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FITTED SIXTH-ORDER TRIDIAGONAL FINITE DIFFERENCE METHOD FOR SINGULAR PERTURBATION PROBLEMS

K. Phaneendra [a], Y.N. Reddy [b] and GBSL.Soujanya [c]

[a,b,c] *Department of mathematics, National Institute of Technology, Warangal 506 004, India*

kollojuphaneendra@yahoo.co.in

Abstract

In this paper, a fitted sixth-order tridiagonal finite difference scheme is presented for solving singularly perturbed two point boundary value problems with the boundary layer at one end point. We consider a six-order tridiagonal finite difference scheme by M.M. Chawla [A sixth-order Tridiagonal Finite Difference Method for General Non-linear Two-point Boundary Value Problems, J.Inst. Maths Appl. 24 (1979), 35-42] and introduced a fitting factor. The fitting factor is obtained from the theory of singular perturbations. Thomas algorithm is used to solve the tridiagonal system. To demonstrate the applicability of the present method, we have solved five linear problems three with left and two with right end boundary layers. Solutions of these problems using the present fitted method are compared with Chawla's solutions. From the numerical results, it is observed that the present method is stable and has better approximation to the exact solution.

Keywords: Singular perturbation problems; Boundary layer; Finite differences; Fitted method

1. Introduction

Singularly perturbation problem containing a small positive parameter ε have appeared in many fields such as fluid mechanics, chemical kinetics, elasticity, aerodynamics, plasma dynamics, magneto-hydrodynamics and other domains of the world of fluid motion. A few notable examples are boundary layer problems, WKB problems. It is well known fact that the solution of these problems exhibits a multiscale character. That is, there is a thin layer(s) where the solution varies rapidly (non uniformly), while away from the layer the solution behaves regularly (uniformly) and varies slowly. Therefore, the numerical treatment for singularly perturbed boundary value problems has always been far from trivial.

A wide verity of papers and books have been published in the recent years, describing various methods for solving singular perturbation problems, among these we mention Bender and Orzag [1], Hemker and Miller [3], Kevorkian and Cole [4], Nayfeh [5], O'Malley [6], Y.N.Reddy and P.Pramod Chakravarthy [8] and Van Dyk [11].

In this paper a sixth-order tridiagonal finite difference scheme is presented for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. We have introduced a fitting factor in M.M.Chawla's [2] sixth-order tridiagonal finite difference scheme. Thomas algorithm is used to solve the tridiagonal system. To demonstrate the applicability of the method, we have solved five linear problems three with left end and two with right end boundary layers. Solutions of these problems using the present fitted method are compared with solutions by Chawla's method. From the results, it is observed that the present method approximates the exact solution very well.

2. Fitted sixth-order scheme

To describe this method we consider a linearly singularly perturbed two point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [0,1] \tag{1}$$

with the boundary conditions $y(0) = \alpha$ (2.1)

and $y(1) = \beta$; (2.2)

where ε is a small positive parameter ($0 < \varepsilon << 1$) and α, β are known constants. We assume that a(x), b(x) and f(x) are sufficiently continuously differentiable functions in [0, 1]. Further more, we assume that b(x) ≤0, a(x) ≥M>0 throughout the interval [0, 1], where M is some positive constant. Under these assumptions, (1) has a unique solution y(x) which in general, displays a boundary layer of width O(ε) at x=0 for small values of ε.

From the theory of singular perturbation s it is known that the solution of (1)-(2) is of the form [cf. O'Malley [103]; pp 22-26]

$$y(x) = y_0(x) + \frac{a(0)}{a(x)}(\alpha - y_0(0))e^{\int_0^x \left(\frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)}\right)dx} + O(\varepsilon) \tag{3}$$

Where $y_0(x)$ is the solution of

$$a(x)y_0'(x)+b(x)y_0(x)=f(x),\quad y_0(1)=\beta \tag{4}$$

By taking Taylor's series expansion for a(x) and b(x) about the point '0' and restricting to their first terms, (3) becomes,

$$y(x)=y_0(x)+(\alpha-y_0(0))e^{-\left(\frac{a(0)}{\varepsilon}-\frac{b(0)}{a(0)}\right)x}+O(\varepsilon) \tag{5}$$

Now we divide the interval [0, 1] into N equal parts with constant mesh length h. Let 0= $x_1, x_2, ...x_N$ =1 be the mesh points. Then we have $x_i = ih : i = 0,1,2,\ldots\ldots$N.

