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Fixed Point Through Contractive and Pseudo Contractive Mappings

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ABSTRACT: In this paper ,we have proved some fixed point theorems for pseudo and strongly pseudo-contractive mappings.

Key Words: Fixed point theorems, pseudo-contractive mappings. , generalized contractive

1. Introduction:

Fixed Point Theorexs for nonlinear semi-contractive mappings in Banach spaces.were discussed by Browder, F.E.(1). Remarks on pseudo-contractive mappings.were discussed by Kirk, W.A (2). For more details one can go through (3),(4),(5) and (6).
This paper consists of two sections. In first section we have proved fixed point theorems for generalized contractive type mappings.

In section two we have extended the notion of pseudo-contractive mappings to 2 – metric space. After that we have proved some fixed point theorems for pseudo and strongly pseudo-contractive mappings.

1.1. In this section we have proved fixed point theorems for generalized contractive type mappings. Before proving the theorem we give some definitions.

1.1 **DEFINITION**: Let (X, d) be a 2-metric space. A mapping $T : X \rightarrow X$ is said to be contractive if for all x, y, a in X

$$d(Tx, Ty, a) < d(x, y, a)$$

Remark: A contractive mapping of a complete 2 – metric space (X,d) into itself need not have affixed point.

1.2. EXAMPLE: Let $X = \{ x \varepsilon R : x \geq 1\}$ with 2-metric defined as

$$d(x, y, z) = \min \{| x - y|, | y - z |, | z - x | \}$$

Let $F(x) = x + \frac{1}{x}$, then $F(1) = 2$, $F(2) = 2.5$, $F(3) = 3.33$ and so on.

$$d(F(1), F(2), 3) = d(2, 2, 5, 3) = \frac{1}{2}$$

$$d(1, 2, 3) = 1$$

But, $\frac{1}{2} < 1$, so F is a contractive but it has no fixed point.

1.3. DEFINITION: Let (X, d) be a 2 – metric space. A mapping $T : X \rightarrow X$ is said to be generalized contractive if for all x, y, a in X.

$$d(Tx, Ty, a) < \max. \{d(x, y, a), d(x, Tx, a), d(y, Ty, a), \frac{1}{2}[d(x, Ty, a) + d(y, Tx, a)]\}.$$

MAIN RESULTS:

1.4. THEOREM: Let S and T be two continuous self mappings on a non-empty compact 2-metric space (X,d).

If for all x, y, a in X

$$d(Sx, Ty, a) < \max. \{ d(x, y, a), d(x, Sx, a), d(y, Ty, a), \frac{1}{2}[d(x, Ty, a) + d(y, Sx, a)]\}$$

Then S and have a unique common fixed point.

Proof: Let for any arbitrary point $x_0 \varepsilon x$, $\{x_n\}$ be a Cauchy sequence defined as

$$X_{2+1} = Sx_{2n}, x_{2+1} = Tx_{2n+1}, n = 0, 1, 2, \ldots.$$

Then from given condition

$d_{2n} = d(x_{2n}, x_{2n+1}, a) = d(Sx_{2n}, Tx_{2n-1}, a)$
$< \max. \{d(x_{2n-1}, x_{2n}, a), d(x_{2n}, Sx_{2n}, a),$
$d(x_{2n-1}, Tx_{2n-1}, a)$
$\frac{1}{2}[d(x_{2n}, Tx_{2n-1}, a)$
$+ d(x_{2n-1}, Sx_{2n}, a)]\}$

i.e., $d(x_{2n}, x_{2n+1}, a) < d(x_{2n-1}, x_{2n}, a)$

$d_{2n+1} = d(x_{2n+1}, x_{2n+2}, a) = d(Sx_{2n}, Tx_{2n+1}, a)$
$< \max. \{d(x_{2n}, x_{2n+1}, a),$
$d(x_{2n}, Sx_{2n}, a), d(x_{2n+1}, Tx_{2+1}, a)$
$\frac{1}{2}[d(x_{2n}, Tx_{2n+1}, a) + d(x_{2n+1}, Sx_{2n}, a)]\}$

i.e., $d(x_{2n+1}, x_{2n+2}, a) < d(x_{2n}, x_{2n+1}, a)$
$< d(x_{2n-1}, x_{2n}, a)$
:
$< d(x_0, x_1, a)$

