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SOME FIXED POINT THEOREMS IN 2 – METRIC SPACES

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Abstract: In this paper two we have done fixed point theorems for three and four operators at a time.

Key words Fixed point theorems, 2 – metric space, 2-Branch space.

1.INTRODUCTION: Let (x, d) be a complete 2 – metric space. A mapping T: X→ X is called a contraction on X if there exists a constant such that

$$d\,(Tx.\ Ty,\ a) \leq h\ d\,(x,\ y,\ a),\ \text{for all } x, y, a \text{ in } X, \text{ where } h\,\varepsilon\ (0.1).$$

nalogue to metric space, In 2 – metric space the theory of fixed points was developed by many authors such as Iseki (1), Lal and Singh (2), Rhoades (7) etc. Further information about metric spaces can be had from (3),(4),(5) and (6).

This paper consists of two sections. In first section we have generalized the results of above authors. In section two we have done fixed point theorems for three and four operators at a time.

PRELIMINARIES: Here we give some basic definitions and results.

DEFINITIONS:

(a) Let X be a non-empty set and $d : X \times X \times X \to R_+$ for all X, y, z, u in X, we have

(i) d (x, y, z) = 0 if at least two of x, y, z are equal

(ii) for all x ≠ y there exists a point Z in X such that d (x, y, z) ≠ 0

(iii) d (x, y, z) = d (x, z, y) = d (y, z, x) = … and so on

(iv) d (x, y, z) ≤ d (x, y, u) * d (x, u, z) + d (u, y, z).

Then d is called a 2–metric on X and the paid (X, d) is called 2–metric space.

(b) Let L be a linear space of dimension greater than one over a field of real numbers, and || .,.||, a real valued function LxL satisfying following conditions : For all a, b, c, ε L and αε R

(i) || a, b || = 0 iff a and b are linearly independent

(ii) || a, b || = || b, a ||

(iii) || a, α b || = | α | || a, b ||

(iv) || a, b + c || ≤ || a, b || + || a, c ||.

Then ||.,.|| is clled a <u>2–norm</u> on L and the paid (L, ||. , .||) is called a linear <u>2–normed</u> space.

(1.0.1) **<u>PROPOSITION</u>**: Every 2 – normed space is a 2 – metric space with 2 – metric d defined by d (a, b, c) = || b – a, 0 – a ||.

(1.0.3) **<u>DEFINITION</u>**: Let (X, d) be a 2 – metric space. A sequence { X_n } of points in X is called a Cauchy sequence if d (x_m, x_n, a) → 0 as m, n → ∞ for all a ε X. The sequence { X_n } is said to Converge to a point in X if $\overset{Lim}{n \to \infty}$ d (x_m, x, a) 0 for every a ε X. A 2 – metric space is <u>complete</u> if every Cauchy sequence converges.

(1.0.4.) **<u>DEFINITION</u>**: Let (L, || . , . ||) be a 2 – normed space. A sequence { X_n } in L is called <u>convergent</u> if there is a x in L such that $\overset{Lim}{n \to \infty}$ || x_n → x, a || = 0 for all a in L. And a sequence { x_n } in a 2 – normed space L is said to be <u>Cauchy</u> sequence if $\overset{Lim}{m, n \to \infty}$ || x_m – xn, a || = 0 for every a in L. A 2 – normed space is complete if every Cauchy sequence converges.

(1.0.5) **DEFINITION** : A complete 2 – normed space is called a <u>2–Banach space.</u>

(1.0.6) **<u>LEMMA</u>** [53] : Let { x_n } be a sequence in a complete 2 – metric space (X, d). If there exists r ε (0, 1) such that d (x_n, x_{n+1}, a) ≤ rd (x_{n-1}, x_n, a), for all non-negative integer n and every a in X, then { x_n } converges to a point in X.

(1.1) Several theorems have been proved for the existence of fixed point in 2- metric space. Lal and Singh [34] proved the following:

(1.1.1) **<u>THEOREM</u>**: Let (X, d) be a complete 2 – metric space and T_1 and T_2 two self-maps on X such that for all x, y, a in X.

(A) $d (T_1 (x), T_2 (y), a) \le a_1 d (x, T_1 (x), a) + a_2 d (y, T_2 (y), a) + a_3 d (x, T_2(y), a) + a_4 d (y, T_1 (x), a) + a_5 d (x, y, a)$

Where a_1, a_2, a_3, a_4, and a_5 are non – negative numbers such that $\sum_{i=1}^{\varepsilon} a_i < 1$ and $(a_1 – a_2) (a_3 – a_4) \ge 0$.

Then T_1 and T_2 have a unique common fixed point.

Here we proved the following more general theorem.

(1.1.2) THEOREM: Let $0 < \alpha < 1$, p and q be non – negative numbers such that $p + q < 1$,

(I) $\alpha \mid p - q \mid < 1 - (p + q)$
and $f : x \to X$ be a mapping of a complete 2 – metric space (X, d) such that whenever x, y, a are distinct elements in X.

(II) $d(f(x), g(y), a) \leq \alpha \max, \{ d(x,y,a), d(x, f(x), a), d(y, g(y), a) \}$
$+ (1-\alpha)\ [pd(x, \varepsilon(y), a) + q\, d(y, f(x), a)]$.

Then f and g have a unique common fixed point.

<u>Proof</u>: Let $x_0\ \varepsilon\ X$, $x_{23n+1} = f(x_{2n})$, $x_{2n+2} = g(x_{2n+2})$, for n = 0, 1, 2, …. First we show that $d(x_{2n}, x_{2n+1}, x_{2n+2}) = d(x_{2n+1}, x_{2n+2}, x_{2n}) = d(f(x_{2n}), g(x_{2+1+}), x_{2n}) = 0$. We have, $d(f(x_{2n}), g(x_{2n+1})\ x_{2n})\ \alpha \max.\{d(x_{2n}, x_{2n+1}, x_{2n}), d(x_{2n}, f(x_{2n}), x_{2n}), d(x_{2n+1}, g(x_{2n+1}), x_{2n})\}$

$+ (1\text{-}\alpha)\ [pd(x_{2n}(x_{2n}, g(x_{2n+1}), x_{2n})$

$+ q\, d(x_{2n+1}, f(x_{2n}), x_{2n})]$

$= \alpha\, d((x_{2n+1}, x_{2n+2}, x_{2n})$

or, $d(x_{2n}, x_{2n+1}, x_{2n+2}) \leq \alpha\, d(x_{2n+1}, x_{2+2}, x_{2n})$, a contradiction as $0, \leq \alpha \leq 1$. Thus, $d(x_{2n}, x_{2n+1}, x_{2n+2}) = 0$

Now, $d(x_{2+1}, x_{2n+2}, a) = d(f(x_{2n}), g(x_{2n+1+}), a)$

$\leq \alpha\ \max, \{d(x_{2n}, f(x_{2n}), a), d(x_{2n+1}),$
$g(x_{2n+1}), a), d(x_{2n+1}, g(x_{2n+1}), a)\}$
$+ (1\text{-}\alpha)\ [pd(x_{2n}, g(x_{2n+1}), a) + qd(x_{2n+1}, x_{2n+2}, a)]$
$= \alpha \max, \{d(x_{2n}, (x_{2n+1,}a), d(x_{2n+1}), x_{2n+1}, x_{2n+2}, a),$
$d(x_{2+1}, x_{2n+2}, a)\}$
$+ (1\text{-}\alpha)\ [pd((x_{2n}, x_{2n+2}, a) + qd(x_{2n+1}, x_{2n+2}, a)]$

or, $d(x_{2n+1}, x_{2n+2}, a) \leq \alpha \max.\{d(x_{2n}, x_{2n+1,}a), d(x_{2n+1}, x_{2n+2}, a)\}$
$+ (1-\alpha)\, pd(x_{2n}, x_{2n+2}, a)$

If $d(x_{2n}, x_{2n+1,}a)$ is maximum then,

$d(x_{2n+1}, x_{2n+2}, a) \leq \alpha\, d(x_{2n}, x_{2n+1}, a) + (1\text{-}\alpha)\, pd(x_{2n}, x_{2n+2}, a)$
$\leq \alpha\, d(x_{2n}, x_{2n+1}, a) + (1\text{-}\alpha)\, p\, [d(x_{2n}, x_{2n+1}, a)$
$+ d(x_{2n+1}, x_{2n+2}, a)]$

or, $1\text{-}(1\text{-}\alpha)\, p\, d(x_{2n+1}, x_{2n+2}, a) \leq \alpha + (1-\alpha)\, pd(x_{2n+1}, x_{2n+2}, a)$

or, $d(x_{2n+1}, x_{2n+2}, a) \leq \dfrac{\alpha + (1\text{-}\alpha)\,p}{1\text{-}(1\text{-}\alpha)\,p}\, d\ (x_{2n}, x_{2n+1}, a)$.

if $d(x_{2n+1}, x_{2n+2}, a)$ is maximum then,

$d(x_{2n+1}, x_{2n+2}, a) \leq \alpha\, d(x_{2n+1}, x_{2n+2}, a) + (1\text{-}\alpha)\, p\, d(x_{2n}, x_{2n+2}, a)$

or, $\left(\dfrac{1}{1-\alpha}\right) d(x_{2n+1}, x_{2n+2}, a) \leq \left(\dfrac{\alpha}{1-\alpha}\right) d\ (x_{2n+1}, x_{2n+2}, a) + pd(x_{2n}, x_{2n+2}, a)$

or, $d(x_{2n+1}, x_{2n+2}, a) \leq pd(x_{2n}, x_{2n+2}, a)$
$\leq p\, [d(x_{2n}, x_{2n+1}, a) + d(x_{2n+1}, x_{2n+2}, a)]$
$\leq \left(\dfrac{p}{1-p}\right) d(x_{2n}, x_{2n+1}, a)$