From (5), we have

$$y(x_i)=y_0(x_i)+(\alpha-y_0(0))e^{-\left(\frac{a(0)}{\varepsilon}-\frac{b(0)}{a(0)}\right)x_i}+O(\varepsilon)$$

i.e., $y(ih)=y_0(ih)+(\alpha-y_0(0))e^{-\left(\frac{a(0)}{\varepsilon}-\frac{b(0)}{a(0)}\right)ih}+O(\varepsilon)$

therefore

$$\lim_{h\to 0} y(ih)=y_0(0)+(\alpha-y_0(0))e^{-\left(\frac{a^2(0)-\varepsilon b(0)}{a(0)}\right)i\rho} \tag{6}$$

Where $\rho=\dfrac{h}{\varepsilon}$

Now rewrite Eq. (1) in the form

$$\varepsilon y''=f(x)-a(x)y'(x)-b(x)y(x)=g(x,y,y') \tag{7}$$

with $y(0)=\alpha$ and $y(1)=\beta$

Now we divide the interval [0, 1] into N equal parts with constant mesh length h. Let 0= $x_1, x_2, ...x_N$ =1 be the mesh points. Then we have $x_i = ih : i = 0,1,2,\ldots\ldots$N. Let us denote the exact solution *y(x)* at the grid points x_i by y_i, similarly, $y_i' = y'(x_i)$.

For i= 1,2,….N-1, let

$$\bar{y}_n'=(y_{n+1}-y_{n-1})/2h \tag{8a}$$

$$\bar{y}_{n+1}'=(3y_{n+1}-4y_n+y_{n-1})/2h \tag{8b}$$

$$\bar{y}_{n-1}'=(-y_{n+1}+4y_n-3y_{n-1})/2h \tag{8c}$$

$$\bar{\bar{y}}_{n+1}'=\bar{y}_n'+\frac{h}{3}(2\bar{f}_n+\bar{f}_{n+1}) \tag{8d}$$

$$\bar{\bar{y}}_{n-1}'=\bar{y}_n'-\frac{h}{3}(2\bar{f}_n+\bar{f}_{n-1}) \tag{8e}$$

$$\bar{y}'_{n+1/2} = \frac{1}{32}(15y_{n+1} + 18y_n - y_{n-1}) - \frac{h^2}{64}(3\bar{g}_{n+1} + 4\bar{g}_n - \bar{g}_{n-1}) \quad (8f)$$

$$\bar{y}'_{n-1/2} = \frac{1}{32}(-y_{n+1} + 18y_n + 15y_{n-1}) - \frac{h^2}{64}(-\bar{g}_{n+1} + 4\bar{g}_n + 3\bar{g}_{n-1}) \quad (8g)$$

$$\bar{\bar{y}}'_{n+1/2} = \frac{1}{4h}(5y_{n+1} - 6y_n + y_{n-1}) - \frac{h}{48}(3\bar{g}_{n+1} + 8\bar{g}_n + \bar{g}_{n-1}) \quad (8h)$$

$$\bar{\bar{y}}'_{n-1/2} = \frac{1}{4h}(-y_{n+1} + 6y_n - 5y_{n-1}) + \frac{h}{48}(\bar{g}_{n+1} + 8\bar{g}_n + 3\bar{g}_{n-1}) \quad (8i)$$

$$\hat{y}'_n = \bar{y}'_n + h\left[\frac{1}{78}(\bar{g}_{n+1} - \bar{g}_{n-1}) - \frac{1}{52}(\bar{\bar{g}}_{n+1} - \bar{\bar{g}}_{n-1}) - \frac{2}{13}(\bar{\bar{g}}_{n+1/2} - \bar{\bar{g}}_{n-1/2})\right] \quad (8j)$$

Then for each x_i, i = 1,2,…..N-1, Eq.(7) can be described as

$$\frac{\varepsilon}{h^2}\delta^2 y_i = \frac{1}{60}\left[26\hat{g}_n + \bar{\bar{g}}_{n+1} + \bar{\bar{g}}_{n-1} + 16(\bar{\bar{g}}_{n+1/2} + \bar{\bar{g}}_{n-1/2})\right] \quad (9)$$

where

$$g_n = g(x_n, y_n, \bar{y}'_n) \quad (10a)$$

$$\bar{g}_{n\pm1} = g(x_{n\pm1}, y_{n\pm1}, \bar{y}'_{n\pm1}) \quad (10b)$$

$$\bar{\bar{g}}_{n\pm1} = g(x_{n\pm1}, y_{n\pm1}, \bar{\bar{y}}'_{n\pm1}) \quad (10c)$$

$$\bar{\bar{g}}_{n\pm1/2} = g(x_{n\pm1/2}, y_{n\pm1/2}, \bar{\bar{y}}'_{n\pm1/2}) \quad (10d)$$

$$\hat{g}_n = g(x_n, y_n, \hat{y}'_n) \quad (10e)$$

Using (8) and (10), in the right hand side expression of (9) can be simplified as

$$\frac{\varepsilon}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = \left(P_1 + P_2 + \frac{ha_i}{15}P_3 + \frac{P_4}{60} + \frac{16P_5}{60}\right)y_{i-1} + \left(P_6 + P_7 + \frac{ha_i}{15}P_8 + \frac{P_9}{60} + \frac{16P_{10}}{60}\right)y_i + \left(P_{11} + P_{12} + \frac{ha_i}{15}P_{13} + \frac{P_{14}}{60} + \frac{16P_{15}}{60}\right)y_{i+1} \quad (11)$$