Thus, $d_{2n+1} < d_{2n} < \ldots. < d_0$. So the sequence $\{d_{2n}\}$ is monotone decreasing and bounded also. Thus $d_{2n} \to 1$ as $n \to \infty$. As X is compact, there exists a cluster point u of $\{x_n\}$ and so there exists a subsequence $\{x_{2n}\} \to u$ as $n \to \infty$.

Also $x_{2n+1} = Sx_{2n} \to Su$ and

$x_{2n+2} = Tx_{2n+1} = T\,Sx_{2n} \to TSu$. When $n \to \infty$.

Thus we get

$$1 = 1 = \underset{n\to\infty}{Lt}\, d(x_{2n}, x_{2n+1}, a) = \underset{n\to\infty}{Lt}\, d(x_{2n}, Sx_{2n}, a) = d(u, Su, a)$$

$$1 = \underset{n\to\infty}{Lt}\, d(x_{2n+1}, x_{2n+2}, a) = \underset{n\to\infty}{Lt}\, d(Sx_{2n}, Tx_{2n+1}, a)$$

$$= \underset{n\to\infty}{Lt}\, d(Sx_{2n}, TS_{2n}, a) = d(Su, TSu, a)$$

Suppose that $u \neq Su$, then

$d(u, Su, a) < d(u, Su, x_{2n}) + d(u, x_{2n}, a) + d(x_{2n}, Su, a)$
$= d(u, Su, x_{2n}) + d(u, x_{2n}, a) + d(x_{2n-1}, Su, a)$
$< d(u, Su, x_{2n}) + d(u, x_{2n}, a) + \max\ \{d(u, x_{2n-1}, a),$
$d(u, Su, a), d(x_{2n-1}, Tx_{2n-1}, a),$
$\frac{1}{2}[d(u, Tx_{2n-1}, a) + d(x_{2n-1}, Su\ a)]\}$
$= d(u, Su, x_{2n}, a) + d(u, x_{2n}, a) + \max. \{d(u, x_{2n-1}, a),$
$d(u, Su, a), d(x_{2n-1}, x_{2n}, a),$

$\frac{1}{2}$ [d (u, x_{2n-1}, a) + d (x_{2n}, Su a)] }

i.e., d (u, u, a) < d (u, u, a) giving a contradiction.

Thus, u = u. Hence u is the unique common fixed point of S and T. //

Remark: If we put S= T = , then we get an analogue of Pal and Maiti with condition (d) [39] in 2- metric space.

1.5. EXAMPLE: Let X = [0, 6] with 2-metric defined as

d (x, y, z) = min { | x – y |, | y – z |, |z – x | } for all x, y, z in X.

we define a mp T : [0, 6] → [0, 6] by

Tx = {5x, x ε [0, 3]
{3x + 18, x ε [3, 6]

Then, d (f(3), T (5), 0) = d (9, 3, 0) = 3

d (3, 5, 0) = 2. We observe that T is not contractive map. But it satisfies the

condition of theorem 3.1.4 by putting S = T = T as

d (T(3), T(5), 0) < max. {d (3, 5, 0), d (3, T(3),0), d (5, T(5), a),

$\frac{1}{2}$ [d(3, T(5), 0) + d (5, T(3), 0)] }.

Or, 3 < max. { 3, 5, 2, 1}

i.e., 3 < 5 and hence have a fixed point and 0 is the fixed point of T. //

A point is unique fixed point of a map T : X → X if it is an unique fixed point of any positive power of T. This fact leads us to the following.