Thus, $d(x_{2n+1}, x_{2n+2}, a) \leq \max.\left\{\dfrac{\alpha + (1\text{-}\alpha)p}{1-(1\text{-}\alpha)p}, \dfrac{p}{1\text{-}p}\right\} d(x_{2n}, x_{2n+1}, a)$
$\leq \beta\, d(x_{2n}, x_{2n+1}, a),$

$$\text{Where } \beta = \max\left\{\frac{\alpha+(1-\alpha)p}{1-(1-\alpha)p}, \frac{p}{1-p}\right\}$$

Again,

$d(x_{2n}, x_{2n+1}, a), = d(g(x_{2-1}), f(x_{2n}), a)$
$= (f(x_{2-n}), g(x_{2n-1}), a)$
$\leq \alpha \max.\{d(x_{2n}, x_{2n-1}, a), d(x_{2n}, f(x_{2n}), a), d(x_{2n-1}, g(x_{2n-1}), a)\}$
$+ (1-\alpha)[pd(x_{2n}, g(x_{2n-1}), a)$
$+ q\, d(x_{2n-1}, f(x_{2n}), a)\}$
$= \alpha \max.\{d(x_{2n}, x_{2n-1}, a), d(x_{2n}, x_{2n-1}, a)\}$
$+ (1-\alpha)\, q\,[d(x_{2n-1}, x_{2n+1}, x)]$

or, $d(x_{2n}, x_{2n+1}, a) \leq y\, d(x_{2n}, x_{2n-1}, a)$,

where $y = \max\left\{\frac{\alpha+(1-\alpha)q}{1-(1-\alpha)q}, \frac{q}{1-q}\right\}$, proceeding in this way,

we have

$d(x_{2n+1}, x_{2n+2}, a) \leq \beta\; d(x_{2n}, x_{2n+1}, a)$
$\leq \beta\; y\, d(x_{2n+1}, x_{2n}, 2)$
$\leq (\beta\, y)^n\; d(x_0, x_1, a)$

Let $c = \beta\, y$, than $d(x_{2n+1}, x_{2n+2}, 2) \leq 0^n d(x_0, x_1, a)$.

Now if p, q $\varepsilon \left[0, \frac{1}{2}\right]$, then $\beta < 1$, $y < 1$ and so $0 < c < 1$.

If $\max\{p,q\} \geq \frac{1}{2}$. Then since $\frac{\alpha+(1-\alpha)x}{1-(1-\alpha)x} \leq \frac{x}{1-x} \Longleftrightarrow \frac{1}{2} \leq x$,

$\forall\, x\; \varepsilon\, [0, 1)$ it follows from (I) that $0 \leq c < 1$.

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges say to z. Then $x_{2n+1} \neq z$ for infinitely many n. If $x_{2n} \neq z$ for infinitely many n, then by considering:

$d(z, g(z), a) \leq d(z, g(x), x_{2+1}) + d(z, x_{2n+1}, a) + d(x_{2n+1}, g(z), a)$
$= d(z, g(z)\, x_{2+1}) + d(z, x_{2n+1}, a) + d(f(x_{2n}), g(z), a)$
$\leq d(z, g(z)\, x_{2+1}) + d(z, x_{2n+1}, a) + \alpha \max\{d(x_{2n}, z, a),$
$d(x_{2n}, f(x_{2n}), a), d(z, g(z), a)\}$
$+ (1-\alpha)[p\, d(x_{2n}, g(z), a) + q\, d(x, f(x_{2n}), a)]$
$d(z, g(z), x_{2n+1}) + d(z, x_{2+1}, a) + \alpha\; \max.\{d(x_{2n}, z, a\},$
$d(x_{2n}, x_{2+1}, a), d(z, g(z), a)\}$
$+ (1-\alpha)[p\, d(x_{2n}, g(z), a) + q\, d(z, x_{2+1}, a]$

When $n \to \infty$,

$d(z,g(z),a) \leq d(z, g(z), a) + d(z,z,a) + \alpha\; \max.\{d(z, z, a)\},$
$d(z,z,a), d(z, g(z), a) + (1-\alpha)[p\, d(z, g(z), a)$
$+ q\, d(z, z, a)]$
$= 0 + 0 + \alpha\, d(z, g(z), a) + (1-\alpha)\, p\, d(z, g(z), a)$

or, 1- α – (1- α) p d(z), g (z), a) ≤ 0, which is a contradiction. So, d (z, g (z), a)=0, i.e. z = g (z).

Similarly we can show that f (z) = z. Thus z is the common fixed point of f and g. we claim that z is the unique common fixed point of f and g. For this let w ≠ z and f (w) = g (w) = w. Then by considering

$$d(z,w,a)=d(f(z), g(z), a)$$
$$\leq \alpha \max \{d(z,w,a), d(z, f(z), a), d(w,g(w, a)\}$$
$$+ (1-\alpha)[p\, d(z, g(w), a) + q\, d(w, f(z), a)]$$
$$= \alpha\, d(z,w,a) + (1-\alpha)(p+q)\, d(z,w,a)$$

or, d (z, w, a) ≤ (p+q) d (z,w, a) which is no possible.

So, d (z, w, a) = 0 i.e., z = w. //

(1.1.3) **COROLLARY**: Let (X, d) be a complete 2 – metric space and f : X → X, g : X → X be such that for all x, y, a in X.

$$d(f(x), g(y), a) \leq k_1\, d(x, y, a) + k_2\, d(x, f(x), a) + k_3\, d(y, f(x), a)$$

Where $k_i \geq 0\ \forall$ I, $\sum_{i=1}^{\beta} k_i < 1$ and

(I) $|k_4 - k_5|\ (k_1 + k_2 + k_3) < 1 - \sum_{i-1}^{\beta} k_i$.

Then f and g have a unique common fixed point.

Proof: we take $\alpha = k_1 + k_2\, k_3$, $p = \frac{k_4}{1-\alpha}$ and $q = \frac{k_5}{1-\alpha}$ in theorem 1.1.1. //

Rhoades [43] proved the following:

(1.1.4) **THEOREM**: Let f, g : X → X be a mapping of a complete 2 – metric space (X,d) into itself satisfying

(B) $$d(f(x), g(y), a) \leq h \max. \{d(x, y, a), d(x, f(x), a)\, d(y, g(y), a), \frac{1}{2}[d(x, g(y), a) + d(y, f(y), a)]\}$$

For all x, y, a ε X, where h is a fixed constant such that 0 ≤ h < 1. Then f and g have a common unique fixed point.

Here first we have generalized the above result by proving it for two sequences of mappings and secondly by improving the hypothesis of the above theorem.

(1.1.5) **THEOREM**: Let $\{f_n\}$ (n = 0, 1, 2, ….) be a sequence of mappings of a complete 2 – metric space (X, d) into itself such that for some h with O < < 1 and for every x, y, a in X.

$$d(f_0^P(x), (f_n^Q(y), a) \leq h \max. \{d(x, y, a), d(x, (f_0^P(x), a), d(y, (f_n^Q(y), a), \frac{1}{2}[d(x, f_n^Q(y), a) + d(y, f_0^P(x),a)]\}$$

for each n = 1, 2, … and p, q > 1 hold. Then there exists a unique common fixed point of f_n (n = 0, 1, 2, …) in X.

Proof: Let x_0 be an arbitrary point of X. Define a sequence $\{x_n\}$ as follows:

x_0, $x_1 = f_0^P (x_0) : x_2 = f_1^q (x_1) : x_3 = f_0^P (x_2)$:
$x_4 = f_2^q (x_3)$, ……….. $x_{2n-1} = f_0^P (x_{2n-2}) : x_{2n} = f_n^q (x_{n-1})$.

Now first we show that $d (x_{2n}, x_{2n+1}, x_{2n+2}) = 0$ for n = 0, 1, 2…

$$\begin{aligned} d(x_{2n}, x_{2n+1}, x_{2n+2}) &= d(f_0^P (x_{2n}), f_{n+1}^q (x_{2n+1}), x_{2n}) \\ &\leq h \max. \{d (x_{2n}, x_{2n+1}, x_{2n}), d(x_{2n}, f_0^P (x_{2n}), x_{2n}), \\ &\quad d (x_{2n+1}, f_{n+1}^q (x_{2n+1}), x_{2n}), \\ &\quad \frac{1}{2} [d (x_{2n}, f_{n+1}^q (x_{2n+1}), x_{2n}), + d (x_{2n+1}), f_0^P (x_{2n}), x_{2n})] \} \\ &= h \max \{d (x_{2n}, x_{2n+1}, x_{2n}), d(x_{2n}, x_{2n}, x_{2n+1}, x_{2n}), \\ &\quad d (x_{2n+1}, x_{2n+2}, x_{2n}), \\ &\quad \frac{1}{2} [d (x_{2n}, x_{2n+2}, x_{2n}), + d (x_{2n+1}, x_{2n+1}, x_{2n})] \} \end{aligned}$$

i.e. $d (x_{2n}, x_{2n+1}, x_{2n+2}) \leq h\, d (x_{2n+1}, x_{2n+2}, x_{2n})$ which is impossible. Hence, $d (x_{2n}, x_{2n+1}, x_{2n+2}) = 0$.

$$\begin{aligned} \text{Now, } d (x_1, x_2, a) &= d (f_0^P (x_0), f_0^q (x_1), a) \\ &\leq h \max \{ d (x_0, x_1, a), d (x_0, f_0^P (x_0), a), \\ &\qquad d (x_1, f_1^q (x_1), a), \frac{1}{2} [d (x_0, f_1^q (x_1), a) + \\ &\qquad\qquad d (x_1, f_0^P (x_0), a)] \} \\ &\qquad = h \max \{ d (x_0, x_1, a), d (x_0, x_1, a)\, d (x_1\, x_2, a) \\ &\qquad \frac{1}{2} [d (x_0, f_1^q (x_1), a) + d (x_1, f_0^P (x_0), a)] \} \\ &\qquad = h \max. [d (x_0, x_1 ,a), d (x_1, x_2, a), \frac{1}{2} d (x_0, x_2, a] \} \\ &\leq h \max \{d (x_0, x_1, a), d (x_1, x_2, a), \frac{1}{2} [d (x_1, x_2, a) \\ &\qquad\qquad + d (x_0, x_1), a)] \} \text{ …………… (2)} \end{aligned}$$

Now if $d (x_1, x_2, a)$ is maximum, then from (2)

$d (x_1, x_2, a) \leq h\, d (x_1, x_2, a) < d (x_1, x_2, a)$, a contradiction.

