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PRIME Γ-RADICALS IN Γ-SEMIGROUPS

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1. INTRODUCTION :

Γ- semigroup was introduced by Sen and Saha [10] as a generalization of semigroup. Anjaneyulu. A [1], [2] and [3] initiated the study of pseudo symmetric ideals and radicals in semigroups. Giri and wazalwar [7] intiated the study of prime radicals in semigroups. Madhusudhana Rao, Anjaneyulu and Gangadhara Rao [8] initiated the study of pseudo symmetric Γ-ideals in Γ-semigroups. In this paper we introduce the notions of Γ-radicals in Γ-semigroup and characterize Γ-radicls in Γ-semigroups.

2. PRELIMINARIES :

DEFINITION 2.1: Let S and Γ be any two non-empty sets. S is called a ***Γ-semigroup*** if there exist a mapping from $S \times \Gamma \times S$ to S which maps $(a, \alpha, b) \rightarrow a\alpha b$ satisfying the condition : $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a,b,c \in$ M and $\gamma,\mu \in \Gamma$.

NOTE 2.2: Let S be a Γ-Semigroup. If A and B are subsets of S, we shall denote the set $\{ a\alpha b : a \in A , b \in B \text{ and } \alpha \in \Gamma \}$ by AΓB.

DEFINITION 2.3: A Γ-Semigroup S is said to be ***commutative*** provided $a\gamma b = b\gamma a$ for all $a,b \in S$ and $\gamma \in \Gamma$.

NOTE 2.4: If S is a commutative Γ-Semigroup then $a\,\Gamma b = b\,\Gamma a$ for all $a, b \in S$.

DEFINITION 2.5: Let S be a Γ-Semigroup. A nonempty subset T of S is said to be a ***Γ- subsemigroup*** of S if $a\gamma b \in$ T, for all $a, b \in$ T and $\gamma \in \Gamma$.

DEFINITION 2.6: An element ***a*** of a Γ-Semigroup S is said to be a ***left identity*** of S provided $a\alpha s = s$ for all $s \in S$ and $\alpha \in \Gamma$.

DEFINITION 2.7: An element ***a*** of a Γ-Semigroup S is said to be a ***right identity*** of S provided $s\alpha a = s$ for all $s \in S$ and $\alpha \in \Gamma$.

DEFINITION 2.8: An element ***a*** of a Γ-Semigroup S is said to be a ***two sided identity*** or an ***identity*** provided it is both a left identity and a right identity of S.

NOTATION 2.9: Let S be a Γ-Semigroup. If S has an identity, let $S^1 = S$ and if S does not have an identity, let S^1 be the Γ-Semigroup S with an identity adjoined, usually denoted by the symbol 1.

DEFINITION 2.10: A nonempty subset A of a Γ-Semigroup S is said to be a ***left Γ- ideal*** provided $S\Gamma A \subseteq A$.

DEFINITION 2.11: A nonempty subset A of a Γ-Semigroup S is said to be a ***right* Γ- *ideal*** provided $A\Gamma S \subseteq A$.

DEFINITION 2.12: A nonempty subset A of a Γ-Semigroup S is said to be a ***two sided* Γ- *ideal*** or simply a **Γ- *ideal*** provided it is both a left and a right Γ –ideal of S.

DEFINITION 2.13: An element ***a*** of Γ- Semigroup S is said to be a ***Γ- idempotent*** provided $a\alpha a = a$ for all $\alpha \in \Gamma$.

NOTE 2.14: If an element ***a*** of Γ- Semigroup S is ***a Γ- idempotent,*** then $a\Gamma a = a$.

DEFINITION 2.15: A Γ- Semigroup S is said to be an ***idempotent Γ- Semigroup*** or ***a band*** provided every element in S is a Γ- idempotent.

DEFINITION 2.16: A Γ- ideal A of a Γ-Semigroup S is said to be a ***principal* Γ- *ideal*** provided A is a Γ- ideal generated by single element ***a***. It is denoted by $J[a] = <a>$.

NOTE 2.17: If S is a Γ-Semigroup and $a \in S$, then $<a> = \{a\} \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S = S^1\Gamma a\,\Gamma S^1$.

DEFINITION 2.18: A Γ-ideal A of a Γ-Semigroup S is said to be ***pseudo symmetric*** provided $x, y \in S$, $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$, for all $s \in S$.

NOTE 2.19: A Γ-ideal A of a Γ-Semigroup S is pseudo symmetric iff $x, y \in S$, $x\Gamma y \subseteq A$ implies $x\Gamma S^1\Gamma y \subseteq A$.