1.6. THEOREM: Let (X, d) be a complete 2-metric space and S, T : X → X such that for all x, y, a in X and positive integers p, q, (p+q).

d (S^p (x), T^q (y), a) < max. {d, (x, y, a), d (x,S^P (x), a), d (y, T^q (y), a)
$\frac{1}{2}$ [d (x,T^q (y), a) + d (y, S^p (x), a)] }

Then S and T have a unique common fixed point.

1.7. THEOREM: Let (X,d) be a complete 2-metric space and {S_n} and {T_n} be sequence of mappings of X into itself and let {S_p} and {t_q} be two sequences of positive integers such that for any positive integers pairs p, q (p+q)

$$d\ (S_p^s p\,(x), T_q^t q\,(y), a) < \max.\{d\,(x, y, a), d\,(x, S_p^s p(x), a),$$
$$d(y, T_q^t q(y), a)\frac{1}{2}[d(x, T_q^t q(y), a)$$
$$+ d(y, S_P^S P(x), a)]\}$$

For all x, y, a in X. Then the sequence $\{S_n\}$ and $\{T_n\}$ have a unique common fixed point in X.

Proof: If we take any pair of positive integers pairs p and q ($p \neq q$) then by theorem (1.6), S_p and T_q have a unique common fixed point in X. Since p and q are arbitrary, this theorem follows. //

1.2. In this section we have extended the notion of pseudo – contractive mappings to 2-Banach space.

2.1 (a) DEFINITION: Let X be a 2- Banach space and $D \subset X$. A mapping $U: D \to X$ is said to be pseudo contractive if for all x, y, a in D and all r > 0.
$\| x - y, a \| \leq \| (1 + r)(x - y) - r\,(U(x) - U(y), a \|$.

Remark: This class of mappings is more general than the class of non-expansive mappings i.e. the mapping U for which $\|U(x) - U(y), a\| \leq \| x - y, a \|$.
2.1.(b) DEFINITION: Let X be a 2-Banach space X and $D \subset X$. A mapping $U: X \to X$ is strongly pseudo – contractive relative to $D \subset X$. If for each $x \in X$ and $r > 0$ there exists a number $\alpha_r(x) < 1$ such that $\| x - y, a \| \leq \alpha_r(x) \| (1 + r)(x-y) - r\,(U(x)-U(y)), a \|$., $y \in D$.

2.2. THEOREM: Let X be a uniformly convex 2-Benach space and B a closed sphere in X. Let U be a lipschitzian pseudo-contractive mapping of B into X such that U also maps the boundary of B into B. Then U has a fixed point in B.

Proof: We may assume without loss of generality that B is a sphere centered at the origin with radius P. Let B denote the boundary of B. For each r > 0, u, v, a ε B,

(1) $\| u-v, a \| \leq \| (1+r)(u - v) - r\,(U(u), U(v)), a \|$ letting $\lambda = \frac{r}{1+r}$ (1) is equivalent to
(2) $(1-\lambda) \| u - v, a \| \leq \| (u - v) - \lambda\,(U(u), U(v)), a \|$, $\lambda > 0$. Let $T_\lambda = T - \lambda U$. Then (2) implies $(1 - \lambda) \| u - v, a \| \leq \| T_\lambda(u) - T_\lambda(v), a \|$: u, v, a ε B.
(3) $\| T_\lambda(u) - T_\lambda(v), a \| \geq (1 - \lambda) \| u - v, a \|$

Since U is lipschitzian, there is a constant M such that

$\| U(u) - U(u), a \| \leq M \| u - v, a \|$

Since $\lambda > 0$ so that $\lambda M < 1$ and $\lambda < 1$, and let $U\lambda = \lambda U$. Then

(4) $\| U_\lambda (u) - U_\lambda (v), a \| \leq \| U(u) - U(v), a \| < \lambda M \| u - v, a \|$

So U_λ is strictly contractive on B. Also since

$\| U(x), a \| \leq \rho$ if $x, a \varepsilon \partial B$
$\| U_\lambda (x), a \| \leq \lambda \rho$ for $x, a \varepsilon \partial B$

Let $y^* \varepsilon B_1 = \{ x, \varepsilon X : \| x, a \| \leq (1 - \lambda) \rho \}$.

