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Fixed Point Theorems through Rational Expression

B.Nageswara Rao,
Department of Basic Science and Humanities (Mathematics)
Costal Institute of Technology and Management, Narapam, Kothavalasa – 535 183
Vijayanagaram district, A.P., India

ABSTRACT: . In this paper, we have established the sufficient condition for existence of unique fixed point by taking non-continuous maps.

Key Words;Fixed point theorems,2-Metric space,non- continuous maps.

1.**INTRODUCTION**: Generalization of fixed point theorems in 2-metric space has been done by many authors in many ways. We know that a contraction mapping is always uniformly continuous but converse is not true in general. In this paper, we have tried to establish the sufficient condition for existence of unique fixed point by taking non-continuous maps.

This paper consists of two sections. In section one, we have extended the result of Khan [2] to 2-metric space in much more general form by increasing the number of constants on the right side of inequality.

In section two, we have extended the result of Jaggi and Das [1] to 2-metric space in more general form by –

(i) increasing constants on the right hand side of inequality.
(ii) taking two or more than two self mappings,
(iii) by taking more points at a time.

1.1. In this section we have extended the result of Khan [2] by taking non-continuous maps.
For details one can go through (3),(4),(5) and (6).

1.1.11**THEOREM:** Let $\{f_m\}$ and $\{g_n\}$ (m, n = 1, 2, 3, ….) be two sequence of mappings on a complete 2-metric space X into itself satisfying.

$$d(f_m^p(x), g_n^q(y), a) \le \frac{\alpha[d(x, f_m^p(x), a)]^{r+w}[d(y, g_n^q(y), a)]^{1-r}}{[d(x,y,a)]^w}$$

$$+\beta[d(x,y,a)]^{1-r-w}[d(f_m^p(x), g_n^q(y), a)]^{r+w}$$

For all x, y, a in X and for some α, β, w ε [0, 1), γ ε (0, 1) with α + β < 1 and 2r + w = 1v when w ≠ 0, p and q are positive integers and f_m and g_n are commutative for each m, n = 1, 2, 3, … . Then f_m and g_n have unique common fixed point in X.

<u>Proof:</u> Let for any arbitrary point x_0 ε X, $\{x_n\}$ be a sequence defined as:

$$x_0, x_1 = f_1^p(x_0) : x_2 = g_1^q(x_1) : x_3 = f_2^p(x_2)....$$
$$z_{2n-1} = f_m^p(x_{2n-2}), x_{2n} = g_n^q(x_{2n-1}),.....$$

Then $d(x_{2n-1}, x_{2n}, a) = d(f_m^p(x_{2n-2}), g_n^q(x_{2n-1}), a)$

$$\le \frac{\alpha[d(x_{2n-2}, f_n^p(x_{2n-2}), a)]^{r+w}[d(x_{2n-1}, g_n^q(x_{2n-1}), a]^{1-r}}{[d(x_{2n-2}, x_{2n-1}, a)]^w}$$

$$+\beta[d(x_{2n-2}, x_{2n-1}, a)]^{1-r-w}[d(f_m^p(x_{2n-2}), g_n^q(x_{2n-1}), a)]^{r+w}$$

$$\le \frac{\alpha[d(x_{2n-2}, f_n^p(x_{2n-2}), a)]^{r+w}[d(x_{2n-1}, g_n^q(x_{2n-1}), a]^{1-r}}{[d(x_{2n-2}, x_{2n}, a)]^w}$$

$$+\beta[d(x_{2n-2}, x_{2n-1}, a)]^{1-r-w}[d(x_{2n-1}, x_{2n}), a)]^{r+w}$$

$$= \alpha[d(x_{2n-2}, x_{2n-1}, a)]^1[d(x_{2n-1}, x_{2n}), a)]^{1-r}$$

$$+\beta[d(x_{2n-2}, x_{2n-1}, a)]^{1-r-w}[d(x_{2n-1}, x_{2n}, a)]^{r+w}$$

Or, $d(x_{2n-2}, x_{2n}, a)] \le \alpha(d_{2n-2})^r(d_{2n-1})^{1-r} + \beta(d_{2n-2})^{1-r-w}(d_{2n-1})^{r+w}$

Where $d_{2n-1} = d(x_{2n-1}, x_{2n}, a)\ d_{2n-2} = (x_{2n-2}, x_{2n-1}, a)$ and so on.

$$d(x_{2n}, x_{2n+1}, a) = d(g_n^q(x_{2n-1}), f_m^p(x_{2n}), a)$$

$$= d(f_m^p(x_{2n}), g_n^q(x_{2n-1}), a)$$

$$\frac{\alpha[d(x_{2n}, f_m^p(x_{2n}),a)]^{r+w}[d(x_{2n-1}, g_n^q(x_{2n-1}),a)]^{1-r}}{[d(x_{2n}, x_{2n-1}, a)]^w}$$

$$+\beta[d(x_{2n},x_{2n-1},a)]^{1-r-w}[d(f_m^p(x_{2n}), g_n^q(x_{2n-1}),a)]^{r+w}$$

$$=\alpha[d(x_{2n},x_{2n+1},a)]^{r+w}[d(x_{2n-1},x_{2n}),a)]^{1-r-w}$$

$$+\beta[d(x_{2n},x_{2n-1},a)]^{1-r-w}[d(x_{2n+1},x_{2n}),a)]^{r+w}$$

$$=(\alpha+\beta)[d(x_{2n},x_{2n+1},a)]^{r+w}[d(x_{2n},x_{2n-1},a)]^{1-r-w}$$

Or, $[d(x_{2n},x_{2n+1},a)]^{1-r-w} \le (\alpha+\beta)[d(x_{2n}, x_{2n-1},a)]^{1-r-w}$

Or, $[d(x_{2n},x_{2n+1},a)] \le (\alpha+\beta)^{\frac{1}{1-r-w}}[d(x_{2n}, x_{2n-1}, a)]$

Or $d_{2n} \le (\alpha+\beta)^{\frac{1}{1-r-w}} d_{2n-1}$.

<u>Case – I:</u> When $w \neq 0$, then we obtain

$$d_{2n} \le (\alpha+\beta)^{\frac{1}{r}} d_{2n-1}.$$

and, $d_{2n-1} \le \alpha\, [d_{2n-2}]^r [d_{2n-1}]^{1-r} + \beta\, [d_{2n-2}]^r [d_{2n-1}]^{1-r}$

$$\le (\alpha+\beta)^{\frac{1}{r}} d_{2n-2}$$

$$= p\, d_{2n-2}\text{, where } p = (\alpha+\beta)^{\frac{1}{r}} < 1$$

Therefore we have,

$$d_{2n} \le p\, d_{2n-1} \le p^2 d_{2n-2} \le \ldots \le p^{2n+1} d_0 \to \infty \text{ as } n \to \infty.$$

<u>Case</u>-II : When $w = 0$, we get

$$d_{2n} \le (\alpha+\beta)^{\frac{1}{1-r}} d_{2n-1},$$

$$\textit{and } d_{2n} \le \alpha\,[d_{2n-2}]^r [d_{2n-1}]^{1-r} + \beta\,[d_{2n-2}]^{1-r} [d_{2-1}]^r.$$

We claim that $d_{2n-1} \leq d_{2n-2}$. If it is not so, suppose that $d_{2n-1} > d_{2n-1}$, then we have:

$$d_{2n-1} \leq \alpha\ [d_{2n-1}]^{r}\, [d_{2n-1}]^{1-r} + \beta\, [d_{2n-1}]^{1-r}\, [d_{2n-1}]^{r}$$

$(\alpha+\beta)[d_{2n-1}]$ which is a contradiction, since $\alpha+\beta<1$, and therefore

$$d_{2n-1} \leq d_{2n-2} \leq q\, d_{2n-3}] \leq q^{2}\, d_{2n-3}\, d_{2n-3} \leq .... \leq q^{\eta-2} d_{0}$$