where $P_1 = \frac{13a_i}{60h} - \frac{ha_i}{180}\left(b_{i-1} - \frac{a_{i+1}}{2h} - \frac{3a_{i-1}}{2h}\right)$

$$P_2 = \frac{ha_i}{120}\left(\frac{a_{i+1}}{2h} - \frac{a_i a_{i+1}}{3} + \frac{a_{i+1}^2}{6} + b_{i-1} - \frac{a_{i-1}}{2h} - \frac{a_i a_{i-1}}{3} + \frac{ha_{i-1}b_{i-1}}{3} - \frac{a_{i-1}^2}{2}\right)$$

$$P_3 = -b_{n+1/2}\left(\frac{-1}{32} - \frac{h^2 b_{i-1}}{64} + \frac{3ha_{i+1}}{128} - \frac{2ha_i}{64} + \frac{3ha_{i-1}}{128}\right) - a_{i+1/2}\left(\frac{hb_{i-1}}{48} + \frac{3a_{i+1}}{96} - \frac{4a_i}{48} - \frac{3a_{i-1}}{96} + \frac{1}{4h}\right) + b_{n-1/2}\left(\frac{15}{32} + \frac{3h^2 b_{i-1}}{64} - \frac{ha_{i+1}}{128} - \frac{2ha_i}{64} - \frac{9ha_{i-1}}{128}\right) + a_{i-1/2}\left(\frac{-3hb_{i-1}}{48} - \frac{a_{i+1}}{96} + \frac{8a_i}{96} + \frac{9a_{i-1}}{96} - \frac{5}{4h}\right)$$

$$P_4 = \frac{a_{i+1}}{2h} - \frac{a_i a_{i+1}}{3} + \frac{a_{i+1}^2}{6} - b_{i-1} + \frac{a_{i-1}}{2h} + \frac{a_i a_{i-1}}{3} - \frac{ha_{i-1}b_{i-1}}{3} + \frac{a_{i-1}^2}{2}$$

$$P_5 = -b_{n+1/2}\left(\frac{-1}{32} - \frac{h^2 b_{i-1}}{64} + \frac{3ha_{i+1}}{128} - \frac{2ha_i}{64} + \frac{3ha_{i-1}}{128}\right) - a_{i+1/2}\left(\frac{hb_{i-1}}{48} + \frac{3a_{i+1}}{96} - \frac{4a_i}{48} - \frac{3a_{i-1}}{96} + \frac{1}{4h}\right)$$
$$- b_{n-1/2}\left(\frac{15}{32} + \frac{3h^2 b_{i-1}}{64} - \frac{ha_{i+1}}{128} - \frac{2ha_i}{64} - \frac{9ha_{i-1}}{128}\right) - a_{i-1/2}\left(\frac{-3hb_{i-1}}{48} - \frac{a_{i+1}}{96} + \frac{8a_i}{96} + \frac{9a_{i-1}}{96} - \frac{5}{4h}\right)$$

$$P_6 = \frac{-26b_i}{60} - \frac{ha_i}{180}\left(\frac{4a_{i+1}}{2h} + \frac{4a_{i-1}}{2h}\right)$$

$$P_7 = \frac{ha_i}{120}\left(\frac{2ha_{i+1}b_i}{3} - \frac{2a_{i+1}^2}{3} + \frac{2ha_{i-1}b_i}{3} + \frac{2a_{i-1}^2}{3}\right)$$

$$P_8 = -b_{n+1/2}\left(\frac{18}{32} + \frac{4h^2 b_i}{64} - \frac{6ha_{i+1}}{64} - \frac{2ha_{i-1}}{64}\right) - a_{i+1/2}\left(\frac{8hb_i}{48} - \frac{6a_{i+1}}{48} + \frac{2a_{i-1}}{48} - \frac{6}{4h}\right)$$
$$+ b_{n-1/2}\left(\frac{18}{32} + \frac{4h^2 b_i}{64} + \frac{2ha_{i+1}}{64} + \frac{6ha_{i-1}}{64}\right) + a_{i-1/2}\left(\frac{-8hb_i}{48} + \frac{4a_{i+1}}{96} - \frac{12a_{i-1}}{96} + \frac{6}{4h}\right)$$

$$P_9 = \left(\frac{2ha_{i+1}b_i}{3} - \frac{2a_{i+1}^2}{3} - \frac{2ha_{i-1}b_i}{3} - \frac{2a_{i-1}^2}{3}\right)$$

$$P_{10} = -b_{n+1/2}\left(\frac{18}{32} + \frac{4h^2 b_i}{64} - \frac{6ha_{i+1}}{64} - \frac{2ha_{i-1}}{64}\right) - a_{i+1/2}\left(\frac{8hb_i}{48} - \frac{6a_{i+1}}{48} + \frac{2a_{i-1}}{48} - \frac{6}{4h}\right)$$
$$- b_{n-1/2}\left(\frac{18}{32} + \frac{4h^2 b_i}{64} + \frac{2ha_{i+1}}{64} + \frac{6ha_{i-1}}{64}\right) - a_{i-1/2}\left(\frac{-8hb_i}{48} + \frac{4a_{i+1}}{96} - \frac{12a_{i-1}}{96} + \frac{6}{4h}\right)$$