Define $\overline{U_\lambda}$ as follows: For $x \varepsilon B$, let $\overline{U_\lambda}(x) = U_\lambda (x) + y^*$ Then if $x \varepsilon \partial B$.

$$\| U_\lambda (x) a \| \leq \| U_\lambda (x) + y^* a \| \leq \| U_\lambda (x), a \| + \| y^*, a \|$$
$$\leq \lambda \rho + (1-\lambda) \rho = \rho.$$

So, $\overline{U_\lambda}$ maps the boundary of B into B. Let define $F = (I + (\overline{U_\lambda})_{/2}$. Thus it follows that F maps B into B.

Also since

$$\overline{U_\lambda}(x) - \overline{U_\lambda}(y), a \| = \| U_\lambda (x) + y^* - U_\lambda (y), y^*, a \|$$
$$= \| U_\lambda (x) - U_\lambda (y), a \| , \text{ y } (4)$$

It implies that $\overline{U_\lambda}$ is strictly contractive. Thus F is also strictly contractive and thus by application of contraction principle to F yield as point $x^* \varepsilon B$ such that $F(x^*) = x^* = \overline{U_\lambda}(x^*)$, and $\overline{U_\lambda}(x^*) = U_\lambda (x^*) + y^* = \lambda U(x^*) + y^*$. Hence $\lambda U(x^*) + y^* = x^*$. Since $\lambda U = T - T_\lambda$,

We have $x^* - T_\lambda, (x^*) + y^* = x^*$, so by $T_\lambda (x^*) = y^*$. Thus we have proved $T_\lambda, [B] = B_1 : T_\lambda^{-1}[B_1] \subset B$. Therefore, $(1- \lambda) T_\lambda [B] \supset B_1 : T_\lambda^{-1}[B_1] \subset B$. Therefore, $(1- \lambda) T_\lambda^{-1} : [B_1 \to B$. By (3) $(1 - \lambda)\ T_\lambda^{-1}$ is non-expensive and so has a fixed point $z \varepsilon B_1$. Thus letting $z = \frac{z}{1-\lambda}$, $T_\lambda (z') = z$, from which

$$z' = \lambda U(z') = (1 - \lambda) U(z') = T_\lambda (x)$$
$$= z' = (1 - \lambda) z,$$

Yielding $U(z') = z'$. //

Now we state a lemma, which is just analogue of Petryshyn [41 (a), theorem 7].

2.3. LEMMA: Let X be a 2 – Benach space, 0 an open bounded subset of X with 0 ε 0 and T: $\overline{G} \to \wedge$ is a 1 - set contraction satisfying.

(i) 1 – set contraction satisfying
(ii) T (x) ≠ λ x if x ε ∂ G and λ > 1
(iii) (I – T) ($\overline{G}$) is closed.

Then T has a fixed point in ($\overline{G}$)

2.4.THEOREM: Let X be a 2 – Benach space, G an open bounded subset of X with 0 ε G, and let U : $\overline{G}$ → X be a lipschitzian-contractive mapping satisfying

i) U (x) ≠ λ x if x ε ∂ and λ > 1
ii) (I – U) $\overline{G}$ is closed.

Then U has a fixed point in $\overline{G}$.

Proof : Let 0 < r < 1 be chosen so that rU is a contraction mapping. We define S, T: $\overline{G}$ → X by S = (1 – r) I, T = I-rU. Then T is one-one, T(G) is open, ∂T (G) = T∂(G), and thus (ii) T ($\overline{G}$) = cl (T (G)).