Since $q = (\alpha+\beta)^{\frac{1}{1-r}<1.} \quad \to 0$ as $n \to \infty$

Thus in both cases the sequence $\{x_n\}$ is a Cauchy sequence. Let it converges to some u in X as X is complete. Now we prove that u is a unique common fixed point of $\{f_n\}$ and $\{g_n\}$ (m, n = 1, 2, 3, …). For this first we prove that u = $f_m^{p}(u) = g_n^{q}(u), m, n = 1,2,.....$

$$d(u, f_m^{p}(u), a) \leq d\,(u, f_m^{p}(u), x_{2n}) + d(x_{2n}, f_m^{p}(u), a)$$

$$= d\ d(u, f_m^{p}(u), x_{2n}) + d(u, x_{2n}, a) + d\,(f_m^{p}(u), g_n^{q}(x_{2n-1}), a)$$

$$\leq d(u, f_m^{p}(u), x_{2n}) + d(u, x_{2n}, a)\ +$$

$$+\alpha\, \frac{[d\,(u, f_m^{p}(u), a)]^{r+w} [d(x_{2n-1}, g_n^{q}(x_{2n-1}), a)]^{1-r}}{[d(u, x_{2n-1}, a)]^{w}}$$

$$+\beta\, [d(u, x_{2n-1}, a)]^{1-r-w}\, [d\,(f_n^{p}(u), g_n^{q}(x_{2n-1}), a)]^{r+w}$$

$$= d(u, f_m^{p}(u), x_{2n}) + d(u, x_{2n}, a)$$

$$+ \frac{\alpha\, [d\,(u, f_m^{p}(u), a)]^{r+w} [d(x_{2n-1}, g_n^{q}(x_{2n-1}), a)]^{1-r}}{[d(u, x_{2n-1}, a)]^{w}}$$

$$+\beta\, [d(u, x_{2n-1}, a)]^{1-r-w}\, [d\,(f_n^{p}(u), g_n^{q}(x_{2n}), a)]^{r+w}$$

When $n \to \infty$

d $(u, f_m^p(u), a) \leq 0$ which implies that d $(u, f_m^p(u), a) = 0$ and hence u = $(u = f_m^p(u)$. Similarly d $g_n^q(u) = \mathrm{u}$. Therefore u is the common fixed point of both d f_m^p and g_n^q. To show that u is unique, let v be another common fixed point of f_m^p and g_n^q. So, v = $f_m^p(\mathrm{v}) = \mathrm{g}_n^q(\mathrm{v})$. Then by given condition,

$$d(f_m^p(\mathrm{u}), \mathrm{g}_n^q(\mathrm{v}), \mathrm{a}) \leq \frac{\alpha\,[\mathrm{d}\,(\mathrm{u}, \mathrm{f}_m^p(u), a)]^{r+w}[d(v, g_n^q(v), \mathrm{a})]^{1-r}}{[\mathrm{d}(\mathrm{u}, \mathrm{v}, \mathrm{a})]^{w}}$$

$$+\beta\,[d(u,v,a)]^{1-r-w}\,[\mathrm{d}\,(f_n^p(\mathrm{u}), g_n^q(\mathrm{v}), a)]^{r+w}$$

$$= \frac{\alpha\,[\mathrm{d}\,(\mathrm{u}, u, a)]^{r+w}[d(v, v, \mathrm{a})]^{1-r}}{[\mathrm{d}(\mathrm{u}, \mathrm{v}, \mathrm{a})]^{w}}$$

$$+\beta\,[d(u,v,a)]^{1-r-w}\,[\mathrm{d}\,(\mathrm{u}, \mathrm{v}), a)]^{r+w}$$

Or, d (u, v, a)≤βd (u, v, a) which is not possible since β<1 and hence u= v.

Next we show that u is the unique common fixed point of f_m and g_n (m, n = 1, 2, 3, …..)

For, $f_m^p(f_m(\mathrm{u})) = \mathrm{f}_m(\mathrm{f}_m^p(\mathrm{u}))$ gives $\mathrm{f}_m^p(\mathrm{f}_m(\mathrm{u})) = \mathrm{f}_m(u)$ i.e., f_m (u) = u, by uniqueness of u as the fixed point of f_m^p. Similarly g_n (u) = u.

Finally we show that u is the only fixed point common to f_m and g_n (m, n = 1, 2, ….)

For if z were the point such that z ≠ u, then d (u, z, a) = d (f_m(u), g_n (z), a)

$$= d(f_m^p(u), \mathrm{g}_n^q(\mathrm{z}), a)$$

$$\leq \frac{\alpha\,[\mathrm{d}(\mathrm{u}, \mathrm{f}_m^p(u), \mathrm{a})]^{\mathrm{r+w}}\,[\mathrm{d}(\mathrm{z}, \mathrm{g}_n^q(\mathrm{z}), \mathrm{a})]^{1-\mathrm{r}}}{\mathrm{d}(\mathrm{u}), \mathrm{z}, \mathrm{a})^{\mathrm{w}}}$$

$$+\beta\,[d(u,z,a)]^{1-r-w}\,[d\,(f_m^p\,(u)\,,g_n^q\,(z),a)\,]^{r+w}$$

$$=\frac{\alpha\,[d(u,u,a)]^{r+w}\,[d(z,z,a)]^{1-r}}{d(u,z,a)^{w}}$$

$$+\beta\,[d(u,z,a)]^{1-r-w}\,[d\,(u,z,a)\,]^{r+w}$$

i.e. d (u,z,a) ≤ β d (u, z, a) which is impossible and then u = z.

Remark: If we put β = 0 and w = 0, we get the generalized version of Khan [31] by putting $f_m^p = g_n^q = f$ in 2 – metric space.

(2.1.2) COROLLARY: Let f : X → X satisfying

$$d(f(x),f(y),a)\le\frac{\alpha\,[d\,(x,f(x),a)]^{r+w}[d(y,f(y),a)]^{1-r}}{[d(x,y,a)]^{w}}$$

$$+\beta\,[d(x,y,a)]^{1-r-w}\,[d\,(f\,(x)\,,f\,(y),a)\,]^{r+w}$$

For all x, y, a in X for some α, β, w ε [0, 1), r ε (0, 1) with α + β < 1 and r + w = 1, when w ≠ 0. Then f has a unique fixed point in X. //

(2.2) In this section we have made attempt to generalize contraction principle in 2-metric space through rational expression. We proved fixed point theorems for pair of sequence of mappings and also for three self maps. Our result in this section generalizes the result of many authors such as [26]. [34] etc.

(2.2.1) THEOREM: Let $\{f_m\}$ and $\{g_n\}$ (m, n = 1, 2, ….) be two sequences of mappings on a complete 2-metric space X into itself such that

(I) $$d(f_m^p(x),g_n^q(y),a)\le\frac{\alpha\,[d(x,f_m^p(x),a)\,d(y,g_n^q\,(y),a)}{d(x,f_m^p(x),a)+d\,(y,g_n^q(y),a)+d(x,g_n^q(y),a)+d(y,f_m^p(x),a)}$$

$$+\beta\,d\,(x,y,a)]+y\,[d\,(x\,,fg_n^q\,(y),a)+d\,(f_m^p(x),y,a)]$$

For all x, y, a in X, α + β, y ε (0, 1), α + β + y < 1, p and q are positive integers. Then $\{f_m\}$ and $\{g_n\}$ have a unique common fixed point in X.

Proof: Let x_0 ε X be arbitrary, we consider a sequence $\{x_n\}$f as follows:

x_0, $x_1 = f_1^p(x_0) : x_2 = g_1^q(x_1) : x_3 = f_2^p(x_2) : x^4 = g_2^q(x_3)$ and so on. In general,

$$x_{2n-1} = f_n^p(x_{2n-2}) : x_{2n} = g_n^q(x_{2n-1})$$

$$d(x_{2n+1}, x_{2n+2}, x_{2n}) = d\, f_m^p(x_{2n}), g_n^q\ (x_{2-1}), x_{2n})$$

$$\leq \alpha \frac{\alpha\,[x_{2n}, f_m^p(x_{2n}), d(x_{2n+1}, g_n^q\ (x_{2n+1}), x_{2n})}{d(x_{2n}, f_n^p(x_{2n}), x_{2n}) + d\,(x_{2n+1}, g_n^q(x_{2n+1}), x_{2n}) + d(x_{2n}, g_n^q(x_{2n+1}), x_{2n})}$$

$$+ d\ (x_{2n+1}, f_n^p(x_{2n}), x_{2n})$$

$$+ \beta\, d\,(x_{2n}, x_{2n+1}, x_{2n}) + y\,[d\,(x_{2n}, g_n^q\ (x_{2n+1}), x_{2n}) + d\,(x_{2n+1}, f_n^p(x_{2n}), x_{2n})]$$

$$= 0$$

i.e., $d(x_{2n}, x_{2n+1}, x_{2n+2}) < 0.$ But $d\,(x_{2n}, x_{2n+1}, x_{2n+2}) \leq 0.$

Thus $d(x_{2n}, x_{2n+1}, x_{2n+2}) = 0$.