DEFINITION 2.20: A Γ-Semigroup S is said to be ***pseudo symmetric*** provided every Γ-ideal is a pseudo symmetric Γ-ideal.

THEOREM 2.21[8]: Every commutative **Γ**-semigroup is a pseudo symmetric **Γ**-semigroup.

COROLLARY 2.22 [8]: Let A is a pseudo symmetric **Γ**-ideal in a **Γ**-Semigroup S. Then for any natural number n, $(a\Gamma)^{n-1}a \subseteq A$ implies $(<a>\Gamma)^{n-1} <a> \subseteq A$.

3. PRIME Γ-IDEALS :

DEFINITION 3.1: A Γ- ideal A of a Γ-Semigroup S is said to be a ***prime* Γ- *ideal*** provided $X \Gamma Y \subseteq A$; X, Y are Γ- ideal of S, then either $X \subseteq A$ or $Y \subseteq A$.

DEFINITION 3.2: A Γ- ideal A of a Γ-Semigroup S is said to be a ***completely prime* Γ- *ideal*** provided $x\Gamma y \subseteq A$; $x, y \in$ S implies either $x \in$ A or $y \in$ A.

THEOREM 3.3 [8]: If *P* is a Γ- ideal of a Γ-Semigroup S, then the following conditions are equivalent.

1) If $A\Gamma B \subseteq P$; *A*, *B* are Γ- ideals of S, then either $A \subseteq P$ or $B \subseteq P$.

2) If *a*, *b* ∈ S such that $a\Gamma S^1 \Gamma b \subseteq P$, then either *a* ∈ P or *b* ∈ P.

DEFINITION 3.4: Let S be a Γ-semigroup. A nonempty subset A of S is said to be a ***c-system*** of S if for each $a, b \in$ A there exist an element $\alpha \in \Gamma$ such that $a\alpha b \in$ A.

THEOREM 3.5: Every Γ-subsemigroup of a Γ-semigroup is a *c-system*.

***Proof* :** Let T be a Γ-subsemigroup of S. Then for all $a, b \in$ T and $\alpha \in \Gamma$; $a\alpha b \in$ T $\Rightarrow$ for all $a, b \in$ T there exist $\alpha \in \Gamma$ such that $a\alpha b \in$ T and hence T is a *c*-system.

THEOREM 3.6: A Γ-ideal P of a Γ-semigroup S is completely prime if and only if S\P is either a *c*-system of S or empty.

***Proof* :** Suppose that P is a completely prime Γ-ideal of S and S\P ≠ ∅. Let $a, b \in$ S\P. Then $a \notin$ P, $b \notin$ P. Suppose if possible there exist no $\alpha \in \Gamma$ such that $a\alpha b \in$ S\P. Then $a\Gamma b \subseteq$ P. Since P is completely prime, either $a \in$ P or $b \in$ P. It is a contradiction. Therefore S\P is a *c*-system.

Conversely suppose that S\P is a *c*-system of S or S\P is empty. If S\P is empty then P = S and hence P is completely prime. Assume that S\P is a *c*-system of S. Let $a, b \in$ S and $a\Gamma b \subseteq$ P. Suppose if $a \notin$ P and $b \notin$ P. Then $a \in$ S\P and $b \in$ S\P. Since S\P is a *c*-system, then there exist $\alpha \in \Gamma$ such that $a\alpha b \in$ S\P. Thus $a\alpha b \notin$ P and hence $a\Gamma b \nsubseteq$ P. It is a contradiction. Hence either $a \in$ P or $b \in$ P. Therefore P is a completely prime Γ-ideal of S.

THEOREM 3.5 [8]: Every completely prime Γ-ideal of a Γ-semigroup S is a prime Γ-ideal of S.

THEOREM 3.6 [8]: Every completely prime Γ-ideal of a Γ-semigroup S is a pseudo symmetric Γ-ideal of a Γ-semigroup S.

THEOREM 3.7: Let S be a commutative Γ-semigroup. A Γ-ideal P of S is prime Γ-ideal if and only if P is a completely prime Γ-ideal.

***Proof* :** Suppose that P is a prime Γ-ideal of Γ-semigroup S. Let $x, y \in$ S and $x\Gamma y \subseteq$ P. Now $x\Gamma y \subseteq$ P, P is a prime Γ-ideal $\Rightarrow x\Gamma y\Gamma S^1 \subseteq$ P $\Rightarrow x\Gamma S^1\Gamma y \subseteq$ P. By theorem 3.3, either $x \in$ P or $y \in$ P. Hence P is a completely prime Γ-ideal. Conversely suppose that P is a completely prime Γ-ideal of S. By theorem 3.5, P is a prime Γ-ideal of S.