Thus, $d (x_0, x_1, a) \geq d (x_1, x_2, a)$ ………………………………… (3)

Again if possible, let

$$d (x_0, x_1, a) < \frac{1}{2} [d (x_0, x_1, a) + d (x_1, x_2, a)]$$

also $(x_1, x_2, a) < \frac{h}{2} [d (x_0, x_1, a) + d (x_1, x_2, a)]$

combining these two, we have

$$d(x_0, x_1, a) + d(x_1, x_2, a) < \frac{1+h}{2} [d(x_0, x_1, a) + d(x_1, x_2, a)],$$

thus, we again get a contradiction

Again, $d(x_2, x_3, a) = d(f_0^P(x_2), (f_0^q(x_1), a)$

$$\leq = h \max. \{d(x_2, x_1, a), d(x_2, x_3, a), (x_1, x_2, a), \frac{1}{2}[d(x_2, x_2, a) + d(x_1, x_3, a)]\}$$

$$\leq = h \max. \{d(x_1, x_2, a), d(x_2, x_3, a), \frac{1}{2}[d(x_1, x_3, a)\} \quad \ldots(4)$$

By similar argument we get from (4),

$$d(x_2, x_3, a) \leq h\, d(x_1, x_2, a) \leq h^2 d(x_0, x_1, a).$$

Therefore $d(x_n, x_{n+1}, a) \leq n^n d(x_0, x_1, a)$

Now we show that $\{x_n\}$ n ε N is a Cauchy sequence

Since, $d(x_n, x_{n+m}, a) \leq d(x_n, x_{n+1}, x_{n+m}) + d(x_n, x_{n+1}, a)$

$$+ d(x_{n+1}, x_{n+2}, x_{n+m}) + d(x_{n+1}, x_{n+2}, a) +$$
$$+ \ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots$$
$$+ \ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots$$
$$+ d(x_{n+m-2}, x_{n+m-1}, x_{n+m}) + d(x_{n+m-1}, x_{n+m}, a)$$
$$\leq \sum_{k=1}^{n+m-2} d(x_k, x_{k+1}, x_{n+m}) + \sum_{k=n}^{n+m-1} d(x_k, x_{k+1}, a).$$

Now, we have $d(x_n, x_{n+1}, x_{n+m}) \leq h\, d(x_n, x_{n-1}, x_{n+m})$

$$\leq h^n d(x_0, x_1, x_{n+m})$$

and also $d(x_n, x_{n+1}, a) \leq h\, d(x_{n-1}, x_n, a)$

$$\leq h^n d(x_0, x_1, a)$$

Thus,

$$\sum_{k=n}^{n+m-2} d(x_k, x_{k+1}, x_{n+m}) \leq [h^{n+m-2} + h^{n+m-1} + \ldots\ldots + h^n]\, d(x_0, x_1, a)$$

$$< \left(\frac{h^{n+m-2}}{1-h}\right) d(x_0, x_1, a)$$

$$\to 0 \text{ as } n \to \infty$$

and $\sum_{k=n}^{n+m-1} d(x_k, x_{k+1}, a) \leq [h^{n+m-1} + h^{n+m} + \ldots\ldots + h^n]\, d(x_0, x_1, a)$

$$\leq \frac{h^{n+m-1}}{1-h} d(x_0, x_1, a)$$

$$\to 0 \text{ as } n \to \infty$$

Thus $\{x_n\}_{n\,\varepsilon\,N}$ is a Cauchy sequence. Since X is complete there exists a point x_0 in X such that $\lim_{n\to\infty} x_n = x_0$.

Now we show that x_0 is the fixed point of f_0^p

$$d(x_0, f_0^p, (x_0), a) \leq d(x_0, x_{2n}, a) + d(x_0, f_0^p(x_0), x_{2n}) + d(x_0, f_0^p(x_0), a)$$

$$= d(x_0, x_{2n}, a) + d(x_0, f_0^p(x_0), x_{2n})$$

$+ \text{d } f_n^q (x_{2n-1}), \; f_0^p (x_0), a)$

$\leq \text{d } (x_0, x_{2n}, a) + \text{d } (x_0, \; f_0^p (x_0), x_{2n})$

$+ \text{max. } \{\text{d } (x_0, x_{2n-1}, a), \text{d } (x_0, \; f_0^p (x_0), a),$

$\text{d } (x_{2n-1}, x_{2n}, a), \; \frac{1}{2} \; [\text{ d } (x_0, x_{2n}, a)$

$+ \text{d } (x_{2-1}, \; f_0^p (x_0), a) \;] \; \}$

When $n \rightarrow \infty$

$\text{D } (x_0, \; f_0^p (x_0), a) \leq 0 + 0 + \text{h max } \{ 0, \text{d } (x_0, \; f_0^p (x_0), a), 0,$

$\frac{1}{2} \; [\; 0 + \text{d } (x_0, \; f_0^p (x_0), a) \;] \; \}$

$= \text{h max } \{\text{d } (x \; f_0^p (x_0), a), \; \frac{d(x_0), f_0^p(x_0), a)}{2} \}$

Thus, $\text{d } (x_0, \; f_0^p (x_0), a) = 0.$

Hence, $x_0 = f_0^p (x_0)$.

Now we shall show that x_0 is also a fixed point of

$f_n^q (n = 1, 2 ......).$

Since, $\text{d } (x_0, \; f_n^q (x_0), a) = \text{d } (\; f_0^p (x_0), \; f_n^q (x_0), a)$

$\leq \text{h max } \{ \text{ d } (x_0, x_0, a), \text{d } (\; f_0^p (x_0), a)$

$\text{d } (x_0, \; f_n^q (x_0), a), \; \frac{1}{2} \text{ d } (x_0, \; f_n^q (x_0), a)$

$+ \text{d } (x_0, (\; f_0^p (x_0), a) \;] \; \}$

$= \text{h max } \{0,0,\text{d } (x_0, f_n^q (x_0), a),$

$\frac{1}{2} \text{ d } (x_0, f_n^q a) + 0] \; \}$

$= \text{h max. } \{\text{d } (x_0, \; f_n^q a) (x_0), a), \; \frac{d(x_0), f_n^q(x_0), a)}{2} \}$

i.e. $\text{d } (x_0, \; f_n^q a) (x_0), a), \leq \text{h d } (x_0, \; f_n^q a) (x_0), a)$ which is impossible

Thus we get $x_0 = x_0, = f_n^q a (x_0)$ (n = 1, 2, ……)

Now we claim that x_0 is the unique common fixed point of

f_0^p and f_n^q n = 1, 2, ……)

If possible let $y_0 \neq x_0$ be another fixed point.

Then $f_0^p (y_0) = f_n^q (y_0) = y_0$.

Now, $\text{d } (x_0, y_0 \; a) = \text{d } (\; f_0^p (x_0), \; f_n^q (y_0) \; , a)$

$\leq \text{h max } \{ \text{ d } (x_0, y_0, a), \text{d } (x_0, f_0^p (x_0), a),$

$\text{d } (y_0, \; f_n^q (y_0) \; , a), \; \frac{1}{2} [\text{d } (x_0, \; f_n^q (y_0) \; , a)$

$+ \text{d } (y_0, f_0^p (x_0), a) \;] \; \}$

$= \text{h max. } \{ \text{ d } (x_0, y_0, a), \text{d } (x_0, x_0), a), \text{d } (y_0, y_0, a),$

$\frac{1}{2} [\text{d } (x_0, y_0 \; , a) + \text{d } (y_0, x_0, a) \;]$

i.e., $\text{d } (x_0, y_0 \; , a) \leq \text{h max. d } (x_0, y_0, a)$ which is a contradiction.

Thus, d $(x_0, y_0, a) = 0$, So, $x_0 = y_0$.

Now, we show that x_0 is the unique common fixed point of f_n (n = 0, 1, 2…..),

$$f_0^p\ (f_0\ (x_0)) = f_0\ (\ f_0^p\ (x_0)) \text{ gives}$$

$$f_0^p\ (f_0\ (x_0)) = f_0\ (x_0)$$

i.e. $f_0\ (x_0) = x_0$, by uniqueness of x_0 as the fixed point of f_0^p.

Similarly, $f_n^q\ (f_n)\ (x_0)) = f_n\ \ f_n^q\ (x_0))$, gives $f_n^q\ (f_n(x_0)) = f_n(x_0)$

i.e., $f_n\ (x_0) = x_0$, by uniqueness of x_0 as the fixed point of f_n^q.

Thus, x_0 is the common fixed point of f_n (n = 1, 2, …)

Finally, we show that x_0 is the only fixed common point of f_n (n = 0, 1, 2)

For if z_0 were a point such that $z_0 \neq x_0$ and $f_n\ (z_0) = z_0$. Then,

$$\begin{aligned} d\ (x_0, z_0, a) = d\ (f_0(x_0), f_n(z_0), a) &= d\ (\ f_0^p(x_0),\ f_n^q(z_0), a) \\ &\leq h \max. \{d(x_0, z_0, a), d\ (x_0,\ f_0^p(x_0), a), \\ &\quad d\ (z_0,\ f_n^q(z_0), a), \frac{1}{2}\ [d\ (x_0,\ f_n^q(x_0), a) \\ &\qquad + d\ (z_0,\ f_0^p(z_0), a)]\ \} \\ &= h \max. \{d\ (x_0, z_0, a), d\ (x_0, x_0, a), d\ (z_0, z_0, a), \\ &\quad \frac{1}{2}\ [d\ (x_0, (z_0), a) + (z_0, x_0, a)] \end{aligned}$$

i.e., $d\ (x_0, z_0, a) \leq h\ d\ (x_0, z_0, a)$ which is a contradiction.

