-open map if f(G) is an α-open set in (Y, σ) for every open set G of (X, τ). A function f : (X, τ) → (Y,σ) is called semi-closed map if f(G) is a semi-closed set in (Y, σ) for every closed set G of (X, τ). A function f : (X, τ) → (Y,σ) is called pre-closed map if f(G) is a pre-closed set in (Y, σ) for every closed set G of (X, τ). A function f : (X, τ) → (Y,σ) is called α-closed map if f(G) is an α-closed set in (Y, σ) for every closed set G of (X, τ). A function f : (X, τ, ≤) → (X*, τ^*, ≤*) is called I-semi-continuous [5] (resp. D-semi-continuous, B-semi-continuous) map if f^{-1}(G)∈ISO(X) (resp. f^{-1}(G)∈DSO(X), f^{-1}(G)∈BSO(X) whenever G is an open set of (X*, τ^*). A function f : (X, τ, ≤) → (X*, τ^*, ≤*) is called I-pre-continuous [4] (resp. D-pre-continuous, B-pre-continuous) map if f^{-1}(G)∈IPC(X) (resp f^{-1}(G)∈DPC(X), f^{-1}(G)∈BPC(X)) whenever G is an closed set of (X*, τ^*). A function f : (X, τ, ≤) → (X*, τ^*, ≤*) is called I-α-continuous [3] (resp. D-α-continuous, B-α-continuous) map if f^{-1}(G)∈IαC(X) (resp f^{-1}(G)∈DαC(X), f^{-1}(G)∈BαC (X)) whenever G is an closed set of

(X^*, τ^*). A function f : $(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ is called an I-semi-open [5] (resp.a D-semi-open, B-semi-open) map if $f(G) \in ISO(X^*)$ (resp $f(G) \in DSO(X^*)$, $f(G) \in BSO(X^*)$) whenever G is an open subset of (X, τ). A function f : $(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ is called an I-pre-open [4] (resp.a D-pre-open, B-pre-open) map if $f(G) \in IPO(X^*)$ (resp $f(G) \in DPO(X^*)$, $f(G) \in BPO(X^*)$) whenever G is an open subset of (X, τ). A function f : $(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ is called an I-α-open [3] (resp. a D-α-open, B-α-open) map if $f(G) \in I\alpha O(X^*)$ (resp $f(G) \in D\alpha O(X^*)$, $f(G) \in B\alpha O(X^*)$) whenever G is an open subset of (X, τ). A function f : $(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ is called an I-semi-closed [5] (resp. a D-semi-closed, a B-semi-closed) map if $f(G) \in ISC(X^*)$ (resp $f(G) \in DSC(X^*)$, $f(G) \in BSC(X^*)$) whenever G is a closed subset of X. A function f : $(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ is called an I-pre-closed [4] (resp. a D-pre-closed, a B-pre-closed) map if $f(G) \in IPC(X^*)$ (resp $f(G) \in DPC(X^*)$, $f(G) \in BPC(X^*)$) whenever G is a closed subset of X, A function f : $(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ is called an I-α-closed [3] (resp. a D-α-closed, a B-α-closed) map if $f(G) \in I\alpha C(X^*)$ (resp $f(G) \in D\alpha C(X^*)$, $f(G) \in B\alpha C(X^*)$) whenever G is a closed subset of X. A bijection f : $(X,\tau,\leq) \rightarrow (X^*,\tau^*, \leq^*)$ is called an I-semi-homeomorphism [5] (resp. a D-semi-homeomorphism, a B-semi-homeomorphism) if both f and f^{-1} are I-semi-continuous (resp. D-semi-continuous and B-semi-continuous). A bijection f : $(X,\tau,\leq) \rightarrow (X^*,\tau^*, \leq^*)$ is called an I-pre-homeomorphism [4] (resp. a D-pre-homeomorphism, a B-pre-homeomorphism) if both f and f^{-1} are I-pre-continuous (resp. D-pre-continuous and B-pre-continuous). A bijection f : $(X,\tau,\leq) \rightarrow (X^*,\tau^*, \leq^*)$ is called an I-α-homeomorphism [3] (resp. a D-α-homeomorphism, a B-α-homeomorphism) if both f and f^{-1} are I-α-continuous (resp. D-α-continuous and B-α-continuous).

Authors studied Semi-homeomorphisms in Topological Ordered Spaces [5] pre-homeomorphisms in Topological Ordered Spaces [4] and α-homeomorphisms in Topological Ordered Spaces [3].

2. I-β-CONTINUOUS, D-β-CONTINUOUS AND B-β-CONTINUOUS MAPS

We introduce the following definition.

DEFINITION 2.01. A function f : $(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ is called an I-β-continuous (resp. D-β-continuous, B-β-continuous) maps if $f^{-1}(V) \in I\beta C(X)$, (resp. $f^{-1}(V) \in D\beta C(X)$, $f^{-1}(V) \in B\beta C(X)$) whenever V is closed in X.

It is evident that every x-β-continuous map is β-continuous for x = I, D, B and that every B-β-continuous map is both I-β-continuous and D-β-continuous.

The following example shows that a β-continuous map need not be x-β-continuous for x = I, D, B.

EXAMPLE 2.01. Let X = {a, b, c} τ = {ϕ, X, {a}, {b}, {a, b}} and ≤ {(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)}. Clearly $(X, \tau, \leq)$ is a topological ordered space. Let f be the identity map from $(X, \tau, \leq)$ onto itself. {a, c} is closed set in X. But $f^{-1}(\{a, c\})$ is neither an increasing nor a decreasing β-closed set. Thus f is not x-β-continuous for x = I, D,B.

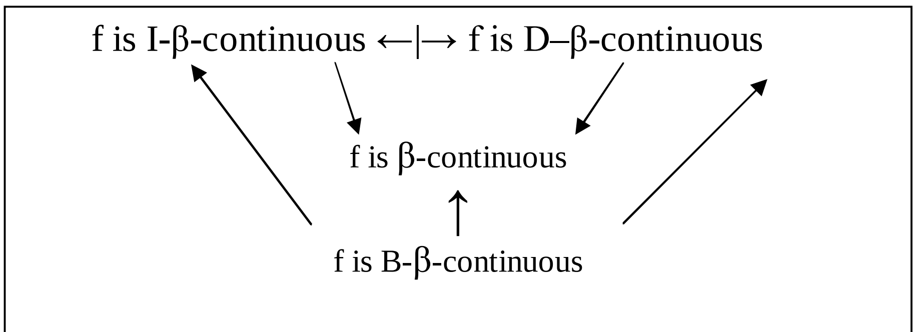
The following example shows that a D-β-continuous map need not be a B-β-continuous.