Now, d (x_1, x_2, a) = d $(f_1^p(x_0), g_1^q(x_1), a)$

$$\leq \alpha \frac{d\,(x_0, f_1^p(x_0), a) + (x_1, g_1^q(x_1), a)}{d(x_0, f_1^p(x_0), a) + d(x_1, g_1^q(x_1), a)}$$

$$+\ \ d\,(x_0, g_1^q(x_1), a) + d\,(x_1, f_1^p(x_0), a)$$

$$+ \beta\, d\,(x_0, x_1, a) + y\,[d\,(x_0, g_1^q\ (x_1), a) + d\,(x_1, f_1^p(x_0),_2)]$$

$$= \alpha \frac{d\,(x_0, x_1, a)\ \ d\,(x_1, x_2, a)}{d(x_0, x_1, a) + d(x_1, x_2, a) + d(x_0, x_2, a) + d(x_1, x_1, a)}$$

$$+\beta\, d\,(x_0, x_1, a) + \mathrm{y}\,[\mathrm{d}\,(\mathrm{x}_0, \mathrm{x}_2, \mathrm{a}) + \mathrm{d}\,(\mathrm{x}_1, x_1),_2\,)]$$

$$\leq \alpha \frac{\mathrm{d}\,(\mathrm{x}_0, x_1, a)\;\; \mathrm{d}\,(\mathrm{x}_1, \mathrm{x}_2, a)}{d(x_0, x_1, a) + d(x_1, x_2, a) + d(x_0, x_2, a)} + \beta\, \mathrm{d}\,(\mathrm{x}_0, x_1, a)$$

$$+ y\,[\mathrm{d}\,(\mathrm{x}_0, \mathrm{x}_1, \mathrm{a}) + \mathrm{d}\,(\mathrm{x}_1, \mathrm{x}_2, \mathrm{a})]$$

i.e., $\mathrm{d}\,(\mathrm{x}_1, \mathrm{x}_2, \mathrm{a}) \leq \leq \alpha \dfrac{\beta + \gamma}{1 - \alpha - \gamma}\; \mathrm{d}\,(\mathrm{x}_0, x_1, a)$

Again, $d(x_2, x_3, a) = d(g_1^q(x_1),\, f_2^p(x_2), a)$

$$= d(\, f_2^p(x_2), g_1^q(x_1), a)$$

$$\leq \alpha \frac{\mathrm{d}\,(\mathrm{x}_2,\, f_2^p\;(\mathrm{x}_2), a)\, \mathrm{d}\,(\mathrm{x}_1, g_1^q\;(\mathrm{x}_1), \mathrm{a}}{d(x_2,\, f_2^p(x_2), a) + d(x_1, g_1^q(x_1), a)}$$

$$+\beta\;\; (x_2, g_1^q, (x_1), a) + \mathrm{d}\,(\mathrm{x}_1,\, f_2^p\;(\mathrm{x}_2), \mathrm{a})$$

$$+\beta\, d\,(x_1, x_2, a) + \gamma\,[\mathrm{d}\,(\mathrm{x}_2, g_1^q(\mathrm{x}_1), a) + \mathrm{d}\,(\mathrm{x}_1,\, f_2^p(x_2),_2\,)]$$

$$= \alpha \frac{\mathrm{d}\,(\mathrm{x}_2, x_3, a)\;\; \mathrm{d}\,(\mathrm{x}_1, \mathrm{x}_2, a)}{d(x_2, x_3, a) + d(x_1, x_2, a) + d(x_2, x_2, a) + d\,(\mathrm{x}_1, x_3, a)}$$

$$+\beta\, d\,(x_1, x_2, a) + \gamma\,[\mathrm{d}\,(\mathrm{x}_2, x_2, a) + \mathrm{d}\,(\mathrm{x}_1, x_3, a)]$$

$$= \alpha \frac{\mathrm{d}\,(\mathrm{x}_2, x_3, a)\;\; \mathrm{d}\,(\mathrm{x}_1, \mathrm{x}_2, a)}{d(x_2, x_3, a)} + \beta\, \mathrm{d}\,(\mathrm{x}_1, x_2, a)$$

$$+ y\,[\mathrm{d}\,(\mathrm{x}_1, \mathrm{x}, \mathrm{a}) + \mathrm{d}\,(\mathrm{x}_2, \mathrm{x}_3, \mathrm{a})]$$

i.e., $\mathrm{d}(\mathrm{x}_2, \mathrm{x}_3, \mathrm{a}) \leq \;\; (\dfrac{\beta + \gamma}{1\text{-}\alpha\text{ -}\gamma}) d(x_1, x_2, a)$

$$\leq (\frac{\beta + \gamma}{1\text{-}\alpha\text{ -}\gamma}) d(x_0, x_1, a)$$

(II) $d(x_n, x_{n+1}, a) = K^n d(x_0, x_1, a)$, where $K = K(\frac{\beta+\gamma}{1-\alpha-\gamma})$

Now we claim that :

(III) $d(x_n, x_{n+1}, a) = 0$ for $n = 0, 1, 2, \ldots.$

We observe that this is true for $n = 0$, and $n = 1$.

Suppose that it is true for $2 < n < m$. Using simplex inequality and inequality (II) we have:

$$0 < d(x_0, x_1, x_{m+1}) \le d(x_0, x_1, x_m) + d(x_0, x_m, x_{m+1}) + d(x_m, x_1, x_{m+1})$$

$$\le K^n [d(x_0, x_0, x_1) + d(x_0, x_1, x_1)].$$

$$= 0.$$

Since $d(x_n, x_{n+1}, x_{n+p}) \le K^n d(x_0, x_1, x_{n+p})$ it follows from (III) that

(IV) $d(x_n, x_{n+1}, x_{n+p}) = 0$ for all non-negative integers m and n.

We now show that $\{x_n\}_n \varepsilon N$ is a Cauchy sequence. For arbitrary $a \varepsilon X$, we have

$$d(x_n, x_{n+p}, a) \le d(x_n, x_{n+1}, x_{n+p}) + d(x_n, x_{n+1}, a)$$

$$+ (x_{n+1}, x_{n+2}, x_{n+p}) + d(x_{n+1}, x_{n+2}, a)$$

$$+ \ldots\ldots\ldots\ldots$$

$$+ d(x_{n+p-2}, x_{n+p-1}, x_{n+p}) + d(x_{n+p}, x_{n+p}, a)$$

$$= [K^n + k^{n+1} + \ldots. k^{n+p-1}] d(x_0, x_1, a)$$

$$= (\frac{k^n}{1-k}) d(x_0, x_1, a) \to 0 \text{ as } n \to \infty \ (k < 1)$$

This shows that $\{x_n\}_{n\ \varepsilon}$ N is a Cauchy sequence. Since X is complete. Therefore there exists a point u in X such that $\underset{n\to\infty}{Lt}\ x_n = u$. Now we prove that $f_m(u) = u$, (m = 1, 2, ….)