Since r U satisfies (i) and (ii) on $\overline{G}$, there exists x ε $\overline{G}$ such that x = r U (x), (this follows by lemma (3.2.3). Hence x ε G, because U satisfies (i) and so 0 = Tx ε T(G) yielding 0 ε int B, where B = cl (T (G)).

Since U is pseudo – contractive, for each x, y, a ε G.

$$\| x - y, a \| \le \| (1 + r) (x - y) - r (U(x) - U (y)), a \|,$$ which gives

$$\| Sx - Sy, a \| \le \| Tx - Ty, a \|, \; x,y, a \; \varepsilon \; G.$$

Now we define H + B → X by Hz = ST $^{-1}$ (z),

If z_1, z_2 ε B, then for every α ε B

$$\begin{aligned} \| H (z_1) - H (z_2), a \| &\le \| ST^{-1} (z_1) - ST^{-1} (z_2), a \| \\ &\le \| TT^{-1} (z_1) = TT^{-1} (z_2), a \| \\ &= \| z_1 - z_2, a \| . \end{aligned}$$

So, H is non-expansive on B. Now we claim that (T – H) (B) is closed. For this suppose z_n – H (z_n) → y, z_n ε B. Then z_n – (1-r) T $^{-1}$ (z_n) → y yielding

$$\frac{z_n}{1-r} - r^{-1} (z_n) \to \frac{y}{1-r}.$$

Let z = $\left(\frac{y}{1-r}\right)$ and let $x_n = T^{-1} (z_n)$. Then

$$\frac{r\,[x_n - U(x_n)]}{(1-r)} = \frac{x_n - r\,U(x)\,]}{(1-r)}\, x_n = \frac{T(x_n)}{1-r} - x_n \to z.$$

and thus $x_n – U(x_n) \to (1\text{-}r)\ \frac{z}{r}$.

Since (I – U) $\overline{G}$ is closed, there exists x ε $\overline{G}$ such that x– U (x) = (1-r) $\frac{z}{r}$.

Then (1 – r) z = r (x – U(x)) = x – r U (x) – (1 – r) x
= Tx – (1 – r) x, which gives

$$\frac{Tx}{1-r} - x = z.$$

Put w = Tx, then we have $\frac{w}{1-r} - T^{-1}(w) = z.$ So

w – (1 – r) T^{-1} (w) = (1 – r) z = y. Hence w - H (w) = y and so we conclude (I – H) B is closed.

Now, we show that H satisfies (i) on B.

Let x ε ∂ B and suppose H (x) = λ x for some λ > 1.

Then T^{-1} (x) = $\frac{\lambda x}{1-r}$ and since T (∂G) = ∂T (G),

we conclude that $\frac{\lambda x}{1-r}$ ε ∂G. Thus we have x = T $(\frac{\lambda x}{1-r})$,

so x = $\frac{\lambda x}{1-r}$ - r U $(\frac{\lambda x}{1-r})$ which implies that U $(\frac{\lambda x}{1-r})$ = $(\frac{(\lambda + r - 1)x}{r(1-r)})$.

Let $\bar{x} = \frac{\lambda x}{1-r}$. Then $\bar{x}$ ε ∂G and U ($\bar{x}$) = μ $\bar{x}$

where μ = $\frac{\lambda + r - 1}{\lambda\, r}$. By μ - 1 = $\frac{(\lambda - 1)\,(1-r)}{\lambda\, r} > 0$

and this contradictions our hypothesis (i) for U on ∂ G.

Therefore we conclude that H satisfies all hypothesis of lemma [3.2.3] on B, so there exists y ε B such that H(y) = y. Letting x = T^{-1} (y),

Sx = ST^{-1} (y) = H (y) = y = Tx, thus (1-r) x = x – r U (x) and we have Ux = x. //

2.5. THEOREM: Let X be a reflexive 2 – Banach space, G a bounded open convex subset of X with O ε G, and suppose U : X → X is a lipschitzian strongly pseudo contractive mapping relative to $\overline{G}$ satisfying (i) U (x) ≠ λ x If x ε ∂ G, λ > 1
Then U has a fixed point in $\overline{G}$.