DEFINITION 3.8: A non empty subset A of a Γ-Semigroup S is said to be an ***m-system*** provided for any $a,b \in A$, there exists an $x \in S$ and some $\alpha, \beta \in \Gamma$ such that $a\alpha x\beta b \in A$.

THEOREM 3.9: A Γ-ideal P of a Γ-semigroup S is a prime Γ-ideal of S if and only if S\P is an *m*-system of S or empty.

Proof **:** Suppose that P is a prime Γ-ideal of a Γ-semigroup S and $S \setminus P \neq \emptyset$. Let $a, b \in S \setminus P$. Then $a \notin P$, $b \notin P$. Suppose if possible there exist no $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a\alpha x\beta b \in S \setminus P$. Then $a\Gamma S\Gamma b \subseteq P$. Since P is prime, either $a \in P$ or $b \in P$. It is a contradiction. Therefore there exist an $x \in S$ and some $\alpha, \beta \in \Gamma$ such that $a\alpha x\beta b \in S \setminus P$ and hence S\P is an *m*-system.

Conversely suppose that S\P is either an *m*-system or $S \setminus P = \emptyset$. If S\P is empty then P = S and hence P is prime. Assume that S\P is an *m*-system of S.
Let $a, b \in S$ and $a\Gamma S\Gamma b \subseteq P$. Suppose if possible $a \notin P$, $b \notin P$. Then $a, b \in S \setminus P$.
Since S\P is an *m*-system, there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a\alpha x\beta b \in S \setminus P$.
Thus $a\alpha x\beta b \notin P$ and hence $a\Gamma S\Gamma b \not\subseteq P$. It is a contradiction.
Hence either $a \in P$ or $b \in P$. Therefore P is a prime Γ-ideal of S.

DEFINITION 3.10: A Γ-semigroup S is said to be a globally idempotent Γ-semigroup if $S\Gamma S = S$.

THEOREM 3.11: If S is a globally idempotent Γ-semigroup then every maximal Γ-ideal of S is a prime Γ-ideal of S.

Proof **:** Let M be a maximal Γ-ideal of S. Let A, B be two Γ-ideals of S such that $A\Gamma B \subseteq M$. Suppose if possible $A \not\subseteq M$, $B \not\subseteq M$. Now $A \not\subseteq M \Rightarrow M \cup A$ is a Γ-ideal of S and $M \subseteq M \cup A \subseteq S$. Since M is maximal, $M \cup A = S$. Similarly $B \not\subseteq M \Rightarrow M \cup B = S$.
Now $S = S\Gamma S = (M \cup A)\Gamma (M \cup B) = (M\Gamma M) \cup (M\Gamma B) \cup (A\Gamma M) \cup (A\Gamma B) \subseteq M \Rightarrow S \subseteq M$.
Hence M = S. It is a contradiction. Therefore either $A \subseteq M$ or $B \subseteq M$. Hence M is prime.

DEFINITION 3.12: An element a of Γ-semigroup S is said to be ***semisimple*** provided $a \in \langle a \rangle \Gamma \langle a \rangle$, i.e., $\langle a \rangle \Gamma \langle a \rangle = \langle a \rangle$.

DEFINITION 3.13: A Γ-semigroup S is said to be a ***semisimple Γ-semigroup*** provided every element is a semisimple.

THEOREM 3.14: If S is a globally idempotent Γ-semigroup with maximal Γ-ideals then S contains semisimple elements.

Proof **:** Suppose S is a globally idempotent Γ-semigroup with maximal Γ-ideals. Let M be a maximal Γ-ideal of S. Then by theorem 3.11, M is prime.
Now if $a \in S \setminus M$ then $\langle a \rangle \Gamma \langle a \rangle \not\subseteq M$ and hence $S = M \cup \langle a \rangle = M \cup (\langle a \rangle \Gamma \langle a \rangle)$.
Therefore $a \in \langle a \rangle \Gamma \langle a \rangle$. Thus a is semisimple. Therefore S contains semisimple elements.

4. SEMIPRIME Γ-IDEALS:

DEFINITION 4.1: A Γ- ideal A of a Γ-Semigroup S is said to be a ***completely semiprime Γ- ideal*** provided $x\Gamma x \subseteq A$; $x \in S$ implies $x \in A$.

DEFINITION 4.2: A Γ- ideal A of a Γ-Semigroup S is said to be a ***semiprime Γ- ideal*** provided $x\Gamma S^1\Gamma x \subseteq A$; $x \in S$ implies $x \in A$.