EXAMPLE 2.02. Let X ={a, b, c} = X*, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and $\leq$ = {(a, a), (b, b), (c, c), (b, a)} and $\leq^*$ = {(a, b), (b, b), (c, c)}. Let g be the identity map from $(X, \tau, \leq)$ onto $(X^*, \tau^*, \leq^*)$. Then g is not a B-β-continuous, however g is a D-β-continuous map.

The following example supports that an I-β-continuous map need not be a B-β continuous map.

EXAMPLE 2.03. Let X = {a, b, c} = X*, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\} = \tau^*$ and $\leq$ = {(a, a), (b, b), (c, c), (a, c), (b, c)} = $\leq^*$. Define $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ is an identity map then f is an I-β-continuous but not a B-β-continuous map.

2.01 Thus we have the following diagram.
For a function $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$



,where p → q (resp. p ←|→ q) represents p implies q but q need not imply p (resp. p and q are independent of each other)

The following theorem characterizes I-β-continuous maps.

THEOREM 2.01. For a function $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ the following statements are equivalent.

1) f is I-β-continuous.
2) $f(i\beta cl(A)) \subseteq cl(f(A))$ for any $A \subseteq X$.
3) $i\beta cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$ for any $B \subseteq X^*$.
4) For any closed subset K of $(X^*, \tau^*, \leq^*)$, $f^{-1}(K)$ is an increasing β-closed subset of $(X, \tau, \leq)$.

Proof. (1) => (2) Since cl(f(A)) is closed in X* and f is I-β-continuous we have that $f^{-1}(cl(f(A)))$ is an increasing β-closed set in X. $f(A) \subseteq cl(f(A)) \Rightarrow A \subseteq f^{-1}(cl(f(A)))$ and iβcl(A) is the smallest increasing β-closed set containing A. Therefore $i\beta cl(A) \subseteq f^{-1}(cl(f(A)))$. Here $f(i\beta cl(A)) \subseteq cl(f(A))$.
(2) =>(3) Put $A = f^{-1}(B)$. Then $f(A) \subseteq B$ and $cl(f(A)) \subseteq cl(B)$. Therefore $i\beta cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$
(3) => (4) Let K be any closed set in X*. From (3) $i\beta cl(f^{-1}(K)) \subseteq f^{-1}(cl(K)) = f^{-1}(K)$. Clearly $f^{-1}(K) \subseteq i\beta cl(f^{-1}(K))$. Thus $f^{-1}(K)$ is increasing β-closed set in $(X, \tau, \leq)$ whenever K is closed in $(X^*, \tau^*, \leq^*)$.
(4) => (1) follows from definition.

THEOREM 2.02. For a function f : (X, τ, ≤) →(X*, τ*, ≤*), the following statements are equivalent.

1) f is D-β-continuous.
2) f (dβcl(A)) ⊆ cl(f(A)) for any A ⊆ X.
3) dβcl(f^{-1}(B)) ⊆ f^{-1}(cl(B)) for any B ⊆ X*.
4) For every closed subset K of (X*, τ*, ≤*), f^{-1}(K) is a decreasing β-closed subset of (X, τ, ≤).

THEOREM 2.03. For a function f : (X, τ, ≤) →(X*, τ,* ≤*), the following statements are equivalent.

1) f is B-β-continuous.
2) f (bβcl(A)) ⊆ cl (f(A)) for any A ⊆ X.
3) bβcl(f^{-1}(B)) ⊆ f^{-1}(cl(B)) for any B ⊆ X*.
4) For every closed subset K of (X*, τ*, ≤*), f^{-1}(K) is a balanced β-closed subset of (X, τ, ≤).

THEOREM 2.04. Every I-semi-continuous map is an I-β-continuous.

Proof. Follows from Lemma 1.1.

The following example shows that an I-β-continuous map need not be an I-semi-continuous map.

EXAMPLE 2.04. Let X = {a, b, c} = X*,τ = {ϕ, X, {a}, {b, c}} = τ* and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f(a) = b, f(b) = a and f(c) = c. Then f is I-β-continuous map, but not a I-semi-continuous map.

THEOREM 2.05. Every I-pre-continuous map is an I-β-continuous.

Proof. Follows from Lemma 1.2.

The following example shows that an I-β-continuous map need not be an I-pre-continuous map.

EXAMPLE 2.05. Let X ={a, b, c} = X*, τ = {ϕ, X, {a}, {b}, {a, b}}, τ*= {ϕ, X, {a}, {b, c}} and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = b, f (b) = a and f(c) = c. Then f is an I-β-continuous map, but not an I-pre-continuous map.

THEOREM 2.06. Every I-α-continuous map is an I-β-continuous map.

Proof. Follows from Lemma 1.3.

The following example shows that an I-β-continuous map need not be an I-α-continuous map.

EXAMPLE 2.06. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {b}, {a, b}}, τ*= {ϕ, X, {a}, {b, c}} and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = b, f (b) = a and f(c) = c. Then f is I-β-continuous map, but not an I-α-continuous map.

THEOREM 2.07. Every D-semi-continuous map is a D-β-continuous.

Proof. Follows from Lemma 1.1

The following example shows that a D-β-continuous map need not be a D-semi-continuous map.

EXAMPLE 2.07. Let X = {a, b, c} = X*, τ = {ϕ, X, {a},{b, c}}= τ* and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f(a) = b, f(b) = a and f(c) = c. Then f is D-β-continuous map, but not a D-semi-continuous map.

THEOREM 2.08. Every D-pre-continuous map is a D-β-continuous.

Proof. Follows from Lemma 1.2.

The following example shows that a D-β-continuous map need not be a D-pre-continuous map.

EXAMPLE 2.08. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {b}, {a, b}}, τ* = {ϕ, X, {a}, {b, c}} and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = b, f (b) = a and f(c) = c. Then f is a D-β-continuous map, but not a D-pre-continuous map.

THEOREM 2.09. Every D-α-continuous map is a D-β-continuous.

Proof. Follows from Lemma 1.3.

The following example shows that a D-β-continuous map need not be a D-α-continuous map.

EXAMPLE 2.09. Let X = {a, b, c}= X*, τ = {ϕ, X, {a}, {b}, {a, b}}, τ* = {ϕ, X, {a}, {b, c}} and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = b, f (b) = a and f(c) = c. Then f is D-β-continuous map, but not a D-α-continuous map.

THEOREM 2.10. Every B-semi-continuous map is a B-β-continuous.

Proof. Follows from Lemma 1.1.

The following example shows that a B-β-continuous map need not be a B-semi-continuous map.

EXAMPLE 2.10. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {b, c}}= τ* and let ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f(a) = b, f(b) = a and f(c) = c. Then f is B-β-continuous map, but not a B-semi-continuous map.

THEOREM 2.11. Every B-pre-continuous map is a B-β-continuous.

Proof. Follows from Lemma 1.2.

The following example shows that a B-β-continuous map need not be a B-pre-continuous map.