$$d(u, f_m^p(u), a) \le d(u, f_m^p(u), x_{2n}) + d(u, x_{2n}, a) + d(x_{2n}, f_m^p(u), a)$$

$$= d = d(u, f_m^p(u), x_{2n}) + d(u, x_{2n}, a) + d(g_n^q(x_{2n-1}), f_m^p(u), a)$$

$$\le d(u, f_m^p(u), x_{2n}) + d(u, x_{2n}, a)$$

$$+ \alpha \frac{d(u, f_m^p(u), a)\, d(x_{2n-1}, g_n^q(x_{2n-1}), a)}{d(u, f_m^p(u), a) + d(x_{2n-1}, g_n^q(x_{2n-1}, a)}$$

$$+ d(u, g_n^q(x_{2n-1}), a) + d(x_{2n-1}, f_n^p(u), a)$$

$$+ \beta\, d(u, x_{2n-1}, a) + \gamma[d(u, g_n^q)(x_{2n-1}, a) + d(x_{2n-1}, f_m^p(u), a)]$$

When n → ∞,

$$d(u, f_m^p, (u), a \le \gamma[d(u, f_m^p(u), a)]$$

Or, (1 – γ) d (u, $f_m^p, (u), a) \le 0$ which implies that

$$d(u, f_m^p(u), a) = 0. \text{ Thus } f_m^p(u) = u, (n = 1, 2, ...)$$

Similarly we can show that $g_n^q(u) = u$

Now we show that u is the unique common fixed point of f_m^p *and* g_n^q. If not suppose v ≠ u be another fixed point such that $f_m^p(v) = g_n^q(v) = v$.

$$d(u, v, a) = d(f_m^p(u), g_n^q(v), a)$$

$$\le \alpha \frac{d(u, f_m^p(u), a)\, d(v, g_n^q(v), a)}{d(u, f_m^p(u), a) + d(u, g_n^q(v)\, a) + d(u, f_m^p(u), a) + d(u, g_n^q(v), a)}$$

$+ \beta \, d(u,v,a) + \gamma \, [d(u, f_m^p(u), a) + d(v, g_n^q(v), a)]$

i.e. $(1 - \alpha - 2\gamma) \, d(u, v, a) \leq 0$ which implies that d (u, v,a) = 0 i.e. u =v. Hence f_m^p and g_n^q (m, n = 1, 2, ...) have a unique common fixed point. Further we show that u is the unique common fixed of $\{f_m\}$ and $\{g_n\}$ (m, n = 1, 2, …..).

For $f_m^p(f_m(u)) = f_m(f_m^p(u))$ gives $f_m^p(f_m(u)) = f_m(u)$

i.e. $f_m(u) = u$ by uniqueness of u as the fixed of f_m^p. Similarly $g_n^q(u) = u$.

Finally we show that u is the only fixed point common to f_m (m= 1, 2, …) f_m) (z) = g_n (z) = z.

Then, d (u, z, a) = d (f_m(u), g_n (z), a) = d (f_m^p(u), g_n^q (z) , a.

$$\leq \alpha \frac{d(u, f_m^p(u), a) \, d(z, g_n^q(z), a)}{d(u, f_m^p(u), a) + d(z, g_n^q(z) \, a) + d(u, g_n^q(z), a) + d(z, f_m^p(z), a)}$$

$$+ \beta \, d(u,z,a) + \gamma \, [d(u, g_n^q(z), a) + d(\varepsilon, f_m^p(u), a)]$$

i.e. $(1 - \beta - 2\gamma_3) d(u,z,a) \leq 0$ which implies that d (u, z, a) = 0 and hence u = a. //

(2.2.2) <u>COROLLARY</u>: Let $\{f_m\}$ and $\{g_n\}$ (where m, n = 1, 2, …) be two sequence of mappings of a complete 2-metric space (X, d) into itself such that for all x, y, a in X

$$d(f_m(x), g_n(y), a) \leq \alpha \frac{d(x, f_m(x), a) \, d(y, g_n(y), a)}{d(x, g_n(y), a) + d(y, f_m(x), a) + d(x, f_m(x), a) + d(y, g_n(y), a)}$$

$$+ \beta \, d(x,y,a) + \gamma \, [d(x, g_n(y), a) + d(y, f_m(x), a)]$$

Where α, β, γ ε (0, 1] such that a + β + 2 γ < 1.

Then $\{f_m\}$ and $\{g_n\}$ (m, n = 1, 2, ….) have a unique common fixed point.

Proof: Putting p = q = 1 in the theorem (2, 2, 1) the result follows.

(2.2.3) COROLLARY: Let f and g be self mappings on a complete 2-metric space (X,d) into itself such that

$$d(f(x), g(y), a) \leq \alpha \frac{d(x, f(x), a)\ d(y, g(y), a)}{d(x, g(y), a) + d(y, f(x), a) + d(x, f(x), a) + d(y, g(y), a)}$$

$$+ \beta\, d(x, y, a) + \gamma\, [d(x, g(y), a) + d(y, f(x), a)]$$

For all x, y, a in X where α, β, γ ε (o, 1] such that α + β + 2 γ < 1. Then f and g have a unique common fixed point.

Proof: By putting m = n = 1 in corollary (2,2,2) the result follows. //

Now we give an example of a fixed of self mappings satisfying the condition of corollary (2.2.3) but is not a contraction mapping.

(2.2.4) EXAMPLE: Let (X, d) be a 2- metric space defined as follows:

d (x, y, z) = min { | x-y |, | y-1 |, | z – x| }

Now let X = [0, 1] with the 2-metric defined as above.

Let f, g : X → X defined as :

Now taking $\frac{1}{2}, \beta = \frac{1}{3}, y = \frac{1}{5}, x = \frac{1}{3}, y = \frac{1}{2}, a = 1$

We find that

$$d(f(x), g(y), 1) = d(\frac{1}{12}, \frac{1}{6}, 1) = \min\{|\frac{1}{12}, -\frac{1}{6}|, |\frac{1}{6} - 1|, |1 - \frac{1}{12}|\}$$

= min { 0, 08, 0.83, 0.91 }

= 0. 08.

Now, $\alpha \dfrac{d(x, f(x), 1)\, d(y, g(y), 1)}{d(x, f(x), 1) + d(y, g(y), 1) + d(x, g(y), 1) + d(y, f(x), 1)}$

$+ \beta\, d(x, y, 1) + \gamma\, [d(x, g(y), 1) + d(y, f(x), 1)]$

$$= \frac{\frac{1}{2} \text{ x } \frac{1}{4} \text{ x } \frac{1}{3}}{\frac{1}{4} + \frac{1}{3} + \frac{1}{6} + \frac{5}{12}} + [\frac{1}{3} \text{ x } \frac{1}{6}] + \frac{1}{5}[\frac{1}{6} + \frac{5}{12}]$$

$$= \frac{\frac{1}{24}}{\frac{3+4+2+5}{12}} + \frac{1}{18} + \frac{1}{5} x \frac{2+5}{12} = \frac{1}{28} + \frac{1}{18} + \frac{7}{60} = \frac{131}{630} \;\underline{\sim} 0.2$$

Again 1f we put a = 0 then also we have

$$d\;\; d(\frac{1}{12}, \frac{1}{6}, 0) = \min\{|\frac{1}{12} - \frac{1}{6}|, |\frac{1}{6} - 0|, |0 - \frac{1}{12}|\}$$

$$= \min\{\frac{1}{12}, \frac{1}{6}, \frac{1}{12}\} = \frac{1}{12} \;\underline{\sim} 0.083$$

Now,

$$\text{R.H.S} = \frac{\frac{1}{2} \text{ x } \frac{1}{12} \text{ x } \frac{1}{6}}{\frac{1}{12} + \frac{1}{6} + \frac{1}{6} + \frac{1}{12}} + [\frac{1}{3} \text{ x } \frac{1}{6}] + \frac{1}{5}[\frac{1}{6} + \frac{1}{12}]$$

$$= \frac{1}{144} \text{ x } \frac{2}{1} + \frac{1}{18} + \frac{1}{20} = \frac{1}{72} + \frac{1}{18} + \frac{1}{120} = \frac{43}{360} \;\underline{\sim} 0.11.$$

Thus we see that the condition of our corollary (2.2.3) is satisfied with 0 as the only fixed point while f and g are not contraction (being discontinuous).