THEOREM 4.3 [8]: Every completely semiprime Γ-ideal of a Γ-semigroup S is a semiprime Γ-ideal of S.

THEOREM 4.4 [8]: Every completely prime Γ-ideal of a Γ-semigroup S is a completely semiprime Γ-ideal of S.

THEOREM 4.5 [8]: Every prime Γ-ideal of a Γ-semigroup S is a semiprime Γ-ideal of S.

THEOREM 4.6: The nonempty intersection of any family of prime Γ-ideals of a Γ-semigroup S is a semiprime Γ-ideal of S.

Proof **:** Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a family of prime Γ-ideals of S such that $\bigcap_{\alpha\in\Delta} A_\alpha \neq \emptyset$. It is clear that $\bigcap_{\alpha\in\Delta} A_\alpha$ is a Γ-ideal. Let $a \in S$, $a\Gamma S\Gamma a \subseteq \bigcap_{\alpha\in\Delta} A_\alpha$. Then $a\Gamma S\Gamma a \subseteq A_\alpha$ for all $\alpha \in \Delta$.
Since A_α is prime, $a \in A_\alpha$ for all $\alpha \in \Delta$ and hence $a \in \bigcap_{\alpha\in\Delta} A_\alpha$.
Therefore $\bigcap_{\alpha\in\Delta} A_\alpha$ is a semiprime Γ-ideal.

THEOREM 4.7: The nonempty intersection of any family completely prime Γ-ideals of a Γ-semigroup S is a completely semiprime Γ-ideal of S.

Proof **:** Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a family of completely prime Γ-ideals of S such that $\bigcap_{\alpha\in\Delta} A_\alpha \neq \emptyset$.
It is clear that $\bigcap_{\alpha\in\Delta} A_\alpha$ is a Γ-ideal. Let $a \in S$, $a\Gamma a \subseteq \bigcap_{\alpha\in\Delta} A_\alpha$. Then $a\Gamma a \subseteq A_\alpha$ for all $\alpha \in \Delta$.
Since A_α is completely prime, $a \in A_\alpha$ for all $\alpha \in \Delta$ and hence $a \in \bigcap_{\alpha\in\Delta} A_\alpha$.
Therefore $\bigcap_{\alpha\in\Delta} A_\alpha$ is a completely semiprime Γ-ideal.

THEOREM 4.8: Let S be a commutative Γ-semigroup. A Γ-ideal A of S is completely semiprime iff semiprime.

Proof **:** Suppose A is a completely semiprime Γ-ideal of S. By theorem 4.3, A is semiprime.
Conversely Suppose that A is a semiprime Γ-ideal of S. Let $x \in S$ and $x\Gamma x \subseteq A$.
Now $x\Gamma x \subseteq A \Rightarrow s\Gamma x\Gamma x \subseteq A$ for all $s \in S \Rightarrow x\Gamma s\Gamma x \subseteq A$ for all $s \in S \Rightarrow x\Gamma S\Gamma x \subseteq A \Rightarrow x \in A$, since A is semiprime. Therefore A is completely semiprime.

DEFINITION 4.9: A non empty subset A of a Γ-Semigroup S is said to be an ***n-system*** provided for any $a \in A$, there exists an $x \in S$ and some $\alpha, \beta \in \Gamma$ such that $a\alpha x\beta a \in A$.

THEOREM 4.10: A Γ- ideal Q of a Γ-Semigroup S is a semiprime Γ- ideal iff S\Q is an *n*-system or empty.

Proof **:** Suppose that Q is a semiprime Γ-ideal of a Γ-semigroup S and $S\backslash Q \neq \emptyset$.
Let $a \in S\backslash Q$. Then $a \notin Q$. Suppose if possible there exist no $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a\alpha x\beta a \in S\backslash Q$. Then $a\Gamma S\Gamma a \subseteq Q$. Since Q is semiprime, $a \in Q$. It is a contradiction.
Therefore there exists an $x \in S$ and $\alpha, \beta \in$ such that $a\alpha x\beta a \in S\backslash Q$ and hence
S\Q is an *n*-system.

Conversely suppose that S\Q is an *n*-system or $S\backslash Q = \emptyset$. If S\Q is empty then Q = S and hence Q is semiprime. Assume that S\Q is an *n*-system of S..
Let $a \in S$ and $a\Gamma S\Gamma a \subseteq Q$. Suppose if possible $a \notin Q$. Then $a \in S\backslash Q$.