EXAMPLE 2.11. Let $X = \{a, b, c\} = X^*$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, $\tau^* = \{\phi, X, \{a\}, \{b, c\}\}$ and $\leq = \{(a, a), (b, b), (c, c)\} = \leq^*$. Define a map $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then f is B-β-continuous map, but not a B-pre-continuous map.

THEOREM 2.12. Every B-α-continuous map is a B-β-continuous.

Proof. Follows from Lemma 1.3.

The following example shows that a B-β-continuous map need not be a B-α-continuous map.

EXAMPLE 2.12. Let $X = \{a, b, c\} = X^*$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, $\tau^* = \{\phi, X, \{a\}, \{b, c\}\}$ and $\leq = \{(a, a), (b, b), (c, c)\} = \leq^*$. Define a map $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then f is a B-β-continuous map but not a B-α-continuous map.

3. I-β-OPEN, D-β-OPEN AND B-β-OPEN MAPS.

DEFINITION 3.01. A function $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ is called an I-β-open map (resp. D-β-open map, B-β-open map) if $f(G) \in I\beta O(X^*)$ (resp. $f(G) \in D\beta O(X^*)$, $f(G) \in B\beta O(X^*)$) whenever G is an open subset of $(X, \tau, \leq)$.

It is evident that every x-β-open map is an β-open map for x= I,D,B and that every B-β-open map is both I-β-open and D-β-open.

The following example shows that a β-open map need not be a x-β-open map for x= I,D,B.

EXAMPLE 3.01. Let $X = \{a, b, c\} = X^*$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}$ and $\leq = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. Let f be the identity map from $(X, \tau, \leq)$ onto itself. $\{b\}$ is an open set in X but $f(\{b\}) = \{b\}$ is not an I-β-open set, not a D-β-open set, not a B-β-open set in X^*. Therefore f is not an I-β-open map, not a D-β-open map and not a B-β-open map. Clearly f is a β-open map.

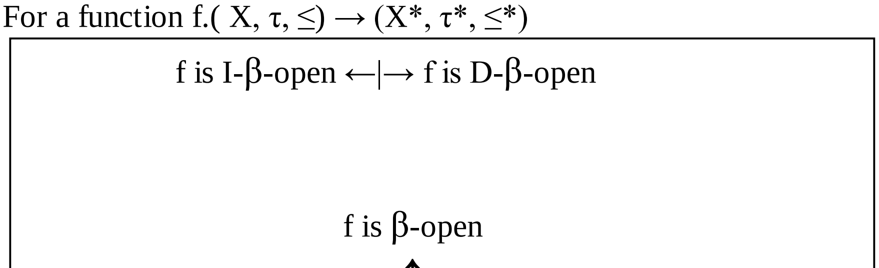
The following example shows that a D-β-open map need not be a B-β-open map.

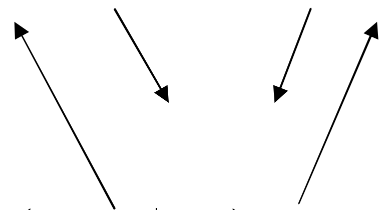
EXAMPLE 3.02. Let $X = \{a, b, c\} = X^*$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\} = \tau^*$, $\leq = \{(a, a), (b, b), (c, c), (a, c)\}$ and $\leq^* = \{(a, a), (b, b), (c, c), (a, c), (b, c)\}$. Let θ be the identity map from $(X, \tau, \leq)$ onto $(X^*, \tau^*, \leq^*)$. θ is D-β-open map but not B-β-open map.

EXAMPLE 3.03. Let $X = \{a, b, c\} = X^*$, $\tau = \{\phi, X, \{a\}, \{a, c\}\} = \tau^*$ and $\leq = \{(a, a), (b, b), (c, c), (c, a), (b, c), (b, a)\} = \leq^*$. Let $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be the identity map. Then f is an I-β-open map but not a B-β-open map.

3.01 Thus we have the following diagram

For a function $f.(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$





,where p→ q (resp.p ←|→ q) represents p implies q but q need not imply p (resp. p and q are independent of each other.

LEMMA 3.01. Let A be any subset of a topological ordered space $(X, \tau, \leq)$. Then

1) $C(d\beta cl(A)) = (C(A))^{i\beta o}$.
2) $C(i\beta cl(A)) = (C(A))^{d\beta o}$.
3) $C(b\beta cl(A)) = (C(A))^{b\beta o}$.

Proof. $C(d\beta cl(A)) = C\{\cap F/F$ is a decreasing β-closed subset of X containing A}
$= \cup\{C(F)/F$ is a decreasing β-closed subset of X containing A}
$= \cup\{G/G$ is an increasing β-open subset of X contained in C(A)}
$= (C(A))^{i\beta o}$.
Proofs of (2) and (3) are analogous to as that of (1) and hence omitted

The following theorem characterizes I-β-open functions.

THEOREM 3.01. For any function $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$, the following statements are equivalent.

1) f is an I-β-open map.
2) $f(A^0) \subseteq [f(A)]^{i\beta o}$ for any $A \subseteq X$.
3) $[f^{-1}(B)]^0 = f^{-1}(B^{i\beta o})$ for any $B \subseteq X^*$.

Proof. (1) => (3). Since $[f^{-1}(B)]^0$ is open in X and f is an I-β-open, $f((f^{-1}(B))^0) \subseteq f(f^{-1}(B)) \subseteq B$, then $f([f^{-1}(^1(B))]^0)$ is I-β-open in X* contained in B. Then $f((f^{-1}(B))^0) \subseteq B^{i\beta o}$ since $B^{i\beta o}$ is the largest increasing β-open set contained in B. Therefore $[f^{-1}(B)]^0 \subseteq f^{-1}(B^{i\beta o})$
(3) => (2). Replacing B by f(A) in (3), we have$[f^{-1}(f(A))]^0 \subseteq f^{-1}([f(A)]^{i\beta o})$. Since$A^0 \subseteq [f^{-1}(f(A)]^0$, we have $A^0 \subseteq f^{-1}([f(A)]^{i\beta o})$. $f(A^0) \subseteq f(f^{-1}([f(A)]^{i\beta o})) \subseteq [f(A)]^{i\beta o}$. Hence $f(A^0) \subseteq [f(A)]^{i\beta o}$.
(2) => (1). Let G be any open set in X. Then $f(G) = f(G^0) \subseteq [f(G)]^{i\beta o} \subseteq f(G)$. Therefore f(G) is an increasing β open set in X* and hence f is an I-β-open map.

The following two theorems give characterizations for D-β-open map and B-β-open maps, whose proofs are similar to as that of the above theorem.

THEOREM 3.02. For any function $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$, the following statements are equivalent.

1) f is D-β-open map.
2) $f(A^0) \subseteq [f(A)]^{d\beta o}$ for any $A \subseteq X$.
3) $[f^{-1}(B)]^0 \subseteq f^{-1}(B^{d\beta o})$ for any $B \subseteq X^*$.