(2.2.4) <u>THOREM</u>: Let $\{f_m\}$ and $\{g_n\}$ (m, n = 1, 2, …..) be two sequences of mappings on a complete 2-metric space X into itself such that

(I) $d(f_m^p g_n^q(x)n, f_m^p g_n^q(y), a) \le \alpha \dfrac{d(x, f_m^p g_n^q(x), a) d(y, f_m^p g_n^q(y), a)}{d(x, f_m^p g_n^q(x), a) + d(y, f_m^p g_n^q(y), a) + 1}$

$+ d\ (x, f_m^p g_n^q(y), a) + d(y, f_m^p g_n^q(x), a).$

$+ \beta\, d\,(x, y, a) + \gamma[d(x, f_m^p\, g_n^q(y), a) +$

$+ d\,(y, f_m^p\, g_n^q(x), a)\,]$

For all x, y a in X, α, β, γ ε (0, 1), p and q are positive integers, f_m and g_n are commutative for each m, n = 1, 2, …. Then $\{f_m\}$ and $\{g_n\}$ have a unique common fixed point.

<u>Proof</u>: Consider a sequence $\{x_n\}$ as follows. Let x_0 ε X be arbitrary. Set

$f_m^p\, g_n^q = T$ and $x_n = Tx_{n-1}$ for n = 1, 2, 3, 4, … Then by condition (I) becomes

$$d\,(Tx, Ty, a) \le \alpha \frac{d\,(x, Tx, a)\, d\,(y, Ty, a)}{d\,(x, Tx, a) + d\,(y, Ty, a) + d\,(x, Ty, a) + d(y, Tx, a)}$$

$+\ \beta\, d\,(x, y, a) + \gamma[d(x, Ty), a) + d\,(y, Tx, a)]$

Write x_{2n} = Tx_{2n-1}, then by corollary (2,2,3) it can be shown that T has a unique fixed point. Let it be u.

i.e., Tu = u.

Now we show that f_m (u) = g_n (u) = u.

Tu = $f_m^p\, g_n^q\,(u) = u,$ which implies that

$f_m\ \ (f_m^p g_n^q\,(u) = f_m\,(u)$

Or, $(f_m^p g_n^q\,(u) = f_m\,(u)$ = f_m (u) [by commutativity]

Or, f_m (u) = u, by uniqueness of fixed point of

T = ($f_m^p g_n^q$ Similarly we can prove g_n (u) = u. //

(2.2.5) EXAMPLE: Let X = [0, 1] with the 2-metric of defined as d (x, y, a) = min { |x-y|, |y – a |, | a – x | }

Let f, g : X → X defined as:

$$f(x) = \begin{cases} \frac{x}{4} : x \,\varepsilon\, [0, \frac{1}{2}) \\ \frac{x}{5} : x \,\varepsilon\, [\frac{1}{2}, 1] \end{cases} \quad \text{and } g(x) = \begin{cases} \frac{x}{2} : x \,\varepsilon\, [0, \frac{1}{2}) \\ \frac{x}{3} : x \,\varepsilon\, [\frac{1}{2}, 1] \end{cases}$$

Now taking $\alpha \frac{1}{2}, \beta = \frac{1}{3}), \gamma = \frac{1}{15}, x = \frac{1}{3}, \gamma = \frac{1}{2},$

a = 1, p = q = 2. If we take f and g are commutative.

Now, d $d\,(f^2 g^2 (\frac{1}{2}), f^2 g^2 (\frac{1}{2}), 1) = \min \{|\frac{1}{192} - \frac{1}{192}|, |\frac{1}{192} - 1|, |1 - \frac{1}{192}|\} = 0.$

$$\text{R.H.S} = \frac{\frac{1}{2} \text{ x } \frac{63}{192} \text{ x } \frac{95}{192}}{\frac{63}{192} + \frac{95}{192} + \frac{95}{192} + \frac{63}{192}} + \frac{1}{18} + \frac{1}{15} \left[\frac{95}{192} + \frac{63}{192}\right]$$

$$d\,(f^2 g^2 (\frac{1}{2}), f^2 = \frac{1}{2} \text{ x } \frac{63}{192} \text{ x } \frac{95}{192} + \text{x } \frac{192}{316} + \frac{1}{18} + \frac{1}{15} (\frac{158}{192}] > 0$$

Thus the condition of our theorem (2,2,4) is satisfied with 0 as the only d fixed point of f and g while f and g are not contraction (being discontinuous).

Now we prove fixed point theorems through rational expressions but taking more than three points at a time. We have proved by taking five points. Our results can be considered extensions of Das and Sharma [11] and others to 2-metric space in more general form.

(2.2.6) <u>THEOREM</u>: Let (X,d) be a complete 2-metric space.

If f and g be two mappings of X into itself such that

$$d(f(z), g(z_2), a) \le \alpha \frac{d\,(z, f^k\,(z_3), a)\;\; d(z_2, g^k\,(x_4), a)}{d(z_1, g^k\,(z_4), a) + d\,(z_2, f^k(z_3), a) + (z_1, z_2, a)}$$

$$+ \beta\, d\,(z_1, z_2, a) + \gamma\, d(z_1, g^k(z_4), a) + \delta\, d\,(z_2, f^k(z_3), a)$$

For an arbitrary z_1, z_2, z_3, z_4, a in X, a fixed integer

k > 1, α, β, γ, ε (0, 1), α + β + γ + < 1, f and g are commutative. Then f and g have a unique common fixed point in X.

<u>Proof</u> : Let x, y, a ε X. On substituting in (1)

$Z_1 = g^k\,(x)$, $z_2 = f^p\,(y)$, $z_3 = y$, $z_4 = x$, we get:

$$d(f\,g^k\,(x), g\,f^k\,(y), a) \le \frac{d\,(g^k\,(x), f^k\,(y), a)\, d\,\,(f^k(y), g^k\,(x), a)}{d(g^k\,(x), g^k\,(x), a) + d\,(f^k(y), f^k\,(y), a) + d\,(g^k(x), f^k\,(y), a)}$$

$$+ \beta\, d\,(z_1, z_2, a) + \gamma\, d(z, g^k(z_4), a) + \delta\, d\,(z_2, f^k(z_3), a)$$

$$= \;\alpha\, d\,(g^k(x), f^k(y), a) + \beta\;\; d\,(g^k(x), f^k(y), a)$$

$$= \;\alpha\, d\,(\propto + \beta)\;\; d\,(g^k(x), f^k(y), a)$$

Suppose x_0 ε X be arbitrary. Define a sequence $\{x_n\}$ as follows:

$$x_n = \begin{cases} \{ f(x_{n-1}) \text{ when n is odd} \\ \{ g\,(x_{n-1}) \text{ when n is even.} \end{cases}$$

As f and g are commutative, we get

$X_{2n} = f^n\, g^n\,(x_0)$; $x_{2n+1} = f^{n+1}\, g^n\,(x_0)$.

Take n > t, which gives n = t+i for some integer i > 1, we have

$d\,(x_{2n}, x_{2n+1}, a) = d\,(f^{t+1}\, g^{t+1}(x_0), g^{t+1} f^{t+i+1}\,(x_0), a)$

$\leq (a+\beta)\, d\,(f^{t+i-1}\, g^{t+1}(x_0), g^{t+i-1} f^{t+i+1}\,(x_0), a)$

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$\leq (a+\beta)^i\, d\,(f^t\, g^{t+i}(x_0), g^t f^{t+i+1}\,(x_0), a)$

$\leq (a+\beta)^i\, d\,(f^{t+i-1}\, g^t(x_0), g^{t+i} f^t\,(x_0), a)$

$\leq (a+\beta)^{i+1}\, d\,(f^{t+i}\, g^t(x_0), g^{t+i-1} f^t\,(x_0), a)$

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$\leq (a+\beta)^{2i}\, d\,(f^{t+i}\, g^t(x_0), g^t f^t\,(x_0), a)$

i.e., $d\,(x_{2n}, x_{2n+1}, a) \leq (\alpha+\beta)^{2n-2t}\, d\,(x_{2n} + x_{2t+1}, a)$

for m > n > t, we have

$d\,(x_{2n}, x_{2m}, a) \leq d\,(x_{2n}, x_{2+1}, a)\, d\,(x_{2n}, x_{2n+1}, x_{2m},)$

$+\, d\,(x_{2n+1}, x_{2n+2}, a) + d\,(x_{2n+1}, x_{2n+2}, x_{2m})$

……………………………………………..