Thus $d\ (x_0, z_0, a) = 0$ showing that $x_0 = z_0$. //

Remark: If we put p = q = 1 in theorem (1.1.5) we get the following.

(1.1.6) **COROLLARY**: Let $\{f_n\}$ (n= 0, 1, 2….) be a sequence of mappings of a complete 2 – metric space X into itself.
If for some h with $0 < h < 1$ and for every x, y, a in X.

$$d\ f_0(x), f_n\ (y), a) \leq h \max. \{d\ (x, y, a), d\ (x, f_0(x), a), d(y, f_n\ (y), a),$$

$$\frac{1}{2}\ [d\ (x, f_n\ (y), a) + d\ (y\ f_0, (x)\ a)]\ \}$$

For each n = 1, 2, … hold. Then there exists a unique common fixed point of f_n (n = 0, 1, 2 ….) in X.

(1.1.7) **THEOREM**: Let $\{f_m\}$, $\{f_n\}$ (m, n = 1, 2, 3, ….) be two sequences of mappings on a complete 2 – metric space X into itself. If for some h with $0 < h < 1$ and for every x, y, a in X.

$$d\ f_m^p(x), g_n^q(y), a) \le h \max.\begin{Bmatrix} d(x, y, a), d(x, f_m^p(x), a), \\ d(y, g_n^q(y), a)\frac{1}{2}[d(x, g_n^q(y), a \\ + d(y, f_m^p(x), a)] \end{Bmatrix}$$

for each m, n = 1, 2, 3 … with p, q ≥ 1 hold. Then f_m, g_n (m, n = 1, 2, 3,) each have a unique common fixed point.

Proof: We consider my x_0 ε X and construct a sequence $\{x_n\}$ as follows:

x_0, $x_1 = f_0^p(x_0)$, : $x_2 = g_1^q(x_1)$

$x_3 = f_2^p(x_2) : x_4 = g_2^q(x_3)$

: :

: :

$x_{2+1} = f_{2n}^p(x_{2n})\ x_{2n+2} = g_{2n+1}^q(x_{2n+1})$

Now, first we show that d $(x_{2n}, x_{2n+1}, x_{2n+2}) = 0$

$d(x_{2n}, x_{2n+1}, x_{2n+2}) = d\ (f_{2n}^p(x_{2n}), g_{2n+1}^q(x_{2n+1}), x_{2n})$

$\le h \max. \{d(x_{2n}, x_{2n+1}, x_{2n}),$

$d(x_{2n}, (f_{2n}^p(x_{2n}), x_{2n}), d(x_{2n+1} g_{2n+1}^q(x_{2+1}) x_{2n})$

$\frac{1}{2}[d(x_{2n}, g_{2n+1}(x_{2n+1}), x_{2n}) + d(x_{2n+1}, f_{2n}^p, (x_{2n}) x_{2n})]\}$

$= h \max. \{d(x_{2n+1}, x_{2n+2}, x_{2n}), dd(x_{2n}, x_{2n+1}, x_{2n}),$

$d(x_{2n+1}, x_{2n+2}, x_{2n}),$

$\frac{1}{2}[d(x_{2n}, x_{2n+2} x_2) + d(x_{2n+1}, x_{2n+1}, x_{2n})]\}$

i.e., $d(x_{2n}, x_{2n+1}, x_{2n+2}) \le h\ d(x_{2n}, x_{2n+1}, x_{2n+2})$, which is impossible. Thus $d(x_{2n}, x_{2n+1}, x_{2n+2}) = 0$.

Now, $d(x_1, x_2, a) = d\ f_0^p(x_0),\ g_1^q(x_1), a)$

$\le h \max. \{d(x_0, x_1, a), d(x_0\ f_0^p(x_0), a),\ d(x_1, g_1^q(x_1), a),$

$\frac{1}{2}[d(x_0, g_1^q(x_1), a) + d(x_1, f_0(x_0), a)]\}$

$\le h \max. \{d(x_0, x_1, a), d(x_0\ x_1, a),\ d(x_1, x_2, a),$

$\frac{1}{2}[d(x_0, x_2, a)]\}$........(2)

$\le h \max. \{d(x_0, x_1, a), d(x_1\ x_2, a),$

$\frac{1}{2}[d(x_1, x_2, a) + d(x_0, x_1, a)]\}$

Now if, $d(x_1, x_2, a)$ is maximum, then from (2)

$d(x_1, x_2, a) \le h\ d(x_1, x_2, a) \le d(x_1, x_2, a)$, a contradiction.

Thus, $d(x_0, x_1, a) \ge d(x_1, x_2, a)$ …….. (3)

Again if possible, let $\frac{1}{2}d(x_0, x_2, a)$ is maximum

Then, $d(x_0, x_1, a) < \frac{1}{2}d(x_0, x_1, a) + d(x_1, x_2, a)$

Also d $(x_1, x_2, a) \le \frac{h}{2}[d(x_0, x_1, a) + d(x_1, x_2, a)]$

Combining these two inequalities, we have :

d (x_0, x_1, a) + d $(x_1, x_2, a) \le \frac{1+h}{2}[d(x_0, x_1, a) + d(x_1, x_2, a)]$,

which is not possible.

So, we get, d $(x_1, x_2, a) \le$ h d (x_0, x_1, x_2).

Again, d $(x_2, x_3, a) \le$ d $g_1^q(x_1), f_2^p(x_2), a) =$ d $(f_2^p(x_2), g_1^q(x_1), a)$

$\le$ h max.. {d (x_2, x_1, a), d $(x_2, f_2^p(x_2), a)$,

d $(x_1, g_1^q(x_1), a), \frac{1}{2}[d(x_2, g_1^q(x_1), a) + d(x_1, f_1^p(x_2), a)]\}$

$=$ h max, $\{d(x_1, x_2, a), d(x_2, x_3, a), \frac{1}{2}d(x_1, x_3, a)]\}$ ….. (4)

By similar argument we get from (4)

d $(x_2, x_3, a) \le$ h d $(x_1, x_2, a), \le h^2 d(x_0, x_1, a)$

:

:

d $(x_n, x_{n+1}, a) \le h^n (x_0, x_1, a)$

Now it is easy to prove that $\{x_n\}_{n\varepsilon}$ N is a Cauchy sequence as done in previous theorem, and since X is complete.

Thus $\lim_{n \to \infty} x_n = x_0$ in X.

Now we claim that x_0 is the common unique fixed point of F_m and g_n.

For this,

d $x_0, f_m^p(x_0), a) \le d(x_{2n}, f_m^p(x_0), a) + d(x_0, x_{2n}, 2) + d(x_0, f_m^p(x_0), x_{2n})$

$=$ d $(x_0, x_{2n}, a) + d(x_0, {}_m^p(x_0), x_{2n})$

$+ d(f_m^p(x_0), g_n^q(x_{2n-1}), a)$ …..(5)

$\le$ d $(x_0, x_{2n}, a) + d(x_0, f_m^p(x_0), x_{2n})$

+ h max. $\{d(x_0, x_{2n-1}, a) d(x_0, f_m^p(x_0), a)$,

$d(x_{2n-1}, g_{2n-1}(x_{2n-1},), a)$,

$\frac{1}{2}[d(x_0, g_{2n-1}^q(x_{2n-1}), a) + d(x_{2n-1}, f_m^p(x_0), a)]\}$

$= d(x_0, x_{2n}, a) + d(x_0, f_m^p(x_0), a_{2n})$

+ h max. $\{d(x_0, x_{2n-1}, a) + d(x_0, f_m^p(x_0), a)$,

$d(x_{2n-1}, x_{2n}, a), \frac{1}{2}[d(x_0, x_{2n}, a) + d(x_{2n-1}, f_m^p(x_0), a)]\}$

When $n \to \infty$

$$d(x_0, f_m^p(x_0), a) \le 0 + 0 + h \max.\{0, d(x_0, f_m^p(x_0), a), 0, \frac{1}{2}[0 + d(x_0, f_m^p(x_0), a)]$$

i.e., $d(x_0, f_m^p(x_0), a) \le h \max.\{d(x_0, f_m^p(x_0), a), 0, \frac{1}{2} d(x_0, f_m^p(x_0), a)\}$

Thus, $d(x_0, f_m^p(x_0), a) \le h\, d(x_0, f_m^p(x_0), a)$ which implies that

$d(x_0, f_m^p(x_0), a) = 0$ showing that $f_m^p(x_0) = x_0$

Similarly $d(x_0, g_n^q(x_0), a) = d(f_m^p(x_0), = g_n^q(x_0), a)$

$$\le h \max.\{d(x_0, x_0, a), d(x_0, f_m^p(x_0), a),$$

$$= d(x_0, g_n^q(x_0), a \frac{1}{2}[d(x_0, g_n^q(x_0), a) + d(x_0, f_m^p(x_0), a)]\}$$

$$= h \max.\ \{0,0, d, (x_0, f_m^p(x_0), a, \frac{1}{2}[d(x_0, g_n^q(x_0), a)\},$$

Which gives that $d(x_0, g_n^q(x_0), a) = 0$ and $\ne 0$, $x_0 = g_n^q x_0$.

Thus x_0 is the common fixed point of $f_m^p and\ g_n^q$,

i.e., $f_m^p(x_0) = g_n^q(y_0) = x_0$,

If possible let $y_0 \ne x_0$ be another common fixed point of

$f_m^p and, g_n^q. Then\ f_m^p(y_0) = g_n^q\ y_0) = y_0$.

Then $d(x_0, y_0, a) = d(f_m^p(x_0), g_n^q(y_0), a)$

$\le h \max.\{d(x_0, y_0, a), d(x_0 f_m^p(x_0), a),$

$d(y_0 g_n^q(y_0), a), \frac{1}{2}[d(x_0, g_n^q(y_0), a) + d(y_0 f_m^p(x_0), a)]\}$

$$= h \max. \begin{array}{l}\{d(x_0, y_0, a), d(x_0 y_0 a), d(y_0, y_0, a), a),\\ \frac{1}{2}[d(x_0, y_0, a) + d(y_0, x_0, a)]\}\end{array}$$

i.e. $d(x_0, y_0, a) \le h\, d(x_0, y_0, a)$ which is not possible.