THEOREM 3.03. For any function f : (X, τ, ≤) → (X*, τ*, ≤*), the following statements are equivalent.

1) f is B-β-open map.
2) $[f(A^0] \subseteq [f(A)]^{b\beta o}$ for any $A \subseteq X$.
3) $[f^{-1}(B)]^0 \subseteq f^{-1}(B^{b\beta o})$ for any $B \subseteq X^*$.

THEOREM 3.04. Let f : $(X, \tau, \leq_1) \rightarrow (Y, \sigma, \leq_2)$ and g: $(Y, \sigma, \leq_2) \rightarrow (Z, \eta, \leq_3)$ be any two mappings. Then gof : $(X, \tau, \leq_1) \rightarrow (Z, \eta, \leq_3)$ is x-β-open if f is open and g is x-β-open for x = I,D,B.

Proof. Omitted.

THEOREM 3.05. Every I-semi-open map is an I-β-open map.

Proof. Follows from Lemma 1.1.

The following example shows that an I-β-open map need not be an I-semi-open map.

EXAMPLE 3.04. Let X = {a, b, c} = X*, τ = {ϕ, X, {a},{b, c}}= τ* and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f(a) = b, f(b) = c and f(c) = a. Then f is an I-β-open map, but not an I-semi-open map.

THEOREM 3.06. Every I-pre-open map is an I-β-open map.

Proof. Follows from Lemma 1.2.

The following example shows that an I-β-open map need not be an I-pre-open map.

EXAMPLE 3.05. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {b, c}}, τ* = {ϕ, X, {a}, {b,}, {a, b}} and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = b, f (b) = c and f(c) = a. Then f is an I-β-open map, but not an I-pre-continuous map.

THEOREM 3.07. Every I-α-open map is an I-β-open map.

Proof. Follows from Lemma 1.3.

The following example shows that an I-β-open map need not be an I-α-open map.

EXAMPLE 3.06. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {b, c}}= τ* and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) =b, f (b) =c and f(c) =a. Then f is an I-β-open map, but not an I-α-open map.

THEOREM 3.08. Every D-semi-open map is a D-β-open map.

Proof. Follows from Lemma 1.1.

The following example shows that a D-β-open map need not be a D-semi-open map.

EXAMPLE 3.07. Let X = {a, b, c}= X*, τ = {ϕ, X, {a}, {b, c}}= τ* and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f(a) = b, f(b) = c and f(c) = a. Then f is a D-β-open map, but not a D-semi-open map.

THEOREM 3.09. Every D-pre-open map is a D-β-open map.

Proof. Follows from Lemma 1.2.

The following example shows that a D-β-open map need not be a D-pre-open map.

EXAMPLE 3.08. Let X={a, b, c}=X*, τ = {ϕ,X,{a},{b, c}}, τ*={ϕ,X,{a},{b,}{a, b}} and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = b, f (b) = c and f(c) = a. Then f is a D-β-open map, but not a D-pre-continuous map.

THEOREM 3.10. Every D-α-open map is a D-β-open map.

Proof. Follows from Lemma 1.3.

The following example shows that a D-β-open map need not be a D-α-open map.

EXAMPLE 3.09. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {b, c}}= τ* and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = b, f (b) = c and f(c) = a. Then f is a D-β-open map, but not a D-α-open map.

THEOREM 3.11. Every B-semi-open map is a B-β-open map.

Proof. Follows from Lemma 1.1.

The following example shows that a B-β-open map need not be a B-semi-open map.

EXAMPLE 3.10. Let X ={a, b, c} = X*, τ = {ϕ, X,{a}, {b, c}}= τ* and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f(a) = b, f(b) = c and f(c) = a. Then f is B-β-open map, but not a B-semi-open map.

THEOREM 3.12. Every B-pre-open map is a B-β-open map.

Proof. Follows from Lemma 1.2.

The following example shows that a B-β-open map need not be a B-pre-open map.

EXAMPLE 3.11. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {b, c}}, τ*= {ϕ, X, {a}, {b,}, {a, b}} and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = b, f (b) = c and f(c) = a. Then f is a B-β-open map, but not a B-pre-continuous map.

THEOREM 3.13. Every B-α-open map is a B-β-open.

Proof. Follows from Lemma 1.3.

The following example shows that a B-β-open map need not be a B-α-open map.

EXAMPLE 3.12. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {b, c}}= τ* and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = b, f (b) = c and f(c) = a. Then f is B-β-open map, but not a B-α-open map.

4. I-β-CLOSED, D-β-CLOSED AND B-β-CLOSED MAPS

DEFINITION 4.01. A function f : (X, τ, ≤) → (X*, τ*, ≤*) is called an I-β-closed (resp. D-β closed, B-β-closed) map if f(G) ∈ IβC(X*) (resp. f(G) ∈ DβC(X*), f(G) ∈ BβC(X*)) whenever G is a closed subset of X. Clearly every x-β-closed map is a β-closed map for x = I,D,B and every B-β-closed map is both I-β-closed and D-β-closed map.

The following example shows that a β-closed map need not be a x-β-closed for x = I,D,B

EXAMPLE 4.01. Let X = {a, b, c}, τ = {ϕ, X, {a}, {b}, {a, b}} and ≤ ={(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)}. Clearly (X, τ, ≤) is a topological ordered space. Let f be the identity map from (X, τ, ≤) onto itself. {a, c} is closed set but f({a, b}) is neither an increasing nor a decreasing β-closed set. Thus f is not x-β-closed map for x = I, D, B. Clearly f is β-closed map.

The following example shows that an I-β-closed map need not be a B-β-closed map.

EXAMPLE 4.02. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {b}, {a, b}} = τ *, ≤ = {(a, a), (b, b), (c, c), (a, c)} and ≤* ={(a, a), (b, b), (c, c), (a, c), (b, c)}. Let θ be the identity map from (X, τ, ≤) onto (X*, τ*, ≤*). θ is an I-β-closed map but not a B-β-closed map.

EXAMPLE 4.03. Let X = {a, b, c} = X*, τ = {ϕ, X, {c}, {b, c}} = τ,*≤ = {(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)} = ≤ * and f : (X, τ, ≤) → (X*, τ*, ≤*) be the identity map. Then f is a D-β-closed map but not a B-β-closed map.