$$P_{11} = \frac{-13a_i}{60h} - \frac{ha_i}{180}\left(-b_{i+1} - \frac{3a_{i+1}}{2h} - \frac{a_{i-1}}{2h}\right)$$

$$P_{12} = \frac{ha_i}{120}\left(\frac{-a_{i+1}}{2h} + \frac{a_i a_{i+1}}{3} + \frac{a_{i+1}^2}{2} - b_{i-1} + \frac{a_{i-1}}{2h} + \frac{a_i a_{i-1}}{3} + \frac{ha_{i+1}b_{i+1}}{3} - \frac{a_{i-1}^2}{6}\right)$$

$$P_{13} = -b_{n+1/2}\left(\frac{15}{32} + \frac{3h^2 b_{i+1}}{64} + \frac{9ha_{i+1}}{128} + \frac{2ha_i}{64} + \frac{ha_{i-1}}{128}\right) - a_{i+1/2}\left(\frac{3hb_{i+1}}{48} + \frac{9a_{i+1}}{96} + \frac{4a_i}{48} - \frac{a_{i-1}}{96} + \frac{5}{4h}\right)$$
$$+ b_{n-1/2}\left(\frac{-1}{32} - \frac{h^2 b_i}{64} - \frac{3ha_{i+1}}{128} + \frac{2ha_i}{64} - \frac{3ha_{i-1}}{128}\right) + a_{i-1/2}\left(\frac{-hb_{i+1}}{48} - \frac{3a_{i+1}}{96} - \frac{8a_i}{96} + \frac{3a_{i-1}}{96} - \frac{1}{4h}\right)$$

$$P_{14} = \left(\frac{-a_{i+1}}{2h} + \frac{a_i a_{i+1}}{3} + \frac{a_{i+1}^2}{2} - b_{i+1} - \frac{a_{i-1}}{2h} - \frac{a_i a_{i-1}}{3} + \frac{ha_{i+1}b_{i+1}}{3} + \frac{a_{i-1}^2}{6}\right)$$

$$P_{15} = -b_{n+1/2}\left(\frac{15}{32} + \frac{3h^2 b_{i+1}}{64} + \frac{9ha_{i+1}}{128} + \frac{2ha_i}{64} + \frac{ha_{i-1}}{128}\right) - a_{i+1/2}\left(\frac{3hb_{i+1}}{48} + \frac{9a_{i+1}}{96} + \frac{4a_i}{48} - \frac{a_{i-1}}{96} + \frac{5}{4h}\right)$$
$$- b_{n-1/2}\left(\frac{-1}{32} - \frac{h^2 b_i}{64} - \frac{3ha_{i+1}}{128} + \frac{2ha_i}{64} - \frac{3ha_{i-1}}{128}\right) - a_{i-1/2}\left(\frac{-hb_{i+1}}{48} - \frac{3a_{i+1}}{96} - \frac{8a_i}{96} + \frac{3a_{i-1}}{96} - \frac{1}{4h}\right)$$

Introducing a fitting factor $\sigma(\rho)$ into Eq. (11), we get

$$\frac{\sigma(\rho)\varepsilon}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = \left(P_1 + P_2 + \frac{ha_i}{15}P_3 + \frac{P_4}{60} + \frac{16P_5}{60}\right)y_{i-1} + \left(P_6 + P_7 + \frac{ha_i}{15}P_8 + \frac{P_9}{60} + \frac{16P_{10}}{60}\right)y_i$$
$$+ \left(P_{11} + P_{12} + \frac{ha_i}{15}P_{13} + \frac{P_{14}}{60} + \frac{16P_{15}}{60}\right)y_{i+1} \qquad (12)$$

Multiplying (12) by h and taking limit as $h \to 0$, we get

$$\therefore \lim_{h\to 0}\left(\frac{\sigma(\rho)}{\rho}\big(y(ih+h)-2y(ih)+y(ih-h)\big)+\frac{1}{2}a(ih)\big(y(ih+h)-y(ih-h)\big)\right)=0 \qquad (13)$$

Substituting (6) in (13) and simplifying, we get

$$\sigma = \frac{\rho}{2}a(0)Coth\left[\left(\frac{a^2(0)-\varepsilon b(0)}{a(0)}\right)\frac{\rho}{2}\right] \qquad (14)$$

which is a constant fitting factor.