$+\, d\,(x_{2n+2}, x_{2n+1}, a) + d\,(x_{2n-3}, x_{2m+2}, x_{2m})$

$+\, d\,(x_{2m-1}, x_{2m,}\, a) + d\,(x_{2m-2}, x_{2m-1}, x_{2m})$

$\leq [(\alpha+\beta)^{2n-2t} + (\alpha+\beta)^{2n+1-2t} + \ldots + (\alpha+\beta)\,]^{2m-1-2t}\; d\,(x_{2t}, x_{2t+1}, a)$

$+[(\alpha+\beta)^{2n-2t} + (\alpha+\beta)^{2n+1-2t} + \ldots.. + (\alpha+\beta)]^{2m-1-2t}\; d\,(x_{2t}, x_{2t+1}, x_{2n})$

$\rightarrow 0$ as $n \rightarrow \infty$ since $\alpha+\beta < 1$.

Thus, $\{x_n\}$ is a Cauchy sequence in the complete 2-metric space X and so there exists a points u in X such that $\underset{n\to\infty}{Lt}\, x_n = u$.

Now, $d(u,g(u),a) \le d(u, g(u), x_{2N+1}) + d(u, x_{2N+1}, a) + d(x_{2N+1}, g(u), a)$

$\le d(u, g(u), x_{2N+1}) + d(u, x_{2N+1}, a) + d(f(x_{2N}), g(u), a)$

Now, putting $z_1 = x_{2N}$, $z_2 = u$, $z_3 = g^k(x_{2N})$, $z_4, = f^k(x_{2N})$ in (I) and using in the above inequality, we get $d(u, g(u), a) \le d(u, g(u), x_{2N+1}) + d(u, x_{2N+1}, a)$

$$= \alpha \frac{d(x_{2N}, f^k g^k (x_{2N}), a)\, d(u, g^k f^k (x_{2N}), a}{d(x_{2N}, u, a) + d(x_{2N}, g^k f^k (x_{2N}), a) + d(u, f^k g^k (x_{2N}), a)}$$

$+ \beta\, d(x_{2N}, u, a) + \gamma\, d(x_{2N}, g^k f^k (x_{2N}), a) +$

$+ \delta\, d(u, g^k f^k (x_{2N}), a)$

$\le d(u, g(u), x_{Xn+1}) + d(u, x_{2N+1}, a)$

$$+ \alpha \frac{d(x_{2N}, x_{2N+2k}, a)\, d(u, x_{2N+2k}, a)}{d(x_{2N}, u, a) + d(x_{2N}, x_{2N+2k}, a) + d(u, x_{2N+2k}, a)}$$

$+ \beta\, d(x_{2N}, u, a) + \gamma\, d(x_{2N}, x_{2N+2k}, a) + \delta\, d(u, x_{2N+2k}, a)$

$\le d(u, g(u), x_{2n+1}) + d(u, x_{2N+1}, a)$

$$+ \alpha \frac{d(x_{2N}, g(x_{2N-2k-1}, a)\, d(u, g(x_{2N+2k-1}), a)}{d(x_{2N}, u, a) + d(x_{2N}, g(x_{2N+2k-1}), a) + d(u, g(x_{2N+2k-1}, a)}$$

$+ \beta\, d(x_{2N}, u, a) + \gamma\, d(x_{2N}, g(x_{2N+2k-1})\, a) + \delta\, d(u, g(x_{2N+2k-1}), a)$

When $N \to \infty$, then x_{2N}, x_{2N+1}, $x_{2N+2K-1}$ all tends to u

$$d(u, g(u), a) \le d(u, g(u), a) + d(u, u, a) + a \frac{d(u, g(u), a)\, d(u, g(U), a}{d(u, u, a) + d(u, g(u), a) + d(u, g(u), a)}$$

$+ \beta\, d(u, u, a) + \gamma\, d(u, g(u), a) + \delta(u, g(u), a)$

$$= \frac{\alpha}{2} \, d\,(u, g(u), a) + \gamma \, d\,(u, g\,(u), a) + \delta \, d\,(u, g(u), a)$$

Or, $(1 - \frac{\alpha}{2} - \gamma - \delta)\, d\,(u, g(u), a) \leq 0.$ But $(1 - \frac{\alpha}{2} - \gamma - \delta) \leq 1.$ So,

D (u, g(u), a) ≤ 0 which implies that d (u, g(u), a) = 0 as d (u, g(u), a) ≠ 0. Thus g (u) = u. Similarly f (u) = u. Now for uniqueness of u, let v ≠ u be a point such that f (v) = g (v) = v.

By substituting $z_1 = u = z_4 : z_2 = v = z_3$ in (I), we get d (u, v, a) = d (f(u), g (v), a)

$$\leq \alpha \frac{d\,(u, f^k(v), a)\, d\,(v, g^k(u), a)}{d\,(u, v, a) + d\,(u, g^k(u), a) + d\,(v, f^k(v), a)}$$

$$+ \beta \, d\,(u, v, a) + \gamma \, d\,(u, g^k(u), a) + d\,(v, f^k(v), a)$$

$$= \alpha \frac{d\,(u, v, a)\, d\,(v, u), a)}{d\,(u, v, a) + d\,(u, u, a) + d\,(v, v), a)} + \beta \, d\,(u, v, a)$$

$$+ \gamma \, d\,(u, u, a) + d\,(v, v), a)$$

$$+ \alpha \, d\,(u, v, a) + \beta \, d\,(u, v, a)$$

i.e., $(1 - \alpha - \beta)\, d\,(u, v, a) \leq 0$ which implies that u = v. Thus f and g have a unique common fixed point.

(2.2.7) EXAMPLE: Let X = [0, 1] with 2-metric d defined as

$$d\,(x, y, z) = \min \{ |x - y|, |y - z|, |z - x| \}$$

Let f, g : X → X defined as

$$f(x) = \begin{cases} \frac{x}{4} : x \,\varepsilon\, [0, \frac{1}{2}) \\ \frac{x}{5} : x \,\varepsilon\, [\frac{1}{2}, 1] \end{cases} \qquad \text{and } g(x) \begin{cases} \frac{x}{2} : x \,\varepsilon\, [0, \frac{1}{2}) \\ \frac{x}{3} : x \,\varepsilon\, [\frac{1}{2}, 1] \end{cases}$$

Now taking $\alpha = \frac{1}{2}, \beta = \frac{1}{3}, \gamma = \frac{1}{5}, \delta = \frac{1}{15}$, k = 2,

$Z_1 = \frac{1}{18}, z_2 = \frac{1}{12}, z_3 = \frac{1}{3}, z_4 = \frac{1}{2}$ and $a = 1$ in Theorem (2.2.6) we find that

d $d(f(\frac{1}{18}), g(\frac{1}{12}), 1) = \min\{|\frac{1}{12} - \frac{1}{48}|, 1\frac{1}{48} - 1|, |1 - \frac{1}{72}|\}$

$$= \min\{|\frac{1}{144} - \frac{47}{48}|, \frac{71}{72}\} = \frac{1}{144}$$

$d(\frac{1}{18}), (\frac{1}{12}), 1) = \min\{|\frac{1}{18} - \frac{1}{12}|, 1\frac{1}{12} - 1|, |1 - \frac{1}{18}|\}$

$= \min\{|\frac{1}{36} - \frac{11}{12}|, \frac{17}{18}\} = \frac{1}{36}$

Now,

$$\text{R.H.S.} = \frac{\frac{1}{2} x \frac{1}{36} \text{ x } \frac{4}{60}}{\frac{7}{180} + \frac{1}{18} + \frac{1}{36}} = \frac{1}{108} + \frac{1}{3} \text{ x } \frac{7}{80} + \frac{1}{15} \text{ x } \frac{1}{18}$$

$$= \frac{1}{132} + \frac{1}{108} + \frac{7}{400} + \frac{1}{270} > \frac{1}{144}$$

Thus, condition of our theorem (2.2.6) is satisfied and 0 is the only fixed point of f and g, while f and g are discontinuous.