So, $d(x_0, y_0, a) = 0$ which gives $x_0 = y_0$.

Thus, we show that x_0 is the unique common fixed point of f_m^p and g_n^q.

Further we show that x_0 is the unique common fixed point of f_m and g_n (m, n = 1, 2, 3, ….)

For, $f_m^p(f_m(x_0)) = f_m(f_m^p(x_0))$ gives $f_m^p(f_m(x_0)) = f_m(x_0)$

i.e., $f_m(x_0) = x_0$, by the uniqueness of x_0 as the fixed point of f_m^p.

Similarly $g_0(x_0) = x_0$.

Finally, we show that x_0 is the only fixed point common to f_m and g_n (m, n = 1, 2, 3, …..)

for if z_0 were the fixed point such that $x_0 \ne z_0$

and $f_m(z_0) = g_n(z_0) = z_0$, then

$$d(x_0, z_0, a) = d(f_m(x_0), g_n(z_0), a) = d(f_m^p(x_0), g_n^q(z_0), a)$$

$$\leq \text{h max.} \begin{matrix}\{d(x_0, z_0, a), d(x_0, f_m^p(x_0), a), d(z_0, g_n^q)(z_0), a), \\ \frac{1}{2}[d(x_0, g_n^q(xz_0), a) + d(z_0, f_m^p(x_0), a)]\}\end{matrix}$$

$$= h.\max.\{d(x_0, z_0, a), d(x_0, x_0, a), d(z_0, z_0, a), \frac{1}{2}[d(x_0, z_0), a) + d(z_0, x_0), a)]\}$$

$$= h.\max.\{d(x_0, z_0, a), d(x_0, z_0, a)\} \text{ which}$$

gives that $d(x_0, z_0, a) = 0$ so $x_0 = z_0$.//

<u>Remark:</u> If we put p = q = 1 in theorem (1.1.7) we get the following:

(1.1.8) <u>COROLLARY:</u> Let $\{f_m\}$, $\{g_n\}$ (m, n = 1, 2, 3, …) be two sequences of mappings on a complete 2 – metric space X into itself. If for some h with 0< h<1 and for every x, y, a in X ;

$$d(f_m(x), g_n(y), a) \leq h \max. \{d(x, y, a), d(x, f_m(x), a). d(y, g_n(y), a), \frac{1}{2}[d(x, g_n(y), a) + d(y, f_m(x), a)]\}$$

Then $\{f_m\}$, $\{g_n\}$ (m, n = 1, 2, 3, …) each have a unique common fixed point.

Next we improve the Theorem 1.1.4 by improving the hypothesis s follows:

(1.1.9) THEOREM: Let f and g be a mapping of a complete 2 – metric space (X,d) into itself such that

$$\text{(I) } d(f(x), g(y), a) \leq \max. \{\propto d(x, y, a), \propto d(x, f(x), a), \propto d(y, g(y), a), pd(x, g(y), a) + q\, d(y, f(x), a)\}$$

For all x, y, a in X where $0 \leq \propto < 1$, p, q > 0, p + q < 1 and max.

$\left\{\frac{p}{1-p}, \frac{q}{1-q}\right\} < 1.$ Then f and g have a unique common fixed point.

<u>Proof</u>: Let x_0 ε X, $x_{2n+1} = f(x_{2n})$ and $x_{2n+2} = g(x_{2n+1})$ for n = 0, 1, 2, ….. . We may assume that $x_n \neq x_{n+1}$ for any n. First we show that $d(x_{2n}, x_{2n+1}, x_{2+2}) = 0$ $d(x_{2n}, x_{2n+1}, x_{2n+2}) = d(f(x_{2n}), g(x_{2+1}), x_{2n})$

$$\leq \max\{\alpha\, d(x_{2n}, x_{2n+1}, x_{2n}), \alpha\, d(x_{2n}, f(x_{2n}), x_{2n}) \alpha\, d(x_{2n+1}, g(x_{2n+1}), x_{2n}), pd(x_{2n}, g(x_{2n+1}), x_{2n}) + q\, d(x_{2n+1}), f(x_{2n}), x_{2n})\}$$

$= \alpha\, d\,(x_{2n+1}, x_{2n+2}, x_{2n})$

or, $(1-\alpha)\, d\,(x_{2n}, x_{2n+1}, x_{2n+2}) < 0$ which implies that $d\,(x_{2n}, x_{2n+1}, x_{2n+2}) = 0$.

Now, $d\,(x_{2n+1}, x_{2n}, a) = d\,(f(x_{2n}), g\,(x_{2n-1}), a)$

$\leq \max.\ \{\alpha\, d\,(x_{2n}, x_{2n-1}, a), \alpha\, d\,(x_{2n}, f(x_{2n}), a),$

$\alpha\, d\,(x_{2n-1}, g\,(x_{2n-1}), a), pd\,(x_{2n}, g\,(x_{2n-1}), a)$

$+ q\, d\,(x_{2n-1}), f\,(x_{2n}), a)\ \}$

$= \max.\ \{\ \alpha\, d\,(x_{2n}, x_{2n-1}, a), \alpha\, d\,(x_{2n}, x_{2n+1}, a),$

$q\, d\,(x_{2n-1}, (x_{2n+1}), a)\ \}$

$= \max.\ \{\alpha\, d\,(x_{2n-1}, x_{2n}, a), q\, d\,(x_{2n-1}, x_{2n+1}, a)\ \}$

Otherwise if $\alpha\, d\,(x_{2n+1}, x_{2n}, a)$ is maximum then we have

$d\,(x_{2n+1}, x_{2n}, a) \leq \alpha\, d\,(x_{2n+1}, x_{2n}, a)$, a contradiction.

Now if (x_{2n-1}, x_{2n}, a) is maximum then we have

$d\,(x_{2n+1}, x_{2n}, a) \leq \alpha\, d\,(x_{2n-1}, x_{2n}, a)$

if $d\,(x_{2n-1}, x_{2n+1}, a)$ is maximum then we have

$d\,(x_{2n+1}, x_{2n}, a) \leq q\, d\,(x_{2n-1}, x_{2n+1}, a)$

$\leq q\,(x_{2n-1}, x_{2n}, a) + d\,(x_{2n}, x_{2n+1}\, a)\ \}$

or, $d\,(x_{2n+1}, x_{2n}, a) \leq \dfrac{q}{1-q}\, d\,(x_{2n-1}, x_{2n}, a)$ and hence,

(II) $d\,(x_{2n+1}, x_{2n}, a) \leq \max.\ \{\alpha \dfrac{q}{1-q}\}\, d\,(x_{2n-1}, x_{2n}, a)$

Similarly we can show that

(III) $d\,(x_{2n+1}, x_{2n+2}, a) \leq \max.\ \{\alpha \dfrac{p}{1-p}\}\, d\,(x_2, x_{2n+1}, a)$

Let $c = (\max,\ \{\alpha \dfrac{p}{1-p}\}).(\max.\ \{\alpha \dfrac{q}{1-q}\})$. Then $0 \leq 0 < 1$.

From II and III, $d\,(x_{2n}, x_{2n+1}, a) \leq c\, d\,(x_{2n-1}, x_{2n-1}, a)$

and $d\,(x_{2n+1}, x_{2n+2}, a) \leq c\, d\,(x_{2n-1}, x_{2n}, a)$ for $n = 1, 2, \ldots$

Hence d $(x_{2n}, x_{2n+1}, a) \leq c^n$ d (x_0, x_1, a) and

d (x_{2n+1}, x_{2n+2}, a) c^n d (x_1, x_2, a) for n = 1, 2, …

Thus $\{x_n\}$ is a Cauchy sequence. Since X is complete $\{x_n\}$ converges say to z. i.e., lim x_n = z.

Then d (f(z), z, a) ≤ d (f(z), z, z_{2n}) + d (f(z), x_{2n}, a) + d (x_{2n}, z, a)

≤ d (f(z), z, x_{2n}) + d (x_{2n}, x, a) + max. {α, d (z, x_{2n} , a)

α, d (z, f (z), a), α, d (x_{2n-1}, ε (x_{2n-1}), a),

p d (z, g(x_{2n-1}), a) + q d (x_{2n-1}, f (z), a) }.

When n → ∞, we have

d f (z), z, a) ≤ max. {α d (z, f (z), a), q d (z, f (z), a) }

= max. {α, q). D (z, f(z), a), a contradiction.

Thus, f (z) = z. Similarly we can prove that g (z) = z. so we have z is the common fixed point of f and g. We claim that z is the unique common fixed point of g f and g. If possible let w is the another common fixed point of f and g such that z ≠ w and f (w) = g (w) = w.

Then

d (z, w, a) = d (f (z), g (w), a)

≤ max. {α d (z, w, a), α d (z, f (z), a), α d (w, g(w), a),

p d (z, g(w), a0 + q d (w, f(z), a) }

= max. { α d (z, w, a), (p+q) d (z, w, a) }

or d (z, w, a) ≤ max. { α, p+q } d (z, w, a), a contradiction.

Thus, d (z, w, a) = 0 which gives z = w. //

<u>Remark</u> : Theorem 1.1.9 is an improvement of Theorem 1.1.4 because by putting

$\frac{p}{\alpha} = \frac{q}{\alpha} = \frac{1}{2}$ we get Theorem 1.1.4.

(1.2) In this section we have proved some fixed point theorems for three and four self maps in 2 – metric spaces.

(1.2.1) <u>THEOREM</u>: If T, T_1 and T_2 are three operators mapping a complete 2-metric space (X,d) to itself if x, y, a, ϵ X, each α_1 is non-negative and $\sum_{i=1}^{5} \alpha_1 < 1$, we have (1) d $(T_1^p(x), T_2^q(y), a) \leq \alpha_1$ d $(T(x), T_1^p(T_x), a)$

+ α_2 d (T(y), $T_2^q(y)), a)$

+ α_3 d (x, y, a) + α_4 d (T(x), $T_2^q(y)), a)$

+ α_5 d (T(y), $T_1^p(T(x)), a)$.