Proof: To prove the theorem we only require to show that (I-U) $\overline{G}$ is closed and then from theorem (3.2.4) proof will follow. As X is reflexive so we need only to show that I – U is demi-closed. Thus if $u_j \to u_0$

Weekly and if $(I - U)u_j \to w$ strongly then we show (I-U) $u_0 = w$. Let us define F: X → X by F (x) = U (x) + w, then F is lipschitzian and strongly pseudo-contractive on X relative to $\overline{G}$ and furthermore $(I-F)u_j \to 0$ strongly. We show that this implies $u_j \to u_0$ strongly and this with continuity of U gives the desired result.

Let λ > 0 so small that λ F is a contraction mapping with lipschitzian constant $\lambda_0 <$ 1 and let r > 0 satisfy $\lambda = \frac{\lambda}{r+1}$ Since F is strongly pseudo contractive on X relative to $\overline{G}$ for each x ε X, there exists this is equivalent to

$(1 - \lambda) \| x - y, a \| \leq \alpha (x) \| (x - y) - \lambda (F(x) = F (y)), a \|$ and since (1–λ F) X = X,

we see that the mapping F λ = I – λ F satisfies

(1) $\| (1 - \lambda)\ F_\lambda^{-1}(x) - (1\text{-}\lambda)\, F_\lambda^{-1}(y), a \| \leq a(x) \| x - y, a \|$, x, a ε X, y ε $\overline{G}$

Now let $z_y = F_\lambda\, u_{j.}$ Then

$$Z_j - (1 - \lambda)\ F_\lambda^{-1}\ (u_j) = U_j - \lambda F (u_j) - (1 - \lambda) u_j$$
$$= - \lambda (F (u_j) - u_j \to 0 \text{ strongly.}$$

Now by inequality (1), proceeding analogue to Kirk [32 (a)] we achieved that $\{z_j\}$ is a Cauchy sequence. And from this we deduce that $\{u_j\}$ is a Cauchy sequence;

$$\| u_i - u_j , a \| \leq \| u_j - \lambda F (u_i) - [u_j - \lambda F (u_j)], a \| + \| \lambda F (u_i) - \lambda F (u_j), a \|$$
$$\leq \| z_i - zj, a \| + \lambda_0 \| u_i - u_j , a \|,$$

Hence $(1 - \lambda_0) \| u_i - u_j, a \| \leq \| z_i - z_j, a \| \to 0$ as i, j → ∞.

Therefore $\{u_j\}$ converges strongly and this completes the proof. //

REFERENCES

(1). Browder, F.E.: Fixed Point Theorexs for nonlinear semi-contractive mappings in Banach spaces. Arch. Rational Math. Anal. 21 (1966) 259 – 269.

(2). Kirk, W.A: Remarks on pseudo-contractive mappings. Proc. Amer. Math. Soc. 25 (4) 1970, 820 – 823.

(3). Nageswara Rao, B: Fixed Point Theorems through Rational Expression International eJournal of Mathematics and Engineering Vol.2:issue.2,166 (2012) 1555 - 1581

(4).Nageswara Rao, B: Fixed Points of Mappings Satisfying Semi-Contractivity Conditions;International Journal of Mathematical Sciences, Technology and Humanities Vol.2:issue.2, 40 (2012) 398 – 407

(5).Nageswara Rao, B: Fixed Point Theorems for Weak Commutating Mapping; International Journal of Mathematical Sciences,Technology and Humanities Vol.2:issue.2, 42 (2012) 422 – 439

(6).Nageswara Rao, B: SOME FIXED POINT THEOREMS IN 2 – METRIC SPACES: International eJournal ofMathematics and Engineering Vol.2:issue.2,164(2012) 1522 – 1545