(2.2.8) <u>THEOREM:</u> Let f and g be two commutative mappings of a complex 2-metric space (X,d) into itself such that

(I) $d(f(z_1), g(z_2), a) \leq \alpha \frac{d(z_2, g^k(z_1), a)[1 + d(z_1, f^k(z_3), a)]}{1 + d(z_2, g^k(z_4), a)}$

$$+ \beta\, d(z_1, z_2, a)$$

For arbitrary z_1, z_2, z_3, z_4 a in X, $\alpha, \beta > 0$, $\alpha + \beta < 1$, k > 1 (k is an integer). Then f and g have a unique common fixed point in X.

Proof: For arbitrary x, y in X. IF we put $z_1 = g^k(x)$,

$Z_2 = f^k(y)$, $z_3 = y$, $z_4 = x$ in (I) we get

$$d(fg^k(x), gf^k(y), a) \le \alpha \frac{d(f^k(y), g^k(x), a)[1 + d(g^k(x), f^k(y), a)]}{1 + d(f^k(y), g^k(x), a)}$$

$$+ \beta\, d(g^k x), f^k(y), a)$$

(II) $$\le (\alpha + \beta)\, d(f^k(y), g^k(x), a)$$

Let x_0 in X arbitrary. Consider a sequence $\{x_n\}$ as follows:

$$x_n = \begin{cases} f(x_{n-1}) & \text{when n is odd} \\ g(x_{n-1}) & \text{when n is even} \end{cases}$$

As f and g are commutative, we observe that

$X_{2n} = f^n g^n(x_0)$ and $x_{2+1} = f^{n+1} g^n(x_0)$: Let n > p, which gives n = p + i some integer I > 1

$$d(x_{2n}, x_{2+1}, a) = d(f^{p+i} g^{p+i}(x_0), g^{p+i} f^{p+i+1}(x_0), a)$$

$$\le (\alpha + \beta)\, d(f^{p+i-1} g^{p+i}(x_0), g^{p+i-1} f^{p+i+1}(x_0), a) \text{ [by using (II)]}$$

…………………………………………………………….

$$\le (\alpha + \beta)^i\, d(f^p g^{p+i}(x_0), g^p f^{p+i+1}(x_0), a)$$

$$= (\alpha + \beta)^i\, d(f^{p+i+1} g^p(x_0), g^{p+i} f^o(x_0), a)$$

$$\le (\alpha + \beta)^{i+1}\, d(f^{p+i} g^p(x_0), g^{p+i-1} f^o(x_0), a)$$

……………………………………………………………….

$$\le (\alpha + \beta)^{2i}\, d(f^{p+i} g^p(x_0), g^p f^p(x_0), a)$$

i.e., $d\,(x_{2n}, x_{2+1}, a) \le (\alpha+\beta)^{2n-2p}\ d\,(x_{2p}), (x_{2p+1}, a)$

For m > n > p. we have

$$d\,(x_{2n}, x_{2m}, a) \le d\,(x_{2n}, x_{2n+1}, a) + d\,(x_{2n}, x_{2n+1}, x_{2m})$$

$$+ d\,(x_{2n+1}, x_{2n+2}, a) + d\,(x_{2n+1}, x_{2n+2}, x_{2m})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ d\,(x_{2m-1}, x_{2m}, a) + d\,(x_{2m-1}, x_{2m-1}, x_{2m})$$

$$= [(\alpha+\beta)^{2n-2p} + (\alpha+\beta)^{2n+1-2p} + \ldots. + (\alpha+\beta)^{2n-1-2p}]\, d\,(x_{2p}, x_{2p+1}, a)$$

$$+ [(\alpha+\beta)^{2n-2p} + (\alpha+\beta)^{2n+1-2p} + \ldots. + (\alpha+\beta)^{2n-1-2p}]\, d\,(x_{2p}, x_{2p+1}, x_{2m})$$

$$= \frac{(\alpha+\beta)^{2n-2p}}{1-\alpha-\beta}\ d\,(x_{2p}, x_{2p+1}, a) + \frac{(\alpha+\beta)^{2n-2p}}{1-\alpha-\beta}\ d(x_{2p}, x_{2p+1}, x_{2m})$$

$$\to 0 \ \text{ as a, n} \to \infty$$

Thus, $\{x_n\}$ is a Cauchy sequence in the complete 2-metric space (X, d), so there exists u in X such that $\underset{n\to\infty}{Lt}\ x_n = u.$

Now for any old positive integer h, we have

$$d\,(u, g\,(u), a) \le d\,(u, g(u), x_n) + d\,(u, x_h, a) + d\,(x_h\ g\,(u), a)$$

$$\le d\,(u, g(u), x_h) + d\,(u, x_h, a) + d\,(f\,(x_{n-1})\ g\,(u), a)$$

Taking $z_1 = x_{h-1}$, $z_2 = u$, $z_3 = g^k\,(x_{n-1})$, $z_4 = f^k\,(x_{h-1})$ in (I) and using the above inequality we get $d\,(u,g\,(u),a) \le d\,(u, g(u), x_h) + d\,(u, x_h, a)$

$$+ \quad \alpha\,\frac{d\,(u, g^k f^k\,(x_{h-1}), a)[1 + d\,(x_{h-1}, f^k\, g^k\,(x_{h-1}), a)]}{1 + d\,(u, g^k\, f^k\,(x_{h-1}), a)}$$

$$+ \beta\, d\,(x_{h-1}, u, a)$$

$$\leq d(u, g(u), x_n) + d(u, x_h, a)$$

$$+ \alpha \frac{d(u, x_{h+2k-1}, a)[1 + d(x_{h-1}, x_{h+2k-1}, a)]}{1 + d(u), g^k f^k (x_{h-1}), a)}$$

$$+ \beta d(x_{h-1}, u, a)$$

$$\leq d(u, g(u), x_h) + d(u, x_h, a)$$

$$+ \alpha \frac{d(u, g(x_{h+2k-1}), a)[1 + d(x_{h-1}, g(x_{h+2k-2}, a)]}{[1 + d(u, g(x_{h+2k-2}), a]}$$

$$+ \beta d(x_{h-1}, u, a)$$

Taking $h \to \infty$, x_h, x_{h-1}, x_{h+2k-2}, all tends to u.

Thus, $d(u, g(u), a) \leq d(u, g(u), u) + d(u, u, a)$

$$+ \alpha \frac{d(u, g(u), a)[1 + d(u, g(u), a)]}{1 + d(u, g(u), a)} + \beta d(u, u, a)$$

Or, $d(u, g(u), a) \leq \alpha\, d(u, g(u), a)$

i.e., $(1- \alpha)\, d(u, gu), a) \leq 0$ which gives $d(u, g(u), a) = 0$ and thus $u = g(u)$. Similarly $u = f(u)$. Thus u is a common fixed point of f and g. To show the uniqueness of u, let $v \neq u$ such that $f(v) = g(v) = v$.

Taking $z_1 = u = z_4 : z_4 = v = z_3$ in (I) we get,

$d(u, v, a) = d(f(u), g(v), a)$

$$\leq = \frac{d(v, u, a)[1 + d(u, f^k (v), a)]}{1 + d(u, g^k (u), a)} + \beta d(u, v, a)$$

$$\leq \alpha \frac{d(v, u, a)[1 + d(u, v, a)]}{1 + d(v, u, a)} + \beta d(u, v, a)$$

i.e., (1-α – β) d (u, v, a) ≤ 0 which gives d (u, v, a) = 0 and thus u = v. So. f and g have a unique common fixed point in X.

(2.2.9) THEOREM: Let E, F, T be three self maps of a complete 2-metric space (X, d) such that T is continuous and E, F, T satisfies the conditions :

(a) {E, T } and {F, T} are commutating pairs

(b) E X ⊂ TX : FX ⊂ TX

(c) ∍ m, n > 0 such that

$$d\,(E^m(x), F^n(y), a) \le \alpha \frac{d\,(T_x, E^m\,(x), a)\, d(T_y F^n(y), a)}{d\,(T_x, F^n(y), a) + d(T_y, E^m(x), a)}$$

$$+\ d\,(Tx, E^m\,(x), a) + d\,(Ty, F^n\,(y), a)$$

$$+\ \beta\, d\,(Tx, Ty, a) + \gamma\, [d(Tx, F^n(y), a) + d\,(Ty, E^m(x), a)\,]$$

For all x, y, a in X, where α, β, γ, ≥ O and α+β +2 y < 1.

