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ON ABSOLUTE NÖRLUND-BANACH SUMMABILITY OF CONJUGATE SERIES OF FOURIER SERIES

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ABSTRACT: A theorem on absolute NÖrlund-Banach summability of Conjugate series of fourier series have been established.

KEY WORDS: Banach summability , NÖrlund summability, NÖrlund-Banach summability.

1. INTRODUCTION:

Let $\{s_n\}$ be the sequence of partial sums of a series $\sum a_n$ .Let the sequence $\{t_k(n)\}$ be defined by

(1.1) $$t_k(n) = \frac{1}{k}\sum_{v=0}^{k-1} s_{n+v} \ ,k \in N$$

Then $t_k(n)$ is said to be kth element of the Banach Transformed sequence .If

(1.2) $$\lim_{k \to \infty} t_k(n) = s \ ,a \ \ finite \ \ number$$

uniformly in $n \in N$, $\sum a_n$ is said to be Banach summable to 's'.

Further , if the series

(1.3) $$\sum_{k=1}^{\infty} \left| t_k(n) - t_{k+1}(n) \right| < \infty,$$

uniformly for all $n \in N$ then the series $\sum a_n$ is said to be absolutely Banach summable or simply $|B|$ summable.

Let $\{p_n\}$ be a sequence of constants real or complex and let us write

(1.4) $$P_n = p_0 + p_1 + \cdots + p_n \ , P_{-k} = p_{-k} = 0 \ for \ k \geq 1.$$

The sequence- $\{t_m\}$ defined by

(1.5) $$t_m = \frac{1}{P_m}\sum_{v=0}^{m} p_{m-v} \ s_v \ ,(P_m \neq 0)$$

defines the NÖrlund mean of the sequence $\{s_n\}$ generated by the sequence of constants $\{p_n\}$.

If

(1.6) $$\sum_m |t_m - t_{m+1}| < \infty$$

then the series $\sum a_n$ is said to be absolutely NÖrlund summable or simply $|N, p_n|$ summable.

Next , let

(1.7) $$T_k(n) = \frac{1}{P_k} \sum_{\nu=0}^{k-1} p_{k-\nu} \ s_{n+\nu} \ , k \in N, P_k \neq 0 \ and \ p_0 = 0$$

then $T_k(n)$ is said to be the k-th element of NÖrlund-Banach transformed sequence. If the series

(1.8) $$\sum_{k=1}^{\infty} |T_k(n) - T_{k+1}(n)| < \infty$$

uniformly in $n \in N$, then the series $\sum a_n$ is said to be absolutely NÖrlund-Banach summable or simply $|(N, p_n) - B|$ summable .

It may be noted here that $|(N, p_n) - B|$-summability reduces to $|B|$-summability if we consider $p_n = 1 \ \forall n \in N$.

Let $f(t)$ be a periodic function with period 2π , integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let

(1.9) $$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x).$$

be the conjugate Fourier series of $f(t)$ **,**then

(1.10) $$B_n(x) = \frac{2}{\pi} \int_0^{\pi} \psi(t) \sin nt \, dt \ , n = 1,2,3...$$

$$= \frac{2}{n\pi} \int_0^{\pi} \cos nt \, d\psi(t) \ ,$$

where

(1.11) $$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}$$

2. KNOWN RESULTS:

Dealing with $|B|$-summability of Conjugate Fourier series , Misra **[2]** established the following theorems:

2.1. THEOREM: Let ‘ f ’ be a periodic function with period 2π and L-integrable in $(-\pi, \pi)$ **.**Then the conjugate Fourier series $\sum B_n(x)$ of f is $|B|$-summable if

(i) $\psi(t) \in BV(0, \pi)$,

(ii) $$\int_0^{\pi} \frac{\psi(t)}{t} dt < \infty .$$

2.2. THEOREM: If $\int_0^{\pi} \frac{\psi(t)}{t} dt < \infty$, then the factored conjugate series $\sum \lambda_n B_n(x)$ is $|B|$- summable for $\{\lambda_n\}$ to be a non-negative convex sequence such that $\sum \frac{\lambda_n}{n} < \infty$.

We wish to prove analogue theorems for $|(N, p_n) - B|$-summability of conjugate and factored conjugate Fourier series.