4.01 Thus we have the following diagram
For a function f : (X, τ, ≤) → (X*, τ*, ≤*)

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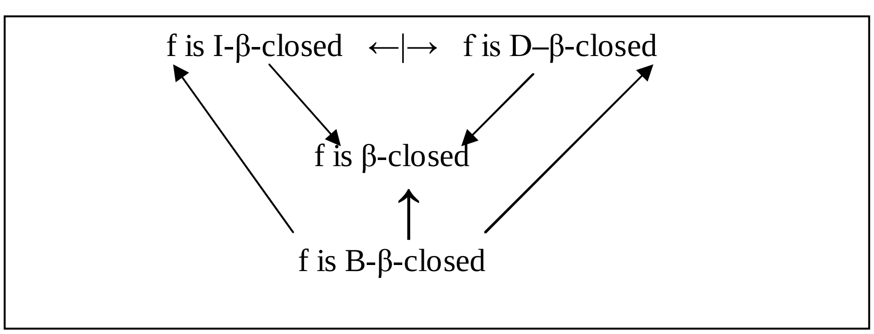


,where p → q (resp. p ←|→ q) represents p implies q, but q need not imply p (p and q are independent of each other)

The following theorem characterizes I-β-closed maps.

THEOREM 4.01. Let $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ be any map. Then f is I-β-closed iff $i\beta cl(f(A)) \subseteq f(cl(A))$ for any $A \subseteq X$.

Proof. Necessity: Since f is I-β-closed, f(cl(A)) is an increasing β-closed subset of X. Clearly $f(A) \subseteq f(cl(A))$. Therefore $i\beta cl(f(A)) \subseteq f(cl(A))$ since iβcl(f(A)) is the smallest increasing β-closed set in X* containing f(A).
Sufficiency: Let F be any β-closed subset of X. Then $f(F) \subseteq i\beta cl(f(F)) \subseteq f(cl(F)) = f(F)$.
Thus f(F) = iβcl(F). So f(F) is an increasing β-closed subset of X*.Therefore f is an I-β-closed map.

The following two theorems characterize D-β-closed maps and B-β-closed maps.

THEOREM 4.02. Let $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ be any map. Then f is D-β-closed iff $d\beta cl(A) \subseteq f(cl(A))$ for every $A \subseteq X$.

Proof. Omitted.

THEOREM 4.03. Let $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ be any map. Then f is B-β-closed iff $b\beta cl(A) \subseteq f(cl(A))$ for every $A \subseteq X$.

Proof. Omitted

THEOREM 4.04. Let $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ be a bijection. Then
1) f is I-β-open iff f is D-β-closed
2) f is I-β-closed iff f is D-β-open
3) f is B-β-open iff f is B-β-closed.

Proof.(1) Necessity. Let F be any closed subset of X. Then f(C(F)) is an increasing β-open subset of X*. Since f(C(F)) = C(f(F)) and C(f(F)) is an increasing β-open subset of X*, f(F) is a decreasing β-closed subset of X*. Therefore f is D-β-closed.
Sufficiency. Let G be any open subset of X. Then f (C(G)) is a decreasing β-closed subset of X*. Since f is a bijection, we have f(C(G)) = C(f(G)). So f(G) is an increasing β-open subset of X*. Therefore f is an I-β-open map.
Proofs of (2) and (3) are similar to that of (1).

THEOREM 4.05. Let $f : (X, \tau, \leq_1) \to (Y, \sigma, \leq_2)$ and $g:(Y,\sigma, \leq_2) \to (Z, \eta, \leq_3)$ be any two mappings. Then $gof : (X, \tau, \leq_1) \to (Z, \eta, \leq_3)$ is x-β-closed if f is closed and g is x-β-closed for x=I,D,B.

Proof. Omitted

THEOREM 4.06. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be a bijection. Then the following statements are equivalent.

1) f is an I-β-open map.
2) f is a D-β-closed map.
3) f^{-1} is a D-β-continuous map.

Proof. (1) =>(2). Let f be I-β-open map. Let F be closed set of X, then C(F) is open. f(C(F)) is an increasing β-open set of X*. =>C(f(F)) is an increasing β-open set of X*. =>f(F) is a decreasing β-closed set of X*. => f is D-β-closed map.
(2)=>(3). Let f be a D-β-closed map. Let F be closed in X, then f(F) is a decreasing β-closed set of X*.
=> $[f^{-1}]^{-1}$ (F) is a decreasing β-closed set of X*. => f^{-1} : X*→ X is D-β-continuous.
(3)=>(1) Let F be open in X. Then C(F) is closed in X. => $[f^{-1}]^{-1}$(C(F)) is a decreasing closed subset of X* => C(f(F)) is a decreasing closed set in X*. =>f(F) is an increasing open set in X*. =>f is I-β-open.

THEOREM 4.07. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be a bijection. Then the following are equivalent.

1) f is a D-β-open map.
2) f is an I-β-closed map.
3) f^{-1} is D-β-continuous map.

Proof. Omitted

THEOREM 4.08. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be a bijection. Then the following statements are equivalent.

1) f is B-β-open map.
2) f is an B-β-closed map.
3) f^{-1} is a B-β-continuous map.

Proof. Omitted

THEOREM 4.09. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be an I-β-closed map and B, C ⊆ X*. Then.

1) If U is an open neighborhood of f^{-1}(B), then there exists a decreasing β-open neighborhood V of B such that $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U$.
2) If f^{-1}(B) and f^{-1}(C) have disjoint neighborhoods, then B and C have disjoint β-open neighborhoods.

Proof. Let U be an open neighborhood of f^{-1}(B). Take C(V) = f(C(U)). Since f is an I-B-closed map and C(V) is closed, then C(V) = f(C(U)) is an increasing β-closed subset of X. Since f^{-1} (B) $\subseteq$ U , then C(V) = f(C(U)) $\subseteq$ f(f^{-1}(C(U))) $\subseteq$ C(B). Therefore B $\subseteq$ V. Thus V is a decreasing β-open neighborhood B. => f^{-1}(B) $\subseteq$ f^{-1}(V). Further C(U) $\subseteq$ f^{-1}(f(C(U))) = f^{-1}(C(V)) = C(f^{-1}(V)) => f^{-1}(V) $\subseteq$ U. Thus f^{-1} (B) $\subseteq$ f^{-1}(V) $\subseteq$ U. Let U_B, U_C be disjoint open neighborhoods of f^{-1}(B), f^{-1}(C), where B, C $\subseteq$ X* From (1) there exists V_B, V_C such that B $\subseteq$ V_B, C $\subseteq$ V_C. Also f^{-1}(B) $\subseteq$ f^{-1}(V_B) $\subseteq$ U_B, f^{-1}(C) $\subseteq$ f^{-1}(V_C) $\subseteq U_C$ where V_B, V_C are decreasing closed neighborhoods of B and C respectively. Since $U_B \cap U_C$ = ϕ; f^{-1}(V_B) ∩ f^{-1}(V_C) = ϕ. => $V_B \cap V_C$ = ϕ.

Similarly we have the following two theorems (proofs are omitted) regarding D-β-closed maps and B-β-closed maps.