Since S\Q is an n-system, there exists an $x \in$ S and $\alpha, \beta \in \Gamma$ such that $a\alpha x\beta a \in$ S\Q.
Thus $a\alpha x\beta a \notin$ Q and hence $a\Gamma S\Gamma a \nsubseteq$ Q. It is a contradiction.
Hence $a \in$ Q. Therefore Q is a semiprime Γ-ideal of S.

THEOREM 4.11: If N is an n-system in a Γ-Semigroup S and $a \in$ N, then there exists an *m*-system M in S such that $a \in$ M and M ⊆N.

Proof: We construct a subset M of N as follows. Define $a_1 = a$. Since $a_1 \in$ N and $a_1\alpha x\beta a_1 \subseteq$ N, for some x∈S, $\alpha, \beta \in \Gamma$ and hence $(a_1\Gamma S\Gamma a_1)\cap N \neq \emptyset$. Let $a_2 \in (a_1\Gamma S\Gamma a_1)\cap N$. Since $a_2 \in$ N, $(a_2\Gamma S\Gamma a_2)\cap N \neq \emptyset$ and so on. In general, if a_i has been defined with $a_i \in$ N, choose a_{i+1} as an element of $(a_i\Gamma S\Gamma a_i)\cap N$. So, let M = { a_1, a_2,--------, a_i, a_{i+1} ------}. Now, $a\in$ M and M ⊆N. We show that M is an m-system. Let a_i, $a_j \in$ M (for i≤ j). Then $a_{j+1} \in a_j\Gamma S\Gamma a_j \subseteq a_i\Gamma S\Gamma a_i$ $\Rightarrow a_{j+1} = a_i\alpha x\beta a_j$, $x \in$ S, $\alpha, \beta \in \Gamma$. But $a_{j+1} \in$ M, so $a_{j+1} = a_i\alpha x\beta a_j \in$ M, for $x \in$ S $\alpha, \beta \in \Gamma$. Therefore M is an m-system.

THEOREM 4.12 [8]: Every completely semiprime Γ-ideal A in a Γ-Semigroup S is a pseudo symmetric Γ-ideal .

THEOREM 4.13 [8]: A Γ-ideal A of a Γ-semigroup S is completely prime Γ-ideal if and only if A is a prime Γ-ideal and A is a pseudo symmetric Γ-ideal of S.

THEOREM 4.14[8]: Every prime Γ-ideal P minimal relative to containing a completely semiprime Γ-ideal A in a Γ-semigroup S is completely prime.

5. PRIME Γ-RADICAL:

NOTATION 5.1: If A is a Γ-ideal of a Γ-semigroup S, then we associate the following four types of sets.

A_1 = The intersection of all completely prime Γ-ideals of S containing A.
$A_2 = \{x \in S : (x\Gamma)^{n-1}x \subseteq A$ for some natural number $n\}$
A_3 = The intersection of all prime ideals of S containing A.
$A_4 = \{x \in S : (<x>\Gamma)^{n-1} <x> \subseteq A$ for some natural number $n\}$

THEOREM 5.2: If A is a Γ- ideal of a Γ-semigroup S, then $A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.

Proof : (i) **$A \subseteq A_4$:** Let $x \in$ A. Then $(<x>\Gamma)^0 <x> \subseteq A$ and hence $x \in A_4$. ∴ $A \subseteq A_4$.

(ii) **$A_4 \subseteq A_3$:** Let $x \in A_4$. Then $(<x>\Gamma)^{n-1} <x> \subseteq A$ for some $n \in$ N. Let P be any prime Γ-ideal of S containing A. Then $(<x>\Gamma)^{n-1} <x> \subseteq A \Rightarrow (<x>\Gamma)^{n-1} <x> \subseteq P$.
Since P is prime, $<x> \subseteq P$ and hence $x \in$ P. Since this is true for all prime Γ-ideals P containing A, $x \in A_3$. Therefore $A_4 \subseteq A_3$.

(iii) **$A_3 \subseteq A_2$:** Let $x\in A_3$. Suppose if possible $x \notin A_2$. Then $(x\Gamma)^{n-1}x \nsubseteq A$ for all $n \in$ N. Consider $T = \cup(x\Gamma)^{n-1}x$, where $x \in$ S and n is a natural number.
Let $a, b \in$ T. Then $a \in (x\Gamma)^{r-1}x$, $b \in (x\Gamma)^{s-1}x$ for some $r, s \in$ N.