3. MAIN THEOREMS:

3.1. THEOREM: Let $\{p_n\}$ be a non-negative non-increasing sequence of numbers such that

(3.1.1) $np_n = O(P_n)$,

(3.1.2) $P_n = O(n\, p_n)$ and

(3.1.3) $\{k\, p_{k-v+1} p_k - (k-v) p_{k-v}\, p_{k+1}\} = O(v\, p_k\, p_{k+1})$.

then the conjugate series $\sum B_n(x)$ of f is summable $|(N, p_n) - B|$ if

(i) $\psi(t) \in BV(0, \pi)$,

(ii) $\int_0^{\pi} \frac{\psi(t)}{t} dt < \infty$.

3.2. THEOREM: Let $\{p_n\}$ be as defined in theorem-3.1 satisfying (3.1.1), (3.1.2) and (3.1.3) . If $\int_0^{\pi} \frac{\psi(t)}{t} dt < \infty$

then the factored conjugate Fourier series $\sum \lambda_n B_n(x)$ is $|(N, p_n) - B|$- summable for $\{\lambda_n\}$ to be a non-negative convex sequence such that $\sum \frac{\lambda_n}{n} < \infty$.

4. REQUIRED LEMMAS:

4.1. LEMMA[1]:

Let $\{p_n\}$ be a non-negative and non-increasing sequence. For $0 < a < b < \infty$, $0 \le t \le \pi$ and for any n and a

$$\sum_{k=a}^{b} p_k e^{i(n-k)t} = O(P_\tau), \text{ where } \tau = \left\{\frac{1}{t}\right\}$$

4.2. LEMMA[3]:

If $\{\lambda_n\}$ is a positive sequence such that $\sum \frac{\lambda_n}{n} < \infty$, then $\{\lambda_n\}$ is a monotonically decreasing sequence.

5. 1. PROOF OF THEOREM - 3.1:

If $T_k(n)$, the k-th element of $(N, p_n) - B$ transform of the series $\sum a_n$,then we have

$$T_k(n) = \frac{1}{P_k} \sum_{v=0}^{k-1} p_{k-v}\; s_{n+v}$$

$$= \frac{1}{P_k} \sum_{\nu=0}^{k-1} p_{k-\nu} \sum_{i=1}^{n+\nu} a_i$$

$$= \frac{1}{P_k} \left[\sum_{i=1}^{n} a_i \sum_{\nu=0}^{k-1} p_{k-\nu} + \sum_{i=n}^{n+k-1} a_i \sum_{\nu=i-n}^{k-1} p_{k-\nu} \right]$$

$$= \frac{1}{P_k} \left[p_k s_n + \sum_{i=n}^{n+k-1} P_{k-i+n} a_i \right]$$

$$= s_n + \frac{1}{P_k} \sum_{\nu=0}^{k-1} P_{k-\nu} \, a_{n+\nu} \quad .$$

Then,

(5.1.1) $$T_k(n) - T_{k+1}(n) = \sum_{\nu=1}^{k} \left(\frac{P_{k-\nu}}{P_k} - \frac{P_{k-\nu+1}}{P_{k+1}} \right) a_{n+\nu}$$

$$= \sum_{\nu=1}^{k} \left(\frac{p_{k+1} P_{k-\nu} - p_{k-\nu+1} P_k}{P_k P_{k+1}} \right) a_{n+\nu} .$$

Thus, for the series $\sum B_n(x)$, we have

$$\sum_{k=1}^{\infty} \left| T_k(n) - T_{k+1}(n) \right| = \sum_{k=1}^{\infty} \left| \sum_{\nu=1}^{k} \left(\frac{p_{k+1} P_{k-\nu} - p_{k-\nu+1} P_k}{P_k P_{k+1}} \right) B_{n+\nu}(x) \right|$$

$$= \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{P_k P_{k+1}} \left| \int_0^{\pi} \sum_{\nu=1}^{k} \left(p_{k+1} P_{k-\nu} - p_{k-\nu+1} P_k \right) \sin(n+\nu)t \; \psi(t) \; dt \right|$$

$$= \frac{2}{\pi} \left[\sum_{k=1}^{\tau} + \sum_{k>\tau} \right] \frac{1}{P_k P_{k+1}} \left| \int_0^{\pi} \sum_{\nu=1}^{k} \left(p_{k+1} P_{k-\nu} - p_{k-\nu+1} P_k \right) \sin(n+\nu)t \; \psi(t) \; dt \right|$$

where $\tau = \left[\frac{1}{t} \right]$

$$= \frac{2}{\pi} \left(\sum\nolimits_1 + \sum\nolimits_2 \right), \text{ say.}$$