THEOREM 4.10. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be a D-β-closed map and B, C, $\subseteq$, X*. Then

1. If U is an open neighborhood of f^{-1} (B), then there exists an increasing β-open neighborhood V of B such that f^{-1} (B) $\subseteq$ f^{-1}(V) $\subseteq$ U
2. If f^{-1}(B) and f^{-1}(C) have disjoint neighborhoods, then f^{-1}(B) and f^{-1}(C) have disjoint increasing β-open neighborhoods.

THEOREM 4.11. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be a B-β-closed map and B, C, $\subseteq$ X*. Then

1. If U is an open neighborhood of f^{-1}(B), then there exists an β-open neighborhood V of B, which is balanced such that f^{-1}(B) $\subseteq$ f^{-1}(V) $\subseteq$ U
2. If f^{-1}(B) and f^{-1}(C) have disjoint neighborhoods, then f^{-1}(B) and f^{-1}(C) have disjoint balanced β-open neighborhoods.

THEOREM 4.12. Every I-semi-closed map is an I-β-closed map.

Proof. Follows from Lemma 1.1.

The following example shows that an I-β-closed map need not be an I-semi-closed map.

EXAMPLE 4.04. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}}, τ* = {ϕ, X, {a}, {b, c}} and ≤ = {(a, a), (b,b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f(a) = b, f(b) = a and f(c) = c. Then f is I-β-closed map, but not an I-semi-closed map.

THEOREM 4.13. Every I-pre-closed map is an I-β-closed map.

Proof. Follows from Lemma 1.2.

The following example shows that an I-β-closed map need not be an I-pre-closed map.

EXAMPLE 4.05. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {b}, {a, b}} = τ* and ≤ = (a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = c, f (b) = b and f(c) = b. Then f is an I-β-closed map, but not an I-pre-closed map.

THEOREM 4.14. Every I-α-closed map is an I-β-closed map.

Proof. Follows from Lemma 1.3.

The following example shows that an I-β-closed map need not be an I-α-closed map.

EXAMPLE4.06. LetX = {a, b, c} = X*,τ = {ϕ, X, {a}, {b},{a, b}}, τ* = {ϕ, X, {a}, {b, c}} and ≤= {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = c, f (b) = a and f(c) = b. Then f is an I-β-closed map, but not an I-α-closed map.

THEOREM 4.15. Every D-semi-closed map is a D-β-closed map.

Proof. Follows from Lemma 1.1.

The following example shows that a D-β-closed map need not be a D-semi-closed map.

EXAMPLE 4.07. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}}, τ* = {ϕ, X, {a}, {b, c}} and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f(a) = b, f(b) = a and f(c) = c. Then f is a D-β-closed map, but not a D-semi-closed map.

THEOREM 4.16. Every D-pre-closed map is a D-β-closed map.

Proof. Follows from Lemma 1.2.

The following example shows that a D-β-closed map need not be D-pre-closed map.

EXAMPLE 4.08. Let X ={a, b, c} = X*, τ = {ϕ, X, {a}, {b}, {a, b}} = τ* and ≤ = (a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = c, f (b) = b and f(c) = b. Then f is a D-β-closed map but not a D-pre-closed map.

THEOREM 4.17. Every D-α-closed map is a D-β-closed map.

Proof. Follows from Lemma 1.3.

The following example shows that a D-β-closed map need not be a D-α-closed map.

EXAMPLE4.09. LetX = {a, b, c} = X*,τ = {ϕ, X, {a}, {b}, {a, b}}, τ* = {ϕ, X, {a}, {b, c}} and ≤={(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = c, f (b) = a and f(c) = b. Then f is a D-β-closed map but not a D-α-closed map.

THEOREM 4.18. Every B-semi-closed map is a B-β-closed map.

Proof. Follows from Lemma 1.1.

The following example shows that a B-β-closed map need not be a B-semi-closed map.

EXAMPLE 4.10. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}}, τ* = {ϕ, X, {a}, {b, c}} and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f(a) = b, f(b) = a and f(c) = c. Then f is a B-β-closed map, but not a B-semi-closed map.

THEOREM 4.19. Every B-pre-closed map is a B-β-closed map..

Proof. Follows from Lemma 1.2.

The following example shows that a B-β-closed map need not be a B-pre-closed map.

EXAMPLE 4.11. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {b}, {a, b}} = τ* and ≤ = (a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = c, f (b) = b and f(c) = b. Then f is a B-β-closed map, but not a B-pre-closed map.

THEOREM 4.20. Every B-α-closed map is a B-β-closed map.

Proof. Follows from Lemma 1.3.

The following example shows that a B-β-closed map need not be B-α-closed map.

EXAMPLE 4.12. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {b}, {a, b}}, τ* = {ϕ, X, {a}, {b, c}} and ≤= {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = c, f (b) = a and f(c) = b. Then f is a B-β-closed map but not a B-α-closed map.

5. I-β-HOMEOMORPHISMS, D-β-HOMEOMORPHISMS AND B-β-HOMEOMORPHISMS

We introduce the following definition

DEFINITION 5.01. A bijection f : (X, τ, ≤) → (X*, τ*, ≤*) is called I-β-homeomorphism (resp. a D-β-homeomorphism, B-β-homeomorphism) if both f and f^{-1} are I-β-continuous (resp. D-β-continuous, B-β-continuous).

Clearly every x-β-homeomorphism is a β-homeomorphism for x = I, D, B and every B-β-homeomorphism is both I-β-homeomorphism and D-β-homeomorphism.

The following example shows that a β-homeomorphism need not be a x-β-homeomorphism for x= I,D,B.

EXAMPLE 5.01. Let X = {a, b, c}, τ ={ϕ, X, {a}, {b}, {a, b}} and ≤ = {(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)}. Clearly (X, τ, ≤) is a topological ordered space. Let f be the identity map from (X, τ, ≤) onto itself. {a, c} is closed set but $f^{-1}(\{a, c\}) = \{a, c\}$ is neither an increasing nor a decreasing β-closed set.Thus f is not x-β-continuous for x= I,D,B. However f is β-continuous. f is a β-homeomorphism but not x-β-homeomorphism for x = I,D,B

.

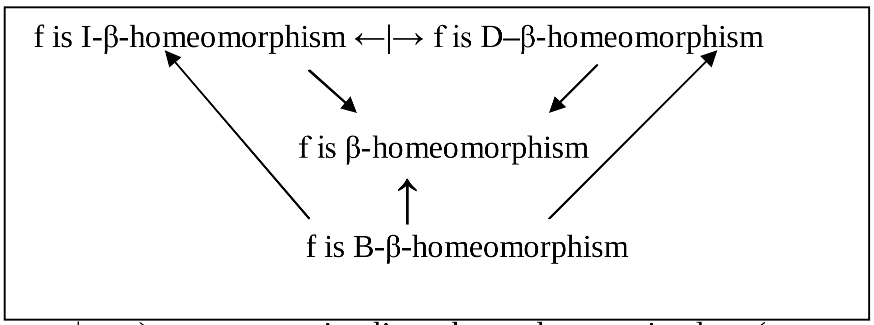
The following example shows that a D-β-homeomorphism need not be a B-β-homeomorphism.