(ii) d $(T_x, T_y, a) \leq$ d (x, y, a)

(iii) T_1 T (x) = T T_1 (x)

T_2 T (x) = T T_2 (x).

Then, there is a unique common fixed point of T, T_1 and T_2.

Proof: Using conditions (ii) and (iii) condition (i) becomes.

$$d(T_1^p(x), T_2^q(y2), a) \le \alpha_1 d(x, T_1^p(x), a) + \alpha_2 d(y, T_2^q(y), a) + \\ + \alpha_3 d(x, y, a) + \alpha_4 d(x\, T_2^q(y), a) + \\ + \alpha_5 d(y, T_1^p(x), a).$$

By theorem (3) of (34) there exists x_0 in X, which is a unique common fixed point of T_1 and T_2

i.e., $x_0 = T_1(x_0) = T_2(x_0)$.

Now, $d(x_0, T(X_0), a) = d(T_1^p(x_0), T_2^q(Tx_0), a)$

$$\le \alpha_1 d(Tx_0, T_1^p Tx_0, a) + \alpha_2 d(T^2 x_0 T_2^q(T^2 x_0), a) + \\ + \alpha_3 d(x_0, Tx_0, a) + \alpha_4 d(Tx_0, T_2^q T^2 x_0, a) + \\ + \alpha_5 d(T^2 x_0, T_1^p(Tx_0), a.$$

or, $d(x_0, Tx_0, a) \le \alpha_1 d(Tx_0, Tx_0, a) + \alpha_2 d(Tx_0, (Tx_0, a) +$

$$+ \alpha_3 d(x_0, Tx_0, a) + \alpha_4 d(Tx_0, T^2 x_0, a) + \\ + \alpha_5 d(Tx_0, (T^2 x_0), a.$$

$$\le \alpha_1 d(Tx_0, Tx_0, a) + \alpha_2 d(Tx_0, (Tx_0, a) + \\ + \alpha_3 d(x_0, Tx_0, a) + \alpha_4 d(x_0, Tx_0, a) + \\ + \alpha_5 d(x_0, (Tx_0, a) \text{ by using (ii)}$$

$$= + \alpha_3 d(x_0, Tx_0, a) + \alpha_4 d(x_0, Tx_0, a), \alpha_3 d(x_c, Tx_0, a)$$

or, $(1 - \alpha_3 - \alpha_4 - \alpha_5) d(x_0, Tx_0, a) \le 0$ which gives

$d(x_0, Tx_0, a) = 0$ showing that $x_0 = Tx_0$.

Hence, x_0 is the unique common fixed point of T_1, T_1 and T_2.//

Remarks:

(a) If we put T = I, theorem (1, 2, 1) reduces to

$$d(T_1^p(x), T_2^q(y), a) \le \alpha_1 d(x, T_1^p(x), a) + \alpha_2 d(y, T_2^q(y), a) + \\ \alpha_3 d(x, y, a) + \alpha_4 d(x\, T_2^q(y), a) + \\ + \alpha_5 d(y, T_1^p(x), a).$$

This shows that T may have more than one fixed point, but there is only one common fixed point for T, T_1 and T_2.

(b) The second condition means T is non-expensive.

This by itself would not ensure a fixed point for T.

(1.2.2) THEOREM: If T, T_1 and T_2 are three operators mapping a complete 2 metric space (X,d) itself and if for all x, y, a in X.

(i) $$d(T_1^p(x), T_2^q(y), a) \le h \max.(d(Tx, T_1^p(Tx), a), \\ d(Ty, T_2^q(Tx), a), d(Tx, Ty, a) \\ \frac{1}{2}[d(Ty, T_1^p(T(x)), a), \\ [d(Tx, T_2^q(T(y)), a)]\},$$

(ii) $d(Tx, Ty, a) \le d(x, y, a)$

(iii) $T_1 T(x) = T T_1(x)$

$T_2\,T(x) = T\,T_2(x)$

Then there is a unique common fixed point of T, T_1 and T_2.

Proof: Using conditions (ii) and (iii), condition (1) becomes

$$d\,[d\,(T_1^p(x), T_2^q(y), a) \le h \max\{ d\,(x, T_1^p(x), a), d\,(y, T_2^q(y), a),$$

$$d\,(x, y, a\ \frac{1}{2}[d\,(y, T_1^p(x), a) + d\,(x, T_2^q(y), a)]$$

By theorem (5) of [43] there exists x_0 in X which is unique common fixed point of T_1 and T_2.

$$\text{Now, } d\,(x_0, Tx_0, a) = d\,(T_1^p(x_0), T_2^q(Tx_0), a)$$

$$\le h \max.\ \{d(T(x_0), T_1^p(Tx_0), a), d(T^2x_0, T_2^q(T^2x_0), a),$$

$$d\,(x_0, Tx_0, a)\ \frac{1}{2}[d\,(T^2x_0, T_1^p(Tx_0), a)$$

$$+\ d(Tx_0, T_2^q(T^2x_0), a)\,],$$

$$\le h \max.\ \{d(Tx_0, (Tx_0, a), d(T^2x_0, T^2x_0), a)$$

$$d\,(x_0, Tx_0, a)\ \frac{1}{2}[d\,(T^2x_0, (Tx_0), a)$$

$$+\ d(Tx_0, (T^2x_0), a)\,]\},$$

$$= h \max.\ \{d(x_0, (Tx_0, a), d(Tx_0, T^2x_0, a)\}$$

$$\le h \max.\ \{d(x_0, (Tx_0, a), d(x_0, Tx_0, a)\} \text{ by (ii)}$$

i.e., $d\,(x_0, Tx_0, a) \le hd\,(x_0, Tx_0, a)$ which gives

$d\,(x_0, Tx_0, a) = 0$. Thus $x_0 = Tx_0$.

Hence, x_0 is the unique common fixed point of T, T_1 and T_2. //

(1.2.3) THEOREM: If T, T_1 and T_2, and T_2 are three operators mapping a complete 2 – metric space (X,d) to itself be sequentially continuous and if for all x, y, a in X

(i) $\min\ \{d(T_1^p(x), T_2^q(y), a), d\,(Tx, T_1^p(Tx), a), d\,(Ty, T_2^q(Ty), a),$

$d(T_1^p(Tx), T_2^q\,T_1^p(Tx), a), d\,(Ty, T_2^q\,T_1^p(Tx), a)\},$

$+\ k \min\ \{d(Tx, T_2^q(Ty), a), d\,(Ty, T_1^p(Tx), a),$

$d(Tx, T_1^p\,T_2^q(Ty), a), d\,(T_2^q(Ty), T_2^q\,T_1^p(x)\,a)\}$

$\le r\,d\,(x, y, a)$, where $r\ \varepsilon\ (0, 1)$ and k is a real number.

(ii) $d\,(Tx, Ty, a) \le d\,(x, y, a)$

(iii) $T\,T_1^p = T_1^p T$

$T\,T_2^q = T_2^q\,T$

Then there is a unique common fixed point of T, T_1 and T_2 if $k > r$.

Proof: Using conditions (ii) and (iii), condition (i) becomes.

$\min\ \{d(T_1^p(x), T_2^q(y), a), d\,(x, T_1^p(x), a), d\,(y, T_2^q(y), a),$

$d\ \{d(T_1^p(x), T_2^q\,T_1^p(x), a), d\,(x, T_2^q\,T_1^p(x), a)\}$

$+\ k \min\ \{d(xT_2^q(y), a),\ d\,(y, T_1^p(x), a), d\,(x, T_1^p\,T_2^q(y), a),$

$d(T_2^q(y), a), T_2^q\,T_1^p(x), a)\}$

$\le r\,d\,(x, y, a)$

Now for given x_0 in X, consider a sequence $\{x_n\}_{n\,\varepsilon\,N}$

as x_0, $x_1 = T_1^P(X_0), x_2 = T_2^q(x_1), \ldots\ldots x_{2_n} = T_2^q(x_{2n-1}), x_{2n+1} = T_1^p(x_{2n})$.

If for some m, x_m x_{n+1}, then T_1^P and T_2^q have a common fixed point x_m in X. Thus we suppose that $x_m \neq x_{m+1}$ for all m = 1, 2, 3, …. From the condition for x = x_{2n}, and y = x_{2n+1}, we have

$$\min\{d(T_1^p x_{2n}, T_2^q x_{2n+1}, a), d(x_{2n}, T_1^p(x_{2n}), a), d(y, T_2^q T_1^p(x_{2n}), a)\}$$
$$+ k\min\{d(x_{2n}, T_1^q(x_{2n+1}), a), d(x_{2n+1}, T_1^p(x_{2n}), a), d(x_{2n}, T_1^p T_2^q(x_{2n+1}), a), d(T_2^q(x_{2n+1}), T_2^q T_1^p(x_{2n}), a)\}$$
$$\leq r\, d(x_{2n}, x_{2+1}, a) \text{ for every non-negative integer n.}$$
$$\text{or, } \min\{d(x_{2n+1}, x_{2n+2}, a), d(x_n, x_{2n+1}, a)$$
$$\leq r\, d(x_{2n}, x_{2+1}, a) \text{ for every non-negative integer n.}$$

Since (X,d) is a 2 – metric space, $\{d(x_{2n}, x_{2n+1}, a) \neq 0$ for some a in X. Hence if $d(x_{2n}, x_{2n+1}, a) < d(x_{2n+1}, x_{2n+2}, a)$

then we have $d(x_{2n}, x_{2n+1}, a) \leq r\, d(x_{2n}, x_{2n+1}, a)$ for r ε (0,1), which is impossible and so

we have $d(x_{2n+1}, x_{2n+2}, a) \leq r\, d(x_{2n}, x_{2n+1}, a)$. Similarly, we have $d(x_{2n}, x_{2n+1}, a) \leq r\, d(x_{2n-1}, x_{2n}, a)$, therefore