From eq. (12) we get a three term recurrence relation of the form

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i; \quad \text{i} = 1,2,.....,N-1 \qquad (15)$$

where the fitting factor σ is given by (14). where

$$E_j = \left(\frac{\varepsilon\sigma}{h^2} - P_1 - P_2 - \frac{ha_i P_3}{15} - \frac{P_4}{60} - \frac{16P_5}{60}\right)$$

$$F_j = \left(\frac{2\varepsilon\sigma}{h^2} + P_6 + P_7 + \frac{ha_i P_8}{15} + \frac{P_9}{60} + \frac{16P_{10}}{60}\right)$$

$$G_j = \left(\frac{\varepsilon\sigma}{h^2} - P_{11} - P_{12} - \frac{ha_i P_{13}}{15} - \frac{P_{14}}{60} - \frac{16P_{15}}{60}\right)$$

$$H_j = F_1 + F_2 + F_3 + F_4 + \frac{16F_5}{60}$$

This gives us the tridiagonal system which can be solved easily by Thomas Algorithm.

3. Numerical Examples

Left Layer Problems: To demonstrate the applicability of the method we have applied it to three linear singular perturbation problems with left-end boundary layer. These examples have been chosen because they have been widely discussed in literature and because exact solutions are available for comparison. The approximate solution is compared with the exact solution.

Example 3.1. Consider the following homogeneous singular perturbation problem from Bender and Orszag [[1], page 480; problem 9.17 with α=0]

$$\varepsilon y''(x) + y'(x) - y(x) = 0; \ \text{x} \in [0, 1]$$

with $y\,(0)$ =1 and y (1) =1.

The exact solution is given by

$$y(x) = \frac{[(e^{m_2}-1)e^{m_1 x} + (1-e^{m_1})e^{m_2 x}]}{[e^{m_2} - e^{m_1}]}$$

where $m_1 = (-1+\sqrt{1+4\varepsilon})/(2\varepsilon)$ and $m_2 = (-1-\sqrt{1+4\varepsilon})/(2\varepsilon)$

The numerical results are given in tables 1(a), 1(b) for $\varepsilon=10^{-3}$, 10^{-4} respectively.

Example 3.2. Now consider the following non-homogeneous singular perturbation problem from fluid dynamics for fluid of small viscosity, Reinhardt [13, example 2]

$$\varepsilon y''(x) + y'(x) = 1 + 2x \quad ; x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 1$.

The exact solution is given by $y(x) = x(x+1-2\varepsilon) + \dfrac{(2\varepsilon-1)(1-e^{-x/\varepsilon})}{(1-e^{-1/\varepsilon})}$

The numerical results are given in tables 2(a), 2(b) for $\varepsilon=10^{-3}$, 10^{-4} respectively.

Example 3.3. Consider the following variable coefficient singular perturbation problem from Kevorkian and Cole [4, p. 33; Eqs. (2.3.26) and (2.3.27) with α = -1/2]

$$\varepsilon y''(x) + (1-\frac{x}{2})y'(x) - y(x) = 0 \quad ; x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 1$.

The exact solution is given by $y(x) = \dfrac{1}{2-x} - \dfrac{1}{2}e^{-(x-x^2/4)/\varepsilon}$

The numerical results are given in tables 3(a), 3(b) for $\varepsilon=10^{-3}$, 10^{-4} respectively.

4. Right-End Boundary Layer Problems

We discuss our method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval. To be specific, we consider a class of singular perturbation problem of the form (1) with (2.1) and (2.2), where ε is a small positive parameter ($0 < \varepsilon << 1$) and α, β are known constants. We assume that a(x), b(x) and f(x) are sufficiently continuously differentiable functions in [0, 1]. Further more, we assume that a(x) $\leq$M <0 throughout the interval [0, 1], where M is some negative constant. Under these assumptions, (1) has a unique solution y(x) which in general, displays a boundary layer of width O(ε) at x=1 for small values of ε.

From the theory of singular perturbation s it is known that the solution of (1)-(2) is of the form [cf. O'Malley [103]; pp 22-26]

$$y(x) = y_0(x) + \frac{a(1)}{a(x)}(\beta - y_0(1))e^{\int_x^1 \left(\frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)}\right)dx} + O(\varepsilon) \tag{16}$$

Where $y_0(x)$ is the solution of

$$a(x)y_0'(x) + b(x)y_0(x) = f(x), \quad y_0(0) = \alpha \tag{17}$$

By taking Taylor's series expansion for a(x) and b(x) about the point '1' and restricting to their first terms, (16) becomes,

$$y(x) = y_0(x) + (\beta - y_0(1))e^{-\left(\frac{a(1)}{\varepsilon} - \frac{b(1)}{a(1)}\right)(1-x)} + O(\varepsilon) \tag{18}$$

Now we divide the interval [0, 1] into N equal parts with constant mesh length h. Let 0= $x_1, x_2, ...x_N$ =1 be the mesh points. Then we have $x_i = ih : i = 0,1,2,\ldots\ldots$N.