Now,

$$\sum\nolimits_1 = \sum_{k=1}^{\tau} \frac{1}{P_k P_{k+1}} \left| \sum_{\nu=1}^{k} \int_0^{\pi} t \left(p_{k+1} P_{k-\nu} - p_{k-\nu+1} P_k \right) \sin(n+\nu)t \, \frac{\psi(t)}{t} dt \right|$$

Since, $\int_0^{\pi} \frac{\psi(t)}{t} dt < \infty$, $\quad \sum_1 < \infty$ if

$$\sum_{k=1}^{\tau} \frac{1}{P_k P_{k+1}} \left| \sum_{\nu=1}^{k} \left(p_{k+1} P_{k-\nu} - p_{k-\nu+1} P_k \right) \sin(n+\nu)t \right| = O(\tau),$$

uniformly for $n \in N$ *and* $0 < t < \pi$.

Now, $$\sum_{k=1}^{\tau} \frac{1}{P_k P_{k+1}} \sum_{\nu=1}^{k} \left(p_{k+1} P_{k-\nu} - p_{k-\nu+1} P_k \right) \left| \sin(n+\nu)t \right|$$

$$= O(1)\sum_{k=1}^{\tau}\frac{1}{P_k P_{k+1}}\sum_{\nu=1}^{k}\left|\left\{p_{k+1}(k-\nu)P_{k-\nu} - p_{k-\nu+1}kP_k\right\}\right|\left|\sin(n+\nu)t\right|$$

$$= O(1)\sum_{k=1}^{\tau}\frac{p_k p_{k+1}}{P_k P_{k+1}}\left|\sum_{\nu=1}^{k}\nu\right|$$

$$= O(1)\sum_{k=1}^{\tau}\frac{k(k+1)}{2k(k+1)}$$

$$= O(\tau).$$

Next , $$\sum\nolimits_2 = \sum_{k>\tau}\frac{1}{P_k P_{k+1}}\left|\int_0^{\pi}\sum_{\nu=1}^{k}\left(p_{k+1}P_{k-\nu} - p_{k-\nu+1}P_k\right)\frac{\cos(n+\nu)t}{n+\nu}\, d\psi(t)\right|$$

Again , since $\int_0^{\pi}|d\psi(t)| < \infty$ as $\psi(t)\in BV(0,\pi)$, in order to show that $\sum_2 < \infty$ it is sufficient to show that $\sum_{k>\tau}\frac{1}{P_k P_{k+1}}\left|\sum_{\nu=1}^{k}\{p_{k+1}P_{k-\nu} - p_{k-\nu+1}P_k\}\frac{\cos(n+\nu)t}{n+\nu}\right| = O(1)$, uniformly for $n\in N$ *and* $0<t<\pi$. Now, by (3.1.3)

$$\sum_{k>\tau}\frac{1}{P_k P_{k+1}}\left|\sum_{\nu=1}^{k}\{p_{k+1}P_{k-\nu} - p_{k-\nu+1}P_k\}\frac{\cos(n+\nu)t}{n+\nu}\right|$$

$$= O(1)\sum_{k>\tau}\frac{p_k p_{k+1}}{P_k P_{k+1}}\left|\sum_{\nu=1}^{k}\frac{\nu}{n+\nu}\cos(n+\nu)t\right|,$$

$$= O(1)\sum_{k>\tau}\frac{p_k p_{k+1}}{P_k P_{k+1}}\frac{k}{n+k} \quad \text{,as } \left\{\frac{\nu}{n+\nu}\right\} \text{ is increasing in } n.$$

$$= O(1).$$

Thus, $\sum < \infty$, uniformly for $n\in N$. Hence $\sum B_n(x)$ is $|(N,p_n)-B|$- summable.