$d(x_m, x_{m+1}, a) \leq r\, d(x_{m-1}, x_m, a)$ for every non-negative integer m and by lemma (1.0.6), the sequence $\{x_n\}$ converges to some point x_0 in X i.e.,

$$\lim_{n\to\infty} x_n = x_0. \text{ Now } d(x_0, T_1^p(x_0), a) \leq d(x_0, T_1^p(x_0), x_{2n}) + d(x_0, x_{2n}, a) + d(x_{2n+1}, T_1^p(x_0), a)$$
$$d(x_0, T_1^p(x_0), x_{2n}) + d(x_0, x_{2n}, a) + d(T_1^p(x_{2n}), T_1^p(x_0), a)$$
$$\to 0 \text{ as } n \to \infty$$

Therefore $d(x_0, T_1^p(x_0), a) = 0$ for all a in X. Thus x_0 is a fixed point of T_1^p. Similarly x_0 is also a fixed point of T_2^q, i.e. x_0 is the common fixed point of T_1^p and T_2^q. Next, let k > r and to prove the uniqueness of a common fixed point of T_1^p and T_2^q, let, x_0 and y_0 be common fixed point of T_1^p and T_2^q with $x_0 \neq y_0$. Then, d (x_0, y_0, a) $\neq 0$. For this a in X,

$$\text{Min}\{d(T_1^p(x_0), T_2^q(y_0), a), d(y_0, T_2^q(y_0), a), d(x_0, T_1^p(x_0), a), d(T_1^p(x_0), T_2^q T_1^p(x_0), a), d(y_0, T_2^q T_1^p(x_0), a)\}$$
$$+ k\min\{d(x_0 T_2^q(y_0), a), d(y_0, T_1^p(x_0), a\, d(x_0, T_1^p T_2^q(y_0), a), d(T_2^q(y_0), T_2^q T_1^p(x_0), a)\},$$
$$\leq r\, d(x_0, y_0, a), \text{ gives}$$

k d (x_0, y_0, a) $\leq$ r d (x_0, y_0, a)
i.e., d (x_0, y_0, a) < r d (x_0, y_0, a), which is impossible.
This proves that T_1^p and T_2^q have a unique common fixed point.
Now, d (x_0, y_0, a) = d $(T_1^p (x_0), T_2^q (Tx_0), a)$

So,

$$\min \{d(T_1^p(x_0), T_2^q(Tx_0), a), d(Tx_0, T_{12}^p(Tx_0), a), d(T^2x_0\, T_2^q(T^2x_0), a), d(T_1^p(Tx_0), T_2^q T_1^p(Tx_0), a), d(T^2x_0, T_2^q T_1^p Tx_0, a)\}$$
$$+ k \min\{d(Tx_0, T_2^q(T^2x_0), a), d(T^2x_0, T_1^p(Tx_0), a)\; d(Tx_0, T_1^p T_2^q(T^2x_0), a), d(T_2^q T^2x_0, T_2^q T_1^p(Tx_0), a)\}$$

$\leq$ r d (x_0, Tx_0, a)
or, k d (Tx_0, T^2 x_0, a) $\leq$ r d (x_0, Tx_0, a)
or, d (Tx_0, T^2 x_0, a) $\leq \frac{r}{k}$ d (x_0, Tx_0, a) which gives
d (x_0, Tx_0, a) = 0, Thus $x_0 = Tx_0$.
Hence, x_0 is the unique common fixed point of T, T_1 and T_2,//

<u>Remarks:</u>

(i) If we take T = I, Theorem (1.2.1) – (1,2,3) reduces to respectively:

(a)
$$d(T_1^p(x), T_2^q(y), a) \leq h \max\{d(x, T_1^p(x), a), d(y, T_2^q(y), a), d(x, y, a), \frac{1}{2}[d(y, T_1^p(x), a) + d(x, T_2^q(y), a)]\}$$

Where $0 \leq h \leq 1$

(b)
$$\min \{d(T_1^p(x), T_2^q(y), a), (x, T_1^p(x), a)\, d(y, T_2^q(y), a), d(T_1^p(x), T_2^q T_1^p(x), a, d(y), T_2^q T_1^p(x), a)\}$$
$$+ k \min\{d(xT_2^q(y), a), d(y, T_1^p(x), a), d(x, T_1^p T_2^q(y), a), d(T_2^q(y), T_2^q T_1^p(x), a)\}$$

$\leq$ r d (x, y, a).

All these shows that T may have more than one fixed point, but there is only one common fixed point for T, T_1 and T_2.

(ii) The second condition in all these theorems means T is non-expensive. This by itself would not ensure a fixed point for 1.

(1.2.4) **<u>THEOREM</u>**: Let (X, d) be a complete 2 – metric space.
Let $T_i : X \rightarrow X : (i = 1, 2, 3, 4)$ satisfying the conditions

$$[d(T_1 T_2(x), T_3, T_4(y), a)]^2 \leq \alpha_1 [d(x, y, a)]^2$$
$$+ \alpha_2[d(x, T_1 T_2(x), a), d(y, T_3T_4(y), a)]$$
$$+ \alpha_3[d(x, T_1 T_2(x), a), d(x, T_3T_4(y), a)]$$
$$+ \alpha_4[d(x, T_1 T_2(x), a), d(y, T_1T_2(x), a)]$$
$$+ \alpha_5[d(y, T_3 T_4(y), a), d(x, T_3T_4(y), a)]$$

$$+ \alpha_6[d(y, T_3 T_4(y), a), d(y, T_1T_2(x), a)]$$
$$+ \alpha_7[d(x, T_3 T_4(y), a), d(y, T_1T_2(x), a)]$$
$$+ \alpha_8[d(x, y, a)\, d(x, T_1T_2(x), a)]$$
$$+ \alpha_9[d(x, y, a)\, d(y, T_1T_2(x), a)]$$
$$+ \alpha_{10}[d(x, y, a)\, d(y, T_3T_4(y), a)]$$
$$+ \alpha_{11}[d(x, y, a)\, d(x, T_3T_4(y), a)] \ldots. (1)$$

For all x, y, a in X, where $\alpha_1 \geq 0$, I = 1, 2, 3, 4, ….. 11,

Further $\sum_{i=1}^{11} \alpha_i < 1, \alpha_1 + \alpha_7 + \alpha_9 + \alpha_{11} < 1$ and $T_1 T_2 = T_2 T_1$ and

$T_3 T_4 = T_4 T_3$, then T_i (I = 1, 2, 1, 4) have a unique common fixed point in X.

<u>Proof :</u> Let x_0 be an arbitrary point in X and we define, $x_{2+1} = T_1 T_2 x_{2n}$ n = 0, 1, 2, ….

$X_{2n} = T_3 T_4 x_{2n-1}$, n = 1, 2, …..

It follows from (1) that

$$[d(T_3 T_4 y, T_1, T_2 x, a)]^2 \leq \alpha_1 [d(x, y, a)]^2$$
$$+ \alpha_2[d(y, T_3 T_4 y, a), d(x, T_1T_2 x, a)]$$
$$+ \alpha_3 d(y, T_3 T_4 y, a), d(y, T_1T_2 x, a)$$
$$+ \alpha_4 d(x, T_1 T_2 x, a), d(y, T_1T_2 x, a)$$
$$+ \alpha_5 d(y, T_3 T_4 y, a), d(x, T_3T_4 y, a)$$
$$+ \alpha_6 d(y, T_3 T_4 y, a), d(y, T_1T_2 x, a)$$
$$+ \alpha_7 d(x, T_3 T_4 y, a), d(y, T_1T_2 x, a)$$
$$+ \alpha_8 d(x, y, a), d(x, T_1T_2(x), a)$$
$$+ \alpha_9 d[d(x, y, a), d(y, T_1T_2 x, a)]$$
$$+ \alpha_{10}[d(x, y, a), d(y, T_3T_4 y, a)]$$
$$+ \alpha_{11}[d(x, y, a), d(x, T_3T_4 y, a)] \ldots.. (2)$$

By symmetric property of 2-metric from (1) and (2) we get

$$[d(T_1T_2x, T_3T_4y, a)]^2 \leq \alpha_1[d(x, y, a)]^2 + \alpha_2[d(x, T_1T_2x, 2) d(y, T_3T_4y, a)]$$
$$+ \alpha_7[d(y, T_1T_2x, a)\, d(x, T_3T_4y, a)$$
$$+ \frac{\alpha_3 + \alpha_6}{2}\{d(x, T_1T_2x, a) d(x, T_3T_4y, a)$$
$$+ d(y, T_1T_2x, a) d(y, T_3T_4y, a)\}$$
$$\frac{\alpha_4 + \alpha_5}{2}\{d(x, T_1T_2x, a) d(y, T_1T_2x, a)$$
$$+ d(x, T_3T_4y, a) d(y, T_3T_4y, a)\}$$
$$\frac{\alpha_8 + \alpha_{10}}{2}\{d(x, y, a)\, d(x, T_1T_2x, a)$$
$$+ d(x, y, a)\, d(y, T_3T_4y, a)\}$$

$$\frac{\alpha_9+\alpha_{10}}{2}\{\mathrm{d}(x,y,a)\,\mathrm{d}\,(y,T_1T_2x,a)$$
$$+\,\mathrm{d}\,(x,y,a)\,d(x,T_3T_4y,a)\}......(3)$$