From (18), we have

i.e., $y(ih) = y_0(ih) + (\beta - y_0(1))e^{-\left(\frac{a(1)}{\varepsilon} - \frac{b(1)}{a(1)}\right)(1-ih)} + O(\varepsilon)$

therefore

$$\lim_{h\to 0} y(ih) = y_0(0) + (\beta - y_0(1))e^{-\left(\frac{a^2(1) - \varepsilon b(1)}{a(1)}\right)(\frac{1}{\varepsilon} - i\rho)} \tag{19}$$

where $\rho = \frac{h}{\varepsilon}$

Now consider the difference scheme (12) and we will get the fitting factor as

$$\sigma = \frac{\rho}{2} a(0) Coth\left[\left(\frac{a^2(1) - \varepsilon b(1)}{a(1)}\right)\frac{\rho}{2}\right] \tag{20}$$

Then from (12) we have the difference scheme (15) where fitting factor is given by (20) and then the three term recurrence relation (15) which gives tridiagonal system which can be solved easily by Thomas Algorithm.

5. Examples with right-end boundary layer

Finally, we discuss our method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval. To illustrate the method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval we considered two examples. The approximate solution is compared with the exact solution.

Example 5.1. Consider the following singular perturbation problem

$\varepsilon y''(x) - y'(x) = 0$; x $\in$[0, 1]

with y (0) =1 and y (1) =0. Clearly, this problem has a boundary layer at x =1. i.e., at the right end of the underlying interval.

The exact solution is given by $y(x) = \frac{\left(e^{(x-1)/\varepsilon} - 1\right)}{\left(e^{-1/\varepsilon} - 1\right)}$

The numerical results are given in tables 4(a), 4(b) for $\varepsilon = 10^{-3}$, 10^{-4} respectively.

Example 5.2. Now we consider the following singular perturbation problem

$\varepsilon y''(x) - y'(x) - (1+\varepsilon)y(x) = 0$; x $\in$[0, 1]

with y (0) = 1+exp (-(1+ε)/ε); and y (1) =1+1/e.

Clearly this problem has a boundary layer at x=1. The exact solution is given by

$y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}$

The numerical results are given in table 5(a), 5(b) for $\varepsilon=10^{-3}$, 10^{-4} respectively.

Table 1(a): Numerical results of example 3.1 with $\varepsilon=10^{-3}$ and $h=10^{-2}$

x	y(x)	Chawla solution	Exact solution
0.00	1.0000000	1.0000000	1.0000000
0.01	0.3732275	-0.0456125	0.3719724
0.02	0.3771380	0.6516672	0.3756784
0.03	0.3809094	0.1970376	0.3794502
0.04	0.3847185	0.5038295	0.3832599
0.05	0.3885658	0.3074202	0.3871079
0.06	0.3924515	0.4436676	0.3909945
0.07	0.3963760	0.3601088	0.3949201
0.08	0.4003398	0.4218970	0.3988851
0.09	0.4043433	0.3876834	0.4028900
0.10	0.4083868	0.4169891	0.4069350
0.20	0.4511136	0.4498496	0.4496879
0.30	0.4983107	0.4969367	0.4969323
0.40	0.5504457	0.5491421	0.5491403
0.50	0.6080352	0.6068349	0.6068334
0.60	0.6716499	0.6705890	0.6705877
0.70	0.7419202	0.7410412	0.7410401
0.80	0.8195425	0.8188950	0.8188941
0.90	0.9052859	0.9049282	0.9049277
1.00	1.0000000	1.0000000	1.0000000

Table 1(b): Numerical results of example 3.1 with $\varepsilon = 10^{-4}$ and $h=10^{-2}$

x	y(x)	Chawla solution	Exact solution
0.00	1.0000000	1.0000000	1.0000000
0.01	0.3731994	-0.2339635	0.3716135
0.02	0.3771380	0.9486007	0.3753479
0.03	0.3809094	-0.1704924	0.3791198
0.04	0.3847185	0.9028042	0.3829296
0.05	0.3885658	-0.1120644	0.3867778
0.06	0.3924515	0.8621077	0.3906645
0.07	0.3963760	-0.0581954	0.3945904
0.08	0.4003398	0.8260553	0.3985557
0.09	0.4043433	-0.0084444	0.4025608
0.10	0.4083868	0.7942340	0.4066062
0.20	0.4511136	0.6866462	0.4493649
0.30	0.4983107	0.6413892	0.4966200
0.40	0.5504457	0.6366383	0.5488445
0.50	0.6080352	0.6592098	0.6065609
0.60	0.6716499	0.7012561	0.6703468
0.70	0.7419202	0.7582348	0.7408404
0.80	0.8195425	0.8276659	0.8187471
0.90	0.9052859	0.9083692	0.9048464
1.00	1.0000000	1.0000000	1.0000000