This completes the proof of theorem 3.1.

5. 2. PROOF OF THEOREM - 3.2:

We have for the series $\sum \lambda_n B_n(x)$,

$$T_k(n) - T_{k+1}(n) = \sum_{\nu=1}^{k}\frac{\left(p_{k+1}P_{k-\nu} - p_{k-\nu+1}P_k\right)}{P_k P_{k+1}}\lambda_{n+\nu}B_{n+\nu}(x)$$

Hence , the series $\sum \lambda_n B_n(x)$ is $|(N,p_n)-B|$- summable if and only if

$$\sum = \sum_{k=1}^{\infty}\left|T_k(n) - T_{k+1}(n)\right|$$

$$= \sum_{k=1}^{\infty}\left|\sum_{\nu=1}^{k}\frac{\left(p_{k+1}P_{k-\nu} - p_{k-\nu+1}P_k\right)}{P_k P_{k+1}}\lambda_{n+\nu}B_{n+\nu}(x)\right| < \infty ,$$

uniformly in $n\in N$.

Now, $$\sum = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{P_k P_{k+1}}\left|\sum_{\nu=1}^{k}\int_0^{\pi}\left(p_{k+1}P_{k-\nu} - p_{k-\nu+1}P_k\right)\lambda_{n+\nu}\sin(n+\nu)t\ \psi(t)\, dt\right|$$

$$= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{P_k P_{k+1}}\left|\sum_{\nu=1}^{k}\int_0^{\pi} t\left(p_{k+1}P_{k-\nu} - p_{k-\nu+1}P_k\right)\lambda_{n+\nu}\ \sin(n+\nu)t\ \frac{\psi(t)}{t}\ dt\right|$$

Since, $\int_0^{\pi}\frac{\psi(t)}{t}dt < \infty$, in order to show that $\sum\ < \infty$ it is enough to show that

$$\sum_{k=1}^{\infty}\frac{1}{P_k P_{k+1}}\left|\sum_{\nu=1}^{k}\left(p_{k+1}P_{k-\nu} - p_{k-\nu+1}P_k\right)\lambda_{n+\nu}\sin(n+\nu)t\right| = O(\tau),$$

uniformly in $n \in N$ and for all't', $0 < t < \pi$.

Now, $$\left[\sum_{k=1}^{\tau} + \sum_{k>\tau}\ \right]\frac{1}{P_k P_{k+1}}\left|\sum_{\nu=1}^{k}\left(p_{k+1}P_{k-\nu} - p_{k-\nu+1}P_k\right)\lambda_{n+\nu}\sin(n+\nu)t\right|,$$

$$= \sum\nolimits_1 + \sum\nolimits_2 \quad \text{,say.}$$

We have

$$\sum\nolimits_1 = \sum_{k=1}^{\tau}\frac{1}{P_k P_{k+1}}\left|\sum_{\nu=1}^{k}\left(p_{k+1}P_{k-\nu} - p_{k-\nu+1}P_k\right)\lambda_{n+\nu}\sin(n+\nu)t\right|$$

$$\le O(1)\sum_{k=1}^{\tau}\frac{p_k p_{k+1}}{P_k P_{k+1}}\sum_{\nu=1}^{k}\nu\left|\lambda_{n+\nu}\ \sin(n+\nu)t\right|$$

$$= O(1)\sum_{k=1}^{\tau}\frac{k(k+1)}{2k(k+1)}$$

$$= O(\tau)$$

By, Abel's Partial summation formula , we have

$$\sum_{\nu=1}^{k}\left(p_{k+1}P_{k-\nu} - p_{k-\nu+1}P_k\right)\lambda_{n+\nu}\sin(n+\nu)t$$

$$=\left[\sum_{\nu=1}^{k-1}\Delta\left\{\left(p_{k+1}P_{k-\nu} - p_{k-\nu+1}P_k\right)\lambda_{n+\nu}\right\}\sum_{p=1}^{\nu}\sin(n+p)t - p_1 P_k\lambda_{n+k}\sum_{p=1}^{k}\sin(n+p)t\ \right]$$

$$=O(\tau)\left[\sum_{\nu=1}^{k-1}\left\{\left(p_{k-\nu}P_{k+1} - p_{k-\nu+1}P_k\right)\lambda_{n+\nu} + \left(p_{k+1}P_{k-\nu+1} - p_{k-\nu}P_k\right)\Delta\lambda_{n+\nu}\right\} - p_1 P_k\lambda_{n+k}\right]$$

$$= S_1 + S_2 + P\ ,say$$

Then $\sum_2 \le \sum_{k>\tau}\left|S_1 + S_2 + P\right| \ = \sum_{21}\ + \sum_{22}\ + \sum_{23}\ ,say$