Thus by (3) we have

$$[\,\mathrm{d}\,(x_{2n+1},x2n_{2n},a)]^2\le[\mathrm{d}(T_1T_2x_{2n},T_3T_4\,x_{2n\text{-}1},a)]^2$$
$$\le\alpha_1\,[\mathrm{d}(x_{2n},x_{2n\text{-}1},a)]^2+\alpha_2 d(x_{2n},x_{2+1},a)d(x_{2n-1}x_{2n},a)$$
$$+\alpha_7\,[\mathrm{d}(x_{2n\text{-}1},x_{2n+1},a)\,d\,(x_{2n},x_{2n},a)$$
$$+\frac{\alpha_3+\alpha_6}{2}\{\mathrm{d}(x_{2n},x_{2n+1},a)\,d\,(x_{2n},x_{2n},a)$$
$$+\,\mathrm{d}(x_{2n\text{-}1},x_{2n+1},a)\,d\,(x_{2n-1},x_{2n},a)\}$$
$$+\frac{\alpha_4+\alpha_5}{2}\{\mathrm{d}(x_{2n},x_{2n+1},a)\,\mathrm{d}\,(x_{2n\text{-}1},x_{2n+1},a)$$
$$+\,\mathrm{d}(x_{2n},x_{2n},a)\,d\,(x_{2n-1},x_{2n},a)\}$$
$$+\frac{\alpha_8+\alpha_{10}}{2}\{\mathrm{d}(x_{2n},x_{2n-1},a)\,\mathrm{d}\,(x_{2n},x_{2n+1},a)$$
$$+\,\mathrm{d}(x_{2n},x_{2n\text{-}1},a)\,d\,(x_{2n-1},x_{2n},a)\}$$
$$+\frac{\alpha_9+\alpha_{11}}{2}\{\mathrm{d}(x_{2n},x_{2n-1},a)\,\mathrm{d}\,(x_{2n\text{-}1},x_{2n+1},a)$$
$$+\,\mathrm{d}(x_{2n},x_{2n\text{-}1},a)\,d\,(x_{2n},x_{2n},a)\}$$
$$\le\frac{2\alpha_1+\alpha_8+\alpha_{10}}{2}[\mathrm{d}(x_{2n},x_{2n-1},a)]^2$$
$$+\frac{2\alpha_2+\alpha_8+\alpha_{10}}{2}[\mathrm{d}(x_{2n+1},x_{2n},a)d\,(x_{2n},x_{2n-1},a)]$$
$$+\frac{\alpha_3+\alpha_6+\alpha_9+\alpha_{11}}{2}[\mathrm{d}(x_{2n},x_{2n-1},a)\{d\,(x_{2n+1},x_{2n},a)+d(x_{2n},x_{2n\text{-}1},a)\}$$
$$+\frac{\alpha_4+\alpha_5}{2}\,\mathrm{d}(x_{2n+1},x_{2n},a)\{d\,(x_{2n+1},x_{2n},a)+d(x_{2n},x_{2n+1},a)\}$$
$$\text{as}\quad \mathrm{d}(x_{2n\text{-}1},x_{2n},x_{2n+1})=0$$

Thus, $\mathrm{d}(x_{2n+1},x_{2n},a)\le k\,\mathrm{d}\,(x_{2n},x_{2n-1},a)$

Where,

$$k^2=\frac{\dfrac{2\alpha_1+\alpha_8+\alpha_{10}+\alpha_3+\alpha_6+\alpha_9+\alpha_{11}}{2}+\dfrac{2\alpha_2+\alpha_8+\alpha_{10}+\alpha_3+\alpha_6+\alpha_9+\alpha_{11}+\alpha_4+\alpha_5}{4}}{\dfrac{1-2\alpha_2+\alpha_8+\alpha_{10}+\alpha_3+\alpha_6+\alpha_9+_{11}+\alpha_4+\alpha_5}{4}-\dfrac{\alpha_4+\alpha_5}{4}}$$

$$<1,\ \text{as}\ \sum_{\substack{i=1\\ i\neq 7}}^{11}\alpha_i<1$$

Similarly, $\mathrm{d}(x_{2n},x_{2n\text{-}1},a)\le k\,\mathrm{d}\,(x_{2n-1},x_{2n-2},a)$

So $\{x_n\}$ is a Cauchy sequence in X. Since X is complete 2-metric space there exists zεX such that $\lim_{n\to\infty} X_n = z$. Now by considering

$[d(T_1T_2\, z, x_{2n}, a)]^2$

We get from (1) letting n→∞ that $[d(T_1T_2a, z, a)]^2 \le 0$

Which implies that $T_1T_2z \neq z$. Similarly by considering $[d(x_{2n+1}, T_3T_4z, a)]^2$ we conclude from (1) by letting $n \to \infty$ that $T_3T_4z = z$.

Thus $T_1T_2z = z = T_3T_4\, z$ ……….. (4)

Lastly,

$$
\begin{aligned}
[d(T_1z, z, a)]^2 = [d\,(T_1T_2z, T_3T_4z, a)]^2 = [d\,(T_1T_2T_1(z)T_3T_4z, a)]^2 \\
&\le \alpha_1[d(T_1z, z, a)]^2 + \alpha_2 d\,(T_1z, T_1T_2T_1z, a)\, d\,(z, T_3T_4z, a) \\
&+ \alpha_3[d(T_1z, T_1T_2T_1z, a)\, d\,(T_1z\;\; T_3T_4z, a) \\
&+ \alpha_4\, d(T_1z, T_1T_2T_1z, a)\, d\,(z, T_1T_2z, a) \\
&+ \alpha_5\, d(z, T_3T_4z, a)\, d\,(T_1z, T_3T_4z, a) \\
&+ \alpha_6\, d(z, T_3T_4z, a)\, d\,(z, T_1\; T_2T_1z, a) \\
&+ \alpha_7\, d(T_1z, T_3T_4z, a)\, d\,(z, T_1T_2T_1z, a) \\
&+ \alpha_8\, d(T_1z, z, a)\, d\,(T_1z\;\; T_1T_2T_1z, a) \\
&+ \alpha_9\, d(T_1z, z, a)\, d\,(z, T_1T_2T_1z, a) \\
&+ \alpha_{10}\, d(T_1z, z, a)\, d\,(z, T_3T_4z, a) \\
&+ \alpha_{11}\, d(T_1z, z, a)\, d\,(T_1z,\; T_3T_4z, a)
\end{aligned}
$$

i.e., $(1-\alpha_1-\alpha_7-\alpha_9-\alpha_{11})[d(T_1z, z, a)]^2 \le 0$ which implies that $T_1z = z$. Since $\alpha_1 + \alpha_7 + \alpha_9 + \alpha_{11} < 1$. Thus from (4), $T_1T_2\, z = T_2T_1\, z = T_2\, z = T_2\, z = z$.

Similarly by considering $[d(z,T_3z,a)]^2$, we conclude from (1) that $T_3\, z = z$, since $\alpha_1 + \alpha_7 + \alpha_9 + \alpha_{11} < 1$ and using (4) we get $T_3T_4z = T_4T_3z = T_4z = z$.

Hence, $T_1z = T_2z = T_3z = T_4z = z$ i.e. z is a common fixed point of T_i (i = 1, 2, 3, 4). Uniqueness of z follows directly from (1), thus here we omit it.

<u>Remark</u>: Since max. $\{\alpha_1, \alpha_2, \ldots \alpha_{11}\} \le \alpha_1 + \alpha_2 + \ldots + \alpha_{11}$ for non-negative reals α_i, i = 1, 2, ….., 11, we obtain the following :

(1.2.5) **<u>COROLLARY</u>**: In a complete 2-metric space (X, d), if there exists four self mappings T_i (i = 1, 2, 3, 4,) and

Satisfy the relations

$[d(T_1T_2xT_3T_4y, \alpha)]^2 \le \max\,[\alpha_1[d(x, y, a)]^2,$

$$\alpha_2\, d(xT_1T_2x,\alpha)]\,\mathrm{d}\,(\mathrm{y}, \mathrm{T}_3\mathrm{T}_4\mathrm{y},\ \mathrm{a}),$$
$$\alpha_3\, d(xT_1T_2x,\alpha)]\,\mathrm{d}\,(\mathrm{x}_3, \mathrm{T}_3\mathrm{T}_4\mathrm{y},\ \mathrm{a}),$$
$$\alpha_4\, d(xT_1T_2zx,\alpha)]\,\mathrm{d}\,(\mathrm{y}, \mathrm{T}_1\mathrm{T}_2\mathrm{x},\ \mathrm{a}),$$
$$\alpha_5\, d(yT_3T_4y,\alpha)]\,\mathrm{d}\,(\mathrm{x}, \mathrm{T}_3\mathrm{T}_4\mathrm{y},\ \mathrm{a}),$$
$$\alpha_6\, d(yT_3T_4y,\alpha)]\,\mathrm{d}\,(\mathrm{y}, \mathrm{T}_1\mathrm{T}_2\mathrm{x},\ \mathrm{a}),$$
$$\alpha_7\, d(xT_3T_4y,\alpha)]\,\mathrm{d}\,(\mathrm{y}, \mathrm{T}_1\mathrm{T}_2\mathrm{x},\ \mathrm{a}),$$
$$\alpha_8\, d(x,y,\alpha)]\,\mathrm{d}\,(\mathrm{x}, \mathrm{T}_1\mathrm{T}_2\mathrm{x},\ \mathrm{a}),$$
$$\alpha_9\, d(x,y,\alpha)]\,\mathrm{d}\,(\mathrm{y}, \mathrm{T}_1\mathrm{T}_2\mathrm{x},\ \mathrm{a}),$$
$$\alpha_{10}\, d(x,y,\alpha)]\,\mathrm{d}\,(\mathrm{y}, \mathrm{T}_3\mathrm{T}_4\mathrm{y},\ \mathrm{a}),$$
$$\alpha_{11}\, d(x,y,\alpha)]\,\mathrm{d}\,(\mathrm{x}, \mathrm{T}_3\mathrm{T}_4\mathrm{y},\ \mathrm{a})\}$$

For all x, y in X, $\alpha_i \geq 0$, i = 1, 2, ….. 11 with

$$\sum_{\substack{i=1\\ i\neq 7}}^{11} \alpha_i < 1 \text{ and } \alpha_1 + \alpha_7 + \alpha_9 + \alpha_{11} < 1. \text{ Further assume } i \neq 7$$

That $T_1T_2 = T_2T_1$ and $T_3T_4 = T_4T_3$, than T_i = i = 1, 2, 3, 4, have a unique common fixed point in X.

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