Therefore $a\Gamma b = (x\Gamma)^{r-1}x\Gamma(x\Gamma)^{s-1}x = (x\Gamma)^{r+s-1}x \subseteq T$. Therefore T is a Γ-subsemigroup of S and T is a c-system of S and $x \in$ T. By theorem 3.4, P = S\T is a completely prime Γ-ideal of S and $x \notin$ P. By theorem 3.5, P is prime Γ-ideal of S and $x \notin$ P.

Therefore $x \notin A_3$. It is a contradiction. $\therefore x \in A_2$ and hence $A_3 \subseteq A_2$.

(iv) **$A_2 \subseteq A_1$:** Let $x \in A_2$. Now $x \in A_2 \Longrightarrow (x\Gamma)^{n-1}x \subseteq A$ for some natural number n. Let P be any completely prime Γ-ideal of S containing A. Then $(x\Gamma)^{n-1}x \subseteq A \subseteq P \Rightarrow (x\Gamma)^{n-1}x \subseteq P \Rightarrow x \in P$. Therefore $x \in A_1$. Therefore $A_2 \subseteq A_1$. Hence $A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.

THEOREM 5.3: In a commutative Γ-semigroup S, $A_1 = A_2 = A_3 = A_4$.

Proof : By theorem 3.6, in a commutative Γ-semigroup S, a Γ-ideal P is a prime Γ-ideal iff P is a completely prime Γ-ideal. So $A_1 = A_3$. By theorem 5.2, $A_4 \subseteq A_2$.
Now $x \in A_2$ then $(x\Gamma)^{n-1}x \subseteq A$ for some $n \in N$. By theorem 2.21, every commutative Γ-semigroup is pseudo symmetric. By theorem 2.22, $(< x >\Gamma)^{n-1}< x > \subseteq A \Rightarrow x \in A_4$. Therefore $A_2 \subseteq A_4$ and hence

$A_2 = A_4$. Now $A_2 \subseteq A_4$, $A_4 \subseteq A_3 \Rightarrow A_2 \subseteq A_3$. We have $A_3 \subseteq A_2$, $\Rightarrow A_2 = A_3$.
Hence $A_1 = A_2 = A_3 = A_4$.

NOTE 5.4: In an arbitrary Γ-semigroup $A_1 \neq A_2 \neq A_3 \neq A_4$.

EXAMPLE 5.5: Let S be the free Γ-semigroup generated by two alphabets a, b. It is clear that $A = S\Gamma a\Gamma a\Gamma S$ is a Γ-ideal in S. Since $(a\Gamma)^3 a \subseteq S\Gamma a\Gamma a\Gamma S = A$, We have $a \in A_2$. Evidently $(a\Gamma b\Gamma)^{n-1}a\Gamma b \nsubseteq S\Gamma a\Gamma a\Gamma S$ for all natural number n and thus $a\Gamma b \nsubseteq A_2$. Thus A_2 is not a Γ-ideal in S. Therefore $A_1 \neq A_2$ and $A_2 \neq A_3$.

EXAMPLE 5.6: Let S be the free Γ-semigroup over the countable infinite alphabet $\{x_1, x_2, \ldots\ldots..\}$ and Γ as $\{ \alpha_1, \alpha_2, \ldots\ldots\ldots\}$. Consider the Γ-ideal $A = \bigcup\limits_{l(s)>1} (< s > \Gamma)^{l(s)-1} < s >$,
where l(s) is the length of the word s. For any $s \in S$, $< x_1\Gamma s\Gamma x_1 >^{l(s)+1} < x_1\Gamma s\Gamma x_1> \subseteq A$ and hence $x_1\Gamma s\Gamma x_1 \subseteq A_4$ for all $s \in S$. If $A_3 = A_4$, then A_4 is a semiprime Γ-ideal and hence $x_1 \in A_4$. Therefore $(< x_1 >\Gamma)^{n-1} < x_1 > \subseteq A$ for some natural number n.
Consider the word $t = x_1\alpha_1x_2\alpha_2x_1\alpha_3x_3\alpha_4x_1\ldots\ldots\ldots\alpha_{n-1}x_1\alpha_nx_{n+1}$. Now $t \in < x_1\Gamma >^{n-1} < x_1 > \subseteq A$.
So $t \in < s\Gamma >^{l(s)-1}< s >$ for some $s \in S$ with $l(s) > 1$. Thus in t, s occurs at least two times, which is a contradiction. So $A_3 \neq A_4$.

DEFINITION 5.7: If A is a Γ-ideal of a Γ-semigroup S, then the intersection of all prime Γ-ideals of S containing A is called ***prime Γ-radical*** or simply ***Γ-radical*** of A and it is denoted by **√A** or ***rad A***.