Table 2(a): Numerical results of example 3.2 with $\varepsilon = 10^{-3}$ and $h = 10^{-2}$

x	y(x)	Chawla solution	Exact solution
0.00	0.0000000	0.0000000	0.0000000
0.01	-0.9789649	-1.6504840	-0.9878747
0.02	-0.9697968	-0.5377655	-0.9776400
0.03	-0.9593979	-1.2591880	-0.9671600
0.04	-0.9487979	-0.7626023	-0.9564800
0.05	-0.9379979	-1.0743120	0.9456000
0.06	-0.9269980	-0.8490668	-0.9345200
0.07	-0.9157980	-0.9799699	-0.9232400
0.08	-0.9043980	-0.8740954	-0.9117600
0.09	-0.8927981	-0.9250834	-0.9000800
0.10	-0.8809981	-0.8715985	-0.8882000
0.20	-0.7519983	-0.7581230	-0.7584000
0.30	-0.6029985	-0.6085948	-0.6086000
0.40	-0.4339987	-0.4387996	-0.4388000
0.50	-0.2449989	-0.2489999	-0.2490000
0.60	-0.0359991	-0.0392001	-0.0392001
0.70	0.1930007	0.1905998	0.1906000
0.80	0.4420004	0.4403998	0.4403999
0.90	0.7110002	0.7101999	0.7102000
1.00	1.0000000	1.0000000	1.0000000

Table 2(b): Numerical results of example 3.2 with $\varepsilon = 10^{-4}$ and $h = 10^{-2}$

x	y(x)	Chawla solution	Exact solution
0.00	0.0000000	0.0000000	0.0000000
0.01	-0.9790097	-1.9707570	-0.9897020
0.02	-0.9697970	-0.0648853	-0.9794040
0.03	-0.9593980	-1.8683510	-0.9689060
0.04	-0.9487981	-0.1217855	-0.9582080
0.05	-0.9379982	-1.7720220	-0.9473100
0.06	-0.9269983	-0.1713059	-0.9362120
0.07	-0.9157983	-1.6811880	-0.9249140
0.08	-0.9043984	-0.2140010	-0.9134160
0.09	-0.8927984	-1.5953210	-0.9017180
0.10	-0.8809985	-0.2503783	-0.8898200
0.20	-0.7519991	-0.3524661	-0.7598400
0.30	-0.6029996	-0.3519358	-0.6098600
0.40	-0.4340000	-0.2782002	-0.4398800
0.50	-0.2450003	-0.1502008	-0.2499000
0.60	-0.0360006	0.0198642	-0.0399201
0.70	0.1929993	0.2241393	0.1900600
0.80	0.4419994	0.4575655	0.4400399
0.90	0.7109997	0.7168850	0.7100199
1.00	1.0000000	1.0000000	1.0000000

Table 3(a): Numerical results of example 3.3 with $\varepsilon = 10^{-3}$ and h=10^{-2}

x	y(x)	Chawla solution	Exact solution
0.00	0.0000000	0.0000000	0.0000000
0.01	0.5052435	0.8359665	0.5024893
0.02	0.5087275	0.2852492	0.5050505
0.03	0.5113007	0.6543077	0.5076142
0.04	0.5138958	0.4146329	0.5102041
0.05	0.5165173	0.5770561	0.5128205
0.06	0.5191656	0.4745143	0.5154639
0.07	0.5218411	0.5462703	0.5181347
0.08	0.5245443	0.5036962	0.5208333
0.09	0.5272756	0.5360276	0.5235602
0.10	0.5300353	0.5194701	0.5263158
0.20	0.5593051	0.5562426	0.5555556
0.30	0.5919852	0.5890134	0.5882353
0.40	0.6287065	0.6257698	0.6249999
0.50	0.6702644	0.6674141	0.6666666
0.60	0.7176770	0.7149906	0.7142856
0.70	0.7722676	0.7698625	0.7692305
0.80	0.8357880	0.8338447	0.8333330
0.90	0.9106061	0.9094072	0.9090905
1.00	1.0000000	1.0000000	1.0000000

Table 3(b): Numerical results of example 3.3 with $\varepsilon = 10^{-4}$ and h=10^{-2}

x	Y(x)	Chawla solution	Exact solution
0.00	0.0000000	0.0000000	0.0000000
0.01	0.5052664	0.9837171	0.5025126
0.02	0.5087277	0.0458975	0.5050505
0.03	0.5113007	0.9495326	0.5076142
0.04	0.5138958	0.0888663	0.5102041
0.05	0.5165172	0.9183809	0.5128205
0.06	0.5191656	0.1291141	0.5154639
0.07	0.5218411	0.8900675	0.5181347
0.08	0.5245443	0.1668359	0.5208333
0.09	0.5272756	0.8644097	0.5235602
0.10	0.5300353	0.2022154	0.5263158
0.20	0.5593049	0.3496649	0.5555556
0.30	0.5919851	0.4604533	0.5882353
0.40	0.6287065	0.5478545	0.6249999
0.50	0.6702645	0.6216480	0.6666666
0.60	0.7176771	0.6891595	0.7142856
0.70	0.7722674	0.7560886	0.7692305
0.80	0.8357878	0.8271949	0.8333330
0.90	0.9106058	0.9069155	0.9090905
1.00	1.0000000	1.0000000	1.0000000