EXAMPLE 5.02. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {a, b}} = τ* and ≤ = {(a, a), (b, b), (c, c), (b, a)}. Let g be the identity map from (X, τ, ≤) onto (X*, τ*, ≤*). .g is a D-β-homeomorphism but not a B-β-homeomorphism.

EXAMPLE 5.03. Let X ={a, b, c} = X*; τ = {ϕ, X, {a}, {b}, {a, b}} and ≤ = {(a, a), (b, b), (c, c), (a, c), (b, c)} = ≤*. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be the identity map. Then f is I-β-homeomorphism but not a B-β-homeomorphism.

5.01 Thus we have the following diagram

For a function f : (X, τ, ≤) → (X*, τ*, ≤*)



,where p→ q (resp. p ←|→ q) represents p implies q but q does not imply p (resp. p and q are independent of each other)

THEOREM 5.01. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be a bijective I-β-continuous map. Then the following are equivalent.

1) f is I-β-homeomorphism.
2) f is D-β-open.
3) f is I-β-closed.

Proof. Follows from the theorem 4.07.

THEOREM 5.02. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be bijection and D-β-continuous map. Then the following are equivalent.

1) f is D-β-homeomorphism.
2) f is I-β-open.
3) f is D-β-closed.

Proof. Follows from the theorem 4.06.

THEOREM 5.03. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be a bijection, B-β-continuous map. Then the following are equivalent.

1) f is B-β-homeomorphism.
2) f is a B-β-open map.
3) f is a B-β-closed map.

Standard separation axioms for topological ordered spaces have been studied systematically by S.D.McCartan [9,10] Now we examine the separation properties of range spaces under some of these mappings.

DEFINITION 5.02. A topological ordered space (X, τ, ≤) is said to be upper strongly T_1-ordered iff for each pair of elements a ≰ b in X, there exists a decreasing τ-open neighborhood W of b such that a ∉W.

DEFINITION 5.03. A topological ordered space (X, τ, ≤) is said to be lower strongly T_1-ordered iff for each pair of elements a ⋬ b in X, there exists an increasing open neighborhood W of a such that b ∉ W. (X, τ, ≤) is said to be strongly T_1 ordered iff it is both lower and upper strongly T_1-ordered.

DEFINITION 5.04. A topological ordered space (X, τ, ≤) is said to be upper strongly β-T_1-ordered iff for each pair of elements a ≰ b in X, there exists a decreasing τ β-open neighborhood W of b such that a ∉ W.

DEFINITION 5.05. A topological order space (X, τ, ≤) is said to be lower strongly β-T_1-ordered iff for each pair of elements a ≰ b in X there exists an increasing τ-β-open neighborhood W of a such that b ∉ W. (X, τ, ≤) is said to be strongly T_1-ordered iff it is both lower and upper strongly β-T_1-ordered
.

THEOREM 5.04. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be a bijective I-β-open map as well as a poset isomorphism (i.e. x ≤ y iff f(x) ≤* f(y) : ∀ x, y ∈ X). If (X, τ, ≤) is a lower strongly T_1-ordered space, then (X*, τ*, ≤*) is a lower strongly β-T_1-ordered space.

Proof. a, b ∈ X* such that a ≰* b. Then f^{-1}(a) ≤| f^{-1}(b). Since (X, τ, ≤) is a lower strongly T_1-ordered space, there exists an increasing open neighborhood U of f^{-1}(a) such that f^{-1}(b) ∉ U. Thus f (U) is an increasing β-open neighborhood of f(f^{-1}(a)) = a such that b= f (f^{-1}(b)) ∈ f (U). Therefore (X*, τ*, ≤*) is a lower strongly β-T_1-ordered space.

THEOREM 5.05. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be a bijective D-β-open map as well as a poset isomorphism. If (X, τ, ≤) is an upper strongly β-T_1-ordered space, then (X*, τ*, ≤*) is upper strongly β-T_1-ordered space.

Proof. Similar to as that of the theorem 5.04.

THEOREM 5.06. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be a bijective B-β-open map. If f is a poset isomorphism and (X, τ, ≤) is strongly T_1-ordered space, then (X*,τ*, ≤*) is a strongly β-T_1-ordered space
.

Proof. Follows from the theorems 5.04 and 5.05.

DEFINITION 5.06. A topological ordered space (X,τ, ≤) is called strongly β-T_2-ordered (or strongly β-Hausdorff ordered or strongly β-Hausdorff closed) if for each pair of elements a $\nleq$ b in X, there exists β-open neighborhoods U and V of a and b respectively such that U is an increasing set and V is a decreasing set.

THEOREM 5.07. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be bijective B-β-open map. If (X,τ) is a Hausdorff space, then (X*,τ*, ≤*) is strongly β-Hausdorff ordered space.

Proof. Let a, b ∈ X* such that a $\nleq^*$ b. Then f^{-1}(a) ≠ f^{-1}(b). Since H is Hausdorff, there exists disjoint τ-open neighborhoods U and V of f^{-1}(a) and f^{-1}(b) respectively. Since f is B-β-open then f(U) and f(V) are two disjoint τ* β-open neighborhoods of a and b respectively such that f(U) is an increasing set and f(V) is a decreasing set. Therefore (X*,τ*, ≤*) is a strongly Hausdorff ordered space.

DEFINITION 5.07. A topological ordered space (X,τ, ≤) is said to be a lower (an upper) strongly regular ordered space iff for each element a ∉ F there exists τ- open neighborhoods U of a and V of F such that U is an increasing (a decreasing) and V is a decreasing (an increasing) set in X and U ∩ V = ϕ.

DEFINITION 5.08. A topological ordered space (X,τ, ≤) is said to be a lower (upper) strongly β-regular ordered space iff for each decreasing (an increasing) τ-closed set F and for each element a of F, there exist β-open neighborhoods U of a and V of F such that U is an increasing (a decreasing) and V is a decreasing (an increasing) set in X and U ∩V = ϕ. X is called strongly β-regular if it is upper and lower strongly β-regular ordered space.

THEOREM 5.08. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be a bijective D-continuous map and a B-β-open map. If (X, τ) is regular space, then (X*, τ*, ≤*) is a lower strongly β-regular ordered space.

Proof. Let F be a decreasing closed subset of X* and a ∈ X* such that a ∉ F. Since f is D-continuous f^{-1} (F) is a decreasing closed set in X. Since f^{-1}(a) ∉ f^{-1}(F) and X is regular there exists two disjoint open neighborhoods U of f^{-1}(a), V of f^{-1}(F) in X. Since f is B-β-open, f^{-1}(U) is an increasing β-open set and f(V) is a decreasing β-open in X*. Also a ∈ f(U), F ⊆ f(V). (X*, τ*, ≤*) is lower strongly
B-regular ordered space.