Then E,F and T have a unique common fixed point in X.

Proof: Using (a) and (b)

$E^m\, T = TE^m : F^n\, T = T\, F^n$ …………… (1)

and

$$E^m X \subset EX \subset TX\}$$
$$F^n\ X \subset FX \subset TX\}\ \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Let x_0 be any arbitrary point of X. Since $E^m\, X \subset TX$,

We can choose a point x_1 in X such that $Tx_1 = E^m x_0$.

Also $F^n X \subset TX$. We can choose a point x_2 in X such that

$Tx_2 = F^n\, x_1$. In general, $Tx_{2p+1}\ \ E^m\, x_{2p}$ and

$Tx_{2p+2} = F^n x_{2p+1}$, for p = 0, 1, 2, ……

Now we consider,

$$d(Tx_{2p+1}, Tx_{2p+2}, a) = d(E^m x_{2p}, F^n x_{2p+1}, a)$$

$$\le \alpha \frac{d(Tx_{2p} E^m x_{2p}, a)\, d(Tx_{p+1}, F^n x_{2p+1}, a)}{d(Tx_{2p}, F^n x_{2p+1}, a) + d(Tx_{2p+1}, E^m x_{2p,}\ a)}$$

$$+ d(Tx_{2p}, E^m x_{2p}, a) + d(Tx_{2p+1}, F^n x_{2p+1}, a)$$

$$+\beta\, d(Tx_{2p}, Tx_{2p+1}, a) + \gamma[d(Tx_{2p}, F^n x_{2p+1}, a) + d(Tx_{2p+1}, F^n x_{2p}, a)$$

$$\le \alpha \frac{d(Tx_{2p} Tx_{2p+1}, a)\, d(Tx_{p+1}, Tx_{2p+2}, a)}{d(Tx_{2p}, Tx_{2p+2}, a) + d(Tx_{2p+1}, Tx_{2p+1,} a)}$$

$$+ d(Tx_{2p}, Tx_{2p+1}, a) + d(Tx_{2p+1}, Tx_{2p+2}, a)$$

$$+\beta\, d(Tx_{2p}, Tx_{2p+1}, a) + \gamma[d(Tx_{2p}, Tx_{2p+2}, a) + d(Tx_{2p+1}, Tx_{2p+1}, a)\ ]$$

$$\le \alpha \frac{d(Tx_{2p} Tx_{2p+1},)\, d(Tx_{2p+1}, Tx_{2p+2},)}{2\, d(Tx_{2p+1}, Tx_{2p+2}, a)} + \beta\, d(Tx_{2p}, Tx_{2p+1}, a)$$

$$+ \gamma\ [d(Tx_{2p}, Tx_{2p+1}, a) + d(Tx_{2p+1}, Tx_{2p+2}, a)$$

or, $(1-\gamma)\, d(Tx_{2p+1}, Tx_{2p+2}, a) \le (\frac{\alpha}{2} + \beta + \gamma)\, d(Tx_{2p}, Tx_{2p+1}, a)$

$\le d(Tx_{2p}, Tx_{2p+1}, \alpha)[\therefore \frac{\alpha}{2} + \beta + \gamma < 1]$

Proceeding in this way, we have

$$d(Tx_{2p+1}, Tx_{2p+2}, \alpha) \le d(Tx_{2p}, Tx_{2p+1}, a) \le ....$$

Thus, $\{Tx_p\}$ is monotone and hence convergent. Let the limit be x. From (2) $\{E^m x_{2p}\}$ and $\{F^n x_{2p}\}$ are subsequence of $\{Tx_{2p}\}$ also converges to some point x. Now,

$$\left.\begin{array}{l}\{E^m Tx_{2p} = T\, E^m x_{2p} = Tx\} \\ F^n Tx_{2p+1} = T\, F^n x_{2p+1} = Tx\}\end{array}\right. \quad \text{……………..} \ (3)$$

Now, we show that $E^m x = Tx = F^n x$.

$$d(E^m x, Tx, a) \le d(E^m x, T\, Tx_{2p+2}, a) + d(E^m x, Tx, T\, Tx_{2p+2}) + d(T\, T_{2p+2}, Tx, a)$$

$= d\,(T\,Tx_{2p+2}, Tx, a) + d\,(E^m x, Tx, T\,Tx_{2p+2}) + d\,(E^m x\, F^n Tx_{2p+1}, a)$

$\leq d\,(TTx_{2p+2}, Tx, \alpha) + d\,(E^m x, Tx, TTx_{2P+2})$

$$\leq \alpha\, \frac{d\,(E^m x\, Tx, a)\, d\,(TTx_{p+1}, F^n Tx_{2p+1}, a)}{d\,(TTx_{2p+1}, E^m x, a) + d\,(Tx, F^n Tx_{2p+1,} a)}$$

$+\, d\,(TTx_{2p+1}, F^n\, Tx_{2p+1}, a) + d\,(E^m x, Tx,\ a)$

$+\, \beta\, d\,(TTx_{2p+1}, Tx, a) + \gamma\,[\, d\,(TTx_{2P+1}, E^m x, a) + d\,(Tx, F^n\, Tx_{2P+1}, a)\,]$

$\to 0$ as $p \to \infty$ as $TTx_{2p+1} \to Tx$

$$L^m\, Tx_{2P} = TE^m\, x_{2P} \to Tx$$

$$F^n\, Tx_{2P+1} = TF^n\, x_{2P+1} = Tx.$$

Thus, $d\,(E^m\, x, Tx, a) = 0$ i.e., $E^m\, x = Tx$. Similarly $F^n\, x = Tx$

$d\,(E^m x, a, a) \leq d\,(E^m x, x, Tx_{2P+2}) + d\,(E^m x, Tx_{2P+2,} a) + d\,(Tx_{2P+2}, x, a)$

$= d\,(E^m x, x, Tx_{2P+2}) + d\,(Tx_{2p+2}, x, a) + d\,(E^m x, F^n x_{2p+1}, a)$

$\leq d\,(E^m x, x, Tx_{2P+2}) + d\,(Tx_{2P,}\ x, a)$

$$\leq \alpha\, \frac{d\,(E^m x\, Tx, a)\, d\,(E^n\ x_{p+1}, Tx_{2p+1}, a)}{d\,(E^m x, Tx\,, a) + d\,(F^n x_{2p+1,} Tx_{2P+1}, a) + d\,(E^m x, Tx_{2P+1}, a) + d\,(F^n x_{2P+1}, Tx, a)}$$

$+\, \beta\, d\,(Tx_{2p+1}, Tx, a) + \gamma\,[\, d\,(E^n x, Tx_{2P+1}, a) + d\,(F^n x_{2P+1}, Tx,\ a)\,]$

When $n \to \infty$, $Tx_{2P+2} \to \alpha$ also $E^m\, x = Tx$, we have $d\,(E^m\, x, x, a)$
$\leq \beta\, d\,(x, E^m x, a) + 2\,\gamma [d\,(E^m\ x, x, a)]$

or, $(1 - \beta - 2\,\gamma)\, d\,(E^m\, x, x, a) \leq 0$ which gives $d\,(E^m\, x, x, a) = 0$

So, E^m, $x = x$. Similarly $F^n\, x = x$. Also $Tx = E^m\, x = x$.

Since $E^m\, x = Tx$. Therefore $Tx = x$. Thus we get $E^m\, x = F^n\, x = x$. Now, $TEx = E\,Tx = Ex = E\,(E^m\, x) = E^m\,(Ex)$ i.e., Ex is a common fixed point of T and E^m. Similarly Fx is a common fixed point of T and F^n. But x is a unique common fixed point of E^m, F^n, and T.

Hence, $Ex = Tx = x = Fx$.

Uniqueness follows directly from (c). //

Remark: If we put m = n = 1 and T = I, identity map, we get corollary (2.2.3).

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