Table 4(a): Numerical results of example 5.1 with $\varepsilon = 10^{-3}$ and h=10^{-2}

x	y(x)	Chawla solution	Exact solution
0.00	1.000000	1.000000	1.000000
0.10	1.000000	1.000000	1.000000
0.20	1.000000	1.000000	1.000000
0.30	1.000000	1.000001	1.000000
0.40	1.000000	1.000001	1.000000
0.50	1.000000	1.000001	1.000000
0.60	1.000000	1.000001	1.000000
0.70	1.000000	0.999996	1.000000
0.80	1.000000	0.999675	1.000000
0.90	1.000000	0.981924	1.000000
0.91	1.000000	1.027008	1.000000
0.92	1.000000	0.959663	1.000000
0.93	1.000000	1.060261	1.000000
0.94	1.000000	0.909990	1.000000
0.95	1.000000	1.134460	1.000000
0.96	1.000000	0.799155	1.000000
0.97	1.000000	1.300023	1.000000
0.98	0.999999	0.551841	1.000000
0.99	1.000953	1.669452	0.999955
1.00	0.000000	0.000000	0.000000

Table 4(b): Numerical results of example 5.1 with $\varepsilon = 10^{-4}$ and h=10^{-2}

x	y(x)	Chawla solution	Exact solution
0.00	1.000000	1.000000	1.000000
0.10	1.000000	0.987831	1.000000
0.20	1.000000	0.970387	1.000000
0.30	1.000000	0.945384	1.000000
0.40	1.000000	0.909545	1.000000
0.50	1.000000	0.858173	1.000000
0.60	1.000000	0.784538	1.000000
0.70	1.000000	0.678990	1.000000
0.80	1.000000	0.527700	1.000000
0.90	1.000000	0.310842	1.000000
0.91	1.000000	1.771609	1.000000
0.92	1.000000	0.257290	1.000000
0.93	1.000000	1.827124	1.000000
0.94	1.000000	0.199740	1.000000
0.95	1.000000	1.886784	1.000000
0.96	1.000000	0.137893	1.000000
0.97	1.000000	1.950899	1.000000
0.98	0.999999	0.071428	1.000000
0.99	1.000998	2.019800	1.000000
1.00	0.000000	0.000000	0.000000

Table 5(a): Numerical results of example 5.2 with $\varepsilon = 10^{-3}$ and h=10^{-2}

x	y(x)	Chawla solution	Exact solution
0.00	1.0000000	1.0000000	1.0000000
0.10	0.9051958	0.9048371	0.9048374
0.20	0.8193794	0.8187299	0.8187308
0.30	0.7416987	0.7408172	0.7408183
0.40	0.6713825	0.6703187	0.6703200
0.50	0.6077328	0.6065293	0.6065307
0.60	0.5501172	0.5488102	0.5488117
0.70	0.4979638	0.4965889	0.4965853
0.80	0.4507548	0.4496263	0.4493290
0.90	0.4080215	0.4238603	0.4065697
0.91	0.4039776	0.3765765	0.4025242
0.92	0.3999738	0.4374473	0.3985191
0.93	0.3960097	0.3361403	0.3945537
0.94	0.3920848	0.4782684	0.3906278
0.95	0.3881989	0.2552384	0.3867410
0.96	0.3843515	0.5801982	0.3828929
0.97	0.3805422	0.0830365	0.3790830
0.98	0.3767758	0.8195024	0.3753111
0.99	0.3707701	-0.2949039	0.3716217
1.00	1.3678790	1.3678790	1.3678790

Table 5(b): Numerical results of example 5.2 with $\varepsilon = 10^{-4}$ and h=10^{-2}

x	y(x)	Chawla solution	Exact solution
0.00	1.0000000	1.0000000	1.0000000
0.10	0.9052769	0.9151292	0.9048374
0.20	0.8195263	0.8434999	0.8187308
0.30	0.7418982	0.7864437	0.7408183
0.40	0.6716233	0.7464665	0.6703200
0.50	0.6080051	0.7277890	0.6065307
0.60	0.5504130	0.7371639	0.5488117
0.70	0.4982761	0.7851078	0.4965853
0.80	0.4510779	0.8877531	0.4493290
0.90	0.4083504	1.0696350	0.4065697
0.91	0.4043069	-0.3023021	0.4025242
0.92	0.4003034	1.1185030	0.3985191
0.93	0.3963396	-0.3692722	0.3945537
0.94	0.3924150	1.1723400	0.3906278
0.95	0.3885292	-0.4410981	0.3867410
0.96	0.3846820	1.2315490	0.3828929
0.97	0.3808728	-0.5182078	0.3790830
0.98	0.3771067	1.2965700	0.3753111
0.99	0.3710582	-0.6010649	0.3715767
1.00	1.3678790	1.3678790	1.3678790

6. Discusión and conclusions

We have presented a fitted sixth-order tridiagonal finite difference method for solving singularly perturbed two-point boundary value problems with boundary layer at one (left or right) end point. We have implemented the present method on three linear examples with left-end boundary layer and two examples with right-end boundary layer by taking different values of ε. Numerical results are presented in tables and compared with the Chawla's [2] solutions. It is observed from the results that the present method approximate the exact solution very well.

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