Now, $$\sum\nolimits_{21} = \sum_{k>\tau}\frac{1}{P_k P_{k+1}}\left|O(\tau)\sum_{\nu=1}^{k-1}\left(p_{k-\nu}P_{k+1} - p_{k-\nu+1}P_k\right)\lambda_{n+\nu}\ \right|$$

$$= O(\tau)\sum_{k>\tau}\frac{p_k p_{k+1}}{P_k P_{k+1}}\sum_{\nu=1}^{k-1}\lambda_{n+\nu}$$

$$= O(\tau)\left[\sum_{\nu=1}^{\tau}\lambda_{n+\nu}\sum_{k=\tau}^{\infty}\frac{1}{k(k+1)} + \sum_{\nu=\tau}^{\infty}\lambda_{n+\nu}\sum_{k=\nu}^{\infty}\frac{1}{k(k+1)}\right]$$

$$= O(\tau)\left[\sum_{\nu=1}^{\tau}\frac{\lambda_{n+\nu}}{\tau} + \sum_{\nu=\tau}^{\infty}\frac{\lambda_{n+\nu}}{\nu}\right]$$

$$= O(\tau)\left[\sum_{\nu=1}^{\infty}\frac{\lambda_{n+\nu}}{\nu}\right]$$
$$= O(\tau).$$

Next, $$\sum\nolimits_{22} = \sum_{k>\tau}\frac{1}{P_k P_{k+1}}\left|O(\tau)\sum_{\nu=1}^{k-1}\left(p_{k+1}P_{k-\nu-1} - p_{k-\nu}P_k\right)\Delta\lambda_{n+\nu}\right|$$
$$= O(\tau)\sum_{k>\tau}\frac{p_k p_{k+1}}{P_k P_{k+1}}\sum_{\nu=1}^{k-1}(\nu+1)\Delta\lambda_{n+\nu}$$
$$= O(\tau)\left[\sum_{\nu=1}^{\tau}(\nu+1)\Delta\lambda_{n+\nu}\sum_{k=\tau}^{\infty}\frac{1}{k(k+1)} + \sum_{\nu=\tau}^{\infty}(\nu+1)\Delta\lambda_{n+\nu}\sum_{k=\nu}^{\infty}\frac{1}{k(k+1)}\right]$$
$$= O(\tau)\left[\sum_{\nu=1}^{\tau}\frac{(\nu+1)\Delta\lambda_{n+\nu}}{\tau} + \sum_{\nu=\tau}^{\infty}\frac{(\nu+1)\Delta\lambda_{n+\nu}}{\nu}\right]$$
$$= O(\tau)\left[\sum_{\nu=\tau}^{\infty}\Delta\lambda_{n+\nu} + \frac{1}{\tau}\left\{(\tau+1)\lambda_{n+1} - (\tau+1)\lambda_{n+\tau+1}\right\}\right]$$
$$= O(\tau).$$

Finally, $$\sum\nolimits_{23} = \sum_{k>\tau}\frac{1}{P_k P_{k+1}}\left|O(\tau)p_1 P_k\lambda_{n+k}\right|$$
$$= O(\tau)\sum_{k>\tau}\frac{p_{k+1}}{P_{k+1}}\lambda_{n+k}$$
$$= O(\tau)\sum_{k>\tau}\frac{\lambda_{n+k}}{k+1}$$
$$= O(\tau).$$

Thus, $\sum_{k=1}^{\infty}\left|T_k(n) - T_{k+1}(n)\right| < \infty$, Uniformly for $n \in N$.

Hence, $\sum \lambda_n B_n(x)$ is $\left|(N, p_n) - B\right|$- summable .

This completes the proof of theorem-3.2.

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