DEFINITION 5.8: If A is a Γ-ideal of a Γ-semigroup S, then the intersection of all completely prime Γ-ideals of S containing A is called ***complete prime Γ-radical*** or simply ***complete Γ-radical*** of A and it is denoted by ***c. rad A***.

NOTE 5.9: If A is a Γ-ideal of a Γ-semigroup S then ***rad*** **A = A_3** and ***c.rad*** **A = A_4.**

THEOREM 5.10: **If $a \in \sqrt{A}$, then there exist a positive integer n such that $(a\Gamma)^{n-1}a \subseteq A$.**

Proof **:** By theorem 5.2, $A_3 \subseteq A_2$ and hence $a \in \sqrt{A} = A_3 \subseteq A_2$. Therefore $(a\Gamma)^{n-1}a \subseteq A$ for some $n \in N$.

THEOREM 5.11: If A is a Γ-ideal of a commutative Γ-semigroup S, then *rad* A = *c.rad* A

Proof **:** By theorem 5.3, *rad* A = *c.rad* A.

THEOREM 5.12: If A is a Γ-ideal of a Γ-semigroup S then *c.rad* A is a completely semiprime Γ-ideal of S.

Proof **:** By theorem 4.7, *c.rad* A is a completely semiprime Γ-ideal of S.

THEOREM 5.13: If A is a Γ-ideal of a Γ-semigroup S then √A is a semiprime Γ-ideal of S.

Proof **:** By theorem 4.5,√A is a semiprime Γ-ideal of S.

THEOREM 5.14: If A and B are any two Γ-ideals of a Γ-Semigroup S, then

(i) A ⊆ B ⇒ √(A) ⊆ √(B).

(ii) √(AΓB) = √(A∩B) = √(A) ∩ √(B).

(iii) √(√(A)) = √(A).

Proof: (i) Suppose that A ⊆ B. If P is a prime Γ-ideal containing B then P is a prime Γ-ideal containing A. Therefore √A ⊆√ B.

(ii) Let P be a prime Γ-ideal containing AΓB. Then AΓB ⊆ P ⇒ A ⊆ P or B ⊆ P
⇒ A ∩ B ⊆ P. Therefore P is a prime Γ-ideal containing A ∩ B.
Therefore √(A ∩ B) ⊆ √(AΓB). Now let P be a prime Γ-ideal containing A ∩ B.
Since A is a Γ-ideal of S, AΓB ⊆ AΓS⊆A. Since B is a Γ-ideal of S, AΓB ⊆ SΓB⊆ B.
Therefore AΓB ⊆ A ∩ B ⊆ P ⇒ AΓB ⊆ P.
Hence P is a prime Γ-ideal containing AΓB. Therefore √(AΓB) ⊆ √(A ∩ B).
Therefore √(AΓB) = √(A ∩ B). Now AΓB ⊆ A, AΓB ⊆ B ⇒ √(AΓB) ⊆ √A, √(AΓB) ⊆ √B, by condition (i). Hence √(AΓB) ⊆ √A ∩ √B.
Let *x* ∈ √A ∩ √B. Then *x* ∈ √A and *x* ∈ √B.
Suppose if possible *x* ∉ √(AΓB). Then there exists a prime Γ-ideal P containing (AΓB) and not containing *x*. AΓB ⊆ P ⇒ either A ⊆ P or B ⊆ P.
If A ⊆ P then P is a prime Γ-ideal containing A and not containing *x* ⇒ *x* ∉ √A.
If B ⊆ P then P is a prime Γ-ideal containing B and not containing *x* ⇒ *x* ∉ √B.
It is a contradiction. Therefore *x* ∈ √(AΓB). Therefore √A ∩ √B ⊆ √(AΓB).
Therefore √A ∩ √B = √(AΓB). Hence √(AΓB) = √(A∩B) = √(A) ∩ √(B).

(iii) √A = The intersection of all prime Γ-ideals of S containing A.

Now $\sqrt{(\sqrt{A})}$ = The intersection of all prime Γ-ideals of S containing $\sqrt{A}$.

= The intersection of all prime Γ-ideals of S containing A = $\sqrt{A}$.

Therefore $\sqrt{(\sqrt{A})} = \sqrt{A}$.

THEOREM 5.15: If P is a prime Γ-ideal of a Γ-semigroup S, then $\sqrt{((P\Gamma)^{n-1}P)} = P$ for all n ∈ N.

***Proof* :** We use induction on *n*, to prove $\sqrt{((P\Gamma)^{n-1}P)} = P$. First we prove that $\sqrt{P} = P$.

Since P is a prime Γ-ideal, $P \subseteq \sqrt{P} \subseteq P \Rightarrow \sqrt{P} = P$.