THEOREM 5.09. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be a bijective D-continuous, B-open map. If (X, τ) is a regular space, then (X*, τ*, ≤*) is an upper strongly β-regular ordered space.

Proof. Analogous to that of the theorem 5.08

THEOREM 5.10. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be a B-β-homomorphism. If (X, τ) is a regular space, then (X*, τ*, ≤*) is a strongly β-regular ordered space.

DEFINITION 5.09. A topological ordered space (X, τ, ≤) is said to be a strongly β-normally ordered space iff for each pair of disjoint closed sets F_1 and F_2 in X, where F_1 is increasing F_2 is decreasing there exists two disjoint β-open neighborhoods U_1 of F_1 and U_2 of F_2 such that U_1 is increasing and U_2 is a decreasing in X.

DEFINITION 5.10. A topological ordered space (X, τ, ≤) is said to be a strongly β-T_3-ordered iff it is both strongly β-T_1-ordered and strongly β-regular ordered

.

DEFINITION 5.11. A topological ordered space (X, τ, ≤) is said to be a strongly β-T_4-ordered space iff it is both strongly β-T_1-ordered and strongly β-normally ordered.

THEOREM 5.11. Let f : (X, τ, ≤) → (X*, τ*, ≤*) be a bijective B-continuous and B-β-open map. Then.

1) If (X, τ,) is normal then (X*, τ*, ≤*) is a strongly β-normally ordered space.
2) If f is a poset isomorphism and (X, τ) is T_3, then (X*, τ*, ≤*) is strongly B-T_3-ordered space (Follows from the 5.06, 5.08)
3) If f is a poset isomorphism and (X, τ) is T_4, then (X*, τ*, ≤*) is strongly B-T4-ordered space. (Follows from 5.06, 5.11(1))

THEOREM 5.12. Every I-semi-homeomorphism is an I-β-homeomorphism.

Proof. Follows from Lemma 1.1.

The following example shows that an I-β-homeomorphism need not be an I-semi-homeomorphism.

EXAMPLE 5.04. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {b, c}} τ* = {ϕ, X, {a}, {b}, {a, b}} and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f(a) = b, f(b) = c and f(c) = a. Then f is I-β-homeomorphism, but not an I-semi-homeomorphism.

THEOREM 5.13. Every I-pre-homeomorphism is an I-β-homeomorphism.

Proof. Follows from Lemma 1.2.

The following example shows that an I-β-homeomorphism needs not be an I-pre-homeomorphism.

EXAMPLE 5.05. Let X = {a, b, c} = X*, τ = {ϕ, X,{a}, {b}, {a, b}}, τ* = {ϕ, X, {a}, {b, c}} and ≤ = (a, a),(b, b),(c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = a, f (b) = c and f(c) = b. Then f is an I-β-homeomorphism, but not an I-pre-homeomorphism.

THEOREM 5.14. Every I-α-homeomorphism is an I-β-homeomorphism.

Proof. Follows from Lemma 1.3.

The following example shows that an I-β-homeomorphism need not be an I-α-homeomorphism.

EXAMPLE 5.06. Let = {a, b, c} = X*,τ = {ϕ, X, {a}, {b, c}} τ* = {ϕ, X, {a}, {b}, {a, b}} and

$\leq$ = (a, a), (b, b), (c, c)} = $\leq$*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = b, f (b) = c and f(c) = a. Then f is an I-β-homeomorphism, but not an I-α-homeomorphism.

THEOREM 5.15. Every D-semi-homeomorphism is a D-β-homeomorphism.

Proof. Follows from Lemma 1.1.

The following example shows that a D-β-homeomorphism need not be D-semi-homeomorphism.

EXAMPLE 5.07. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {b, c}}, τ* = {ϕ, X, {a}, {b}, {a, b}} and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f(a) =b, f(b)=c and f(c)=a. Then f is D-β-homeomorphism, but not an D-semi-homeomorphism.

THEOREM 5.16. Every D-pre-homeomorphism is a D-β-homeomorphism

Proof. Follows from Lemma 1.2.

The following example shows that a D-β-homeomorphism need not be a D-pre-homeomorphism.

EXAMPLE 5.08. Let X ={a, b, c} = X*, τ = {ϕ, X, {a}, {b}, {a, b}}, τ* = {ϕ, X, {a}, {b, c}}and ≤ = (a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = a, f (b) = c and f(c) = b. Then f is a D-β-homeomorphism but not a D-pre-homeomorphism.

THEOREM 5.17. Every D-α-homeomorphism is a D-β-homeomorphism.

Proof. Follows from Lemma 1.3.

The following example shows that a D-β-homeomorphism need not be a D-α-homeomorphism.

EXAMPLE 5.09. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {b, c}}, τ* = {ϕ, X, {a}, {b}, {a, b}} and ≤ = (a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = b, f (b) = c and f(c) = a. Then f is a D-β-homeomorphism but not a D-α-homeomorphism.

THEOREM 5.18. Every B-semi-homeomorphism is a B-β-homeomorphism.

Proof. Follows from Lemma 1.1.

The following example shows that a B-β-homeomorphism need not be a B-semi-homeomorphism.

EXAMPLE 5.10. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {b, c}}, τ* = {ϕ, X, {a}, {b}, {a, b}} and ≤ = {(a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f(a) = b, f(b) = c and f(c) = a. Then f is B-β-homeomorphism, but not a B-semi-homeomorphism.

THEOREM 5.19. Every B-pre-homeomorphism is a B-β-homeomorphism.

Proof. Follows from Lemma 1.2.

The following example shows that a B-β-homeomorphism need not be a B-pre-homeomorphism.

EXAMPLE 5.11. Let X = {a, b, c} = X*, τ = {ϕ, X, {a}, {b}, {a, b}} τ* = {ϕ, X, {a}, {b, c}} and ≤ = (a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = a, f (b) = c and f(c) = b. Then f is a B-β-homeomorphism but not a B-pre-homeomorphism.

THEOREM 5.20. Every B-α-homeomorphism is a B-β-homeomorphism.

Proof. Follows from Lemma 1.3.

The following example shows that a B-β-homeomorphism need not be a B-α-homeomorphism.

EXAMPLE 5.12. Let X = {a, b, c} = X*,τ = {ϕ, X, {a}, {b, c}}, τ* = {ϕ, X, {a}, {b}, {a, b}} and ≤ = (a, a), (b, b), (c, c)} = ≤*. Define a map f : (X, τ, ≤) → (X*, τ*, ≤*) by f (a) = b, f (b) = c and f(c) = a. Then f is a B-β-homeomorphism but not a B-α-homeomorphism.

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