Assume that $\sqrt{((P\Gamma)^{k-1}P)} = P$ for $k \in N$ such that $1 \leq k < n$.

Now $\sqrt{((P\Gamma)^{k}P)} = \sqrt{((P\Gamma)^{k-1}P\Gamma P)} = \sqrt{((P\Gamma)^{k-1}P \cap P)} = \sqrt{((P\Gamma)^{n-1}P)} \cap \sqrt{P}$

$= \sqrt{P} \cap \sqrt{P} = P \cap P = P$. Therefore $\sqrt{((P\Gamma)^{k}P)} = P$.

By induction $\sqrt{((P\Gamma)^{n-1}P)} = P$ for all n ∈ N.

THEOREM 5.16: In a Γ-semigroup S with identity there is a unique maximal Γ-ideal M such that $\sqrt{((M\Gamma)^{n-1}M} = M$ for all *n* ∈ N.

***Proof* :** Since S contains identity, S is a globally idempotent Γ-semigroup. Since M is a maximal Γ-ideal of S, by theorem 3.11, M is prime. By theorem 5.15, $\sqrt{((M\Gamma)^{n-1}M} = M$ for all $n \in N$.

THEOREM 5.17: If A is a Γ-ideal of a Γ-semigroup S then $\sqrt{A}$ = {*x* ∈ S : every *m*-system of S containing *x* meets A} i.e, $\sqrt{A} = \{ x \in S : M(x) \cap A \neq \emptyset\}$.

***Proof* :** Suppose that $x \in \sqrt{A}$. Let M be an *m*-system containing *x*. Then S\M is a prime Γ-ideal of S and $x \notin$ S\M. If $M \cap A = \emptyset$ then $A \subseteq$ S\M. Since S\M is a prime Γ-ideal containing A, $\sqrt{A} \subseteq$ S\M and hence $x \in$ S\M. It is a contradiction. Therefore $M(x) \cap A \neq \emptyset$. Hence $x \in \{x \in S : M(x) \cap A \neq \emptyset\}$. Conversely suppose that $x \in \{ x \in S : M(x) \cap A \neq \emptyset\}$. Suppose if possible $x \notin \sqrt{A}$. Then there exists a prime Γ-ideal P such that $x \notin P$. Now S\P is an *m*-system and $x \in$ S\P. Therefore $A \subseteq P \Rightarrow S\backslash P \cap A = \emptyset \Rightarrow x \notin \{x \in S : M(x) \cap A \neq \emptyset\}$. It is a contradiction. Therefore $x \in \sqrt{A}$. Thus $\sqrt{A} = \{ x \in S : M(x) \cap A \neq \emptyset\}$.

THEOREM 5.18: A Γ- ideal Q of Γ-Semigroup S is a semiprime Γ- ideal of S iff $\sqrt{(Q)} = Q$.

***Proof* :** Suppose that Q is a semiprime Γ-ideal of S. Clearly $Q \subseteq \sqrt{Q}$.

Suppose if possible $\sqrt{Q} \nsubseteq Q$. Let $a \in \sqrt{Q}$ and $a \notin Q$. Now $a \notin Q \Rightarrow a \in$ S\Q and Q is semiprime. By theorem 4.10, S\Q is an *n*-system. By theorem4.11, there exists an *m*-system M such that $a \in M \subseteq$ S\Q. Now $Q \subseteq$ S\M and $a \notin$ S\M. By yheorem 3.9, S\M is a prime Γ-ideal of S, It is a contradiction. Therefore $\sqrt{Q} \subseteq Q$. Hence $\sqrt{Q} = Q$. Conversely

Suppose that Q is a Γ-ideal of S such that √Q = Q. By theorem 5.13, √Q is a semiprime Γ-ideal of S. Therefore Q is semiprime.

COROLLARY 5.19 : If A is a Γ- ideal of a Γ-Semigroup S, then √(A) is the smallest semiprime Γ- ideal of S containig A.

Proof: By the theorem 5.14, √(√(A)) = √A and hence by theorem 5.18, √(A) is a semiprime Γ-ideal of S. Clearly A ⊆ √A. Let Q be any semiprime Γ- ideal of S containing A. By theorem 5.14, A ⊆ Q ⇒ √(A) ⊆ √(Q). Since Q is semiprime. By theorem 5.18, √(Q) = Q. Therefore, √(A) ⊆ Q. Hence √(A) is the smallest semiprime Γ- ideal of S containing A.

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