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ON NEW VERTEX NATURALLY LABELED GRAPHS

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Abstract

Vertices of the graphs are labeled from the set of natural numbers from one to the order of the given graph. Vertex Adjacency Label Set(VALS) is the set of ordered pair of vertices and its corresponding label of the graph. In this paper we introduced a notion of Vertex Adjacency Label Number(VALN). For each VALS, VLN of graph is the sum of labels of all the adjacent pairs of the vertices of the graph. $\aleph$ is the maximum number among all the VALNs of the different labeling of the graph and the corresponding VALS is defined as Max Vertex Adjacency Label Set $MVALS_{\aleph}$. In this paper $\aleph$ for special graphs are computed and also algorithm to find the $\aleph$ of any graph is described.

***Subject Classification**: 05C12, 05C07*

Key Words: Graph labeling, Graceful labeling, Complement.

1. Introduction

The research in graph enumeration and graph labeling started way back in 1857 by Arthur Cayley. Graph enumeration is defined as counting number of different graphs of particular type, subgraphs, etc. with graph variants (the number of vertices and edges of the graph). Labeling of graph is assigning labels to the vertices or edges of a graph. Most graph labelings trace their origins to labelings presented by Alex Rosa in his 1967 paper. Some of graph labeling methods are graceful labeling, harmonious labeling, and coloring of graphs. Graph labeling and enumeration finds the application in chemical graph theory, social networking and computer networking and channel assignment problem. For example, Cayley demonstrated that the number of different trees of n vertices is analogues to number of isomers of the saturated hydrocarbon with n carbon items C_nH_{2n+2} [5]. More such applications can be found in [2,3,4].In this paper vertex labeling of graphs are studied. Let $G(p,q)$ be a graph with p vertices and q edges and V, E are set of vertices and edges of G respectively.

Vertex Natural Labeling of a graph: A vertex natural labeling of G is a mapping function l which assigns each vertex v of G, an unique number $l(u)$ from the set of natural numbers $N = \{1,2,3,......,p\}$. That is all the vertices are having distinct labels from 1 to *p*. Thus *l* is bijective. So there are $p!$ sets of different labeling of a given graph. Each such label set is called *Vertex Adjacency Label Set(VALS). Vertex Adjacency Labeling Number(VALN)* for each *VALS* is defined as the sum of labels of all the adjacent pairs of the vertices of the graph, which is given by $\sum_{for\,each\,edge(u,v \in E(G))} l(u) + l(v)$. Let the set $\{VALS_1, VALS_2, ..., VALS_{p!}\}$ be all the *VALSs* of the given graph $G(p,q)$ and the set $\{VALN_1, VALN_2, ..., VALN_{p!}\}$ be set of corresponding VALNs. Then $\aleph(G) = Max(VALN_1, VALN_2, ..., VALN_{p!})$ is the Maximum Vertex Label Number of a given graph $G(p,q)$ and the corresponding *VALS* is called *Maximal Vertex Adjacency Label Set(MAVLS).*

This paper is organized into several sections. In section 2, important results and observations which are useful for deriving the new results are described. In section 3, we proved Main results and some corollaries; recurrence relations and bounds for $\aleph(G)$ are derived in sections 4, 5 and 6 respectively. In section 7, algorithm to compute $\aleph(G)$ is described. The $\aleph(G)$ for special graphs are derived in sections 8. Conclusions are given in section 9.

1. Propositions

1. For complete graph, K_p , any order vertex natural label of graphs gives $\aleph$. That is all the *p!* Vertex labeling of graphs yields same $\aleph$ value.
2. For a $G(p,r)$ is *r*-regular graph, any order vertex natural label of graphs gives $\aleph$.
3. For any graph G, by assigning the labels from the label set $\{p, p-1, ...1\}$ to the VALS, $\{v_{i_1}, v_{i_2}, ...v_{i_p}\}$ respectively where $d(v_{i_1}) \geq d(v_{i_2}) \geq .... >= d(v_{i_p})$ is one of the MVALS and corresponding MVALN is $\aleph(G)$.

3. Theorems

Theorem1. *F*or any complete graph K_p with *p* vertices, $\aleph(K_p) = \frac{p(p^2-1)}{2}$.

Proof: Since K_p is a complete graph with *p* vertices, every vertex of K_p is adjacent to all other vertices and from the proposition 1, any order labeling of the vertices of graph gives the *Maximum Vertex Adjacency Label Number(MVALN)* $\aleph$. Let vertices of K_p be $v_1, v_2.....v_p$ are assigned the labels $1,2,3,...p$ respectively.

The $\aleph(K_p)$ is given by $\sum_{for\,each\,(u,v) \in E(G)} l(u) + l(v)$ ***(1)***

The contribution of any vertex v_i to the $\aleph(K_p)$ summation is given by $(p-1)i$. ***(2)***

Thus $\aleph(K_p) = \sum_{(u,v) \in E(G)} l(u) + l(v) = (p-1)\sum_{i=1}^{p-1} i$. ***(3)***

$$\therefore \aleph(K_p) = \frac{p(p^2-1)}{2}. \quad \textbf{(4)}$$

□

Theorem2. If $G = K_{m,n}$ is a complete bipartite graph with $m \le n$, then $\aleph(K_{m,n}) = \frac{1}{2}[m(m+n)(m+n+1)-(m-n)m(m+1)]$.

Proof: Let the vertex set V of $K_{m,n}$ is partitioned into two sets X and Y with $|X| = m$ and $|Y| = n$. Then by proposition 3, there exist an *MVALS* whose associating labeling l satisfies $l(u) \in \{1,2,3,....,m\}$ for every $u \in X$ and $l(v) \in \{1,2,3,....,n\}$ for every $v \in Y$. Note that the contributions of $u \in X$ and $v \in Y$ to $\aleph(K_{m,n})$ are $l(u)m$ and $l(v)n$ respectively. Then it follows that $\aleph(K_{m,n}) = \frac{1}{2}[m(m+n)(m+n+1)-(m-n)m(m+1)]$. □

Theorem 3. If $G(p,r)$ is a r-regular graph , then

a. $\aleph(G(p,r)) = r\frac{p(p+1)}{2}$

b. $\aleph(\overline{G(p,r)}) = (p-(1+r))\frac{p(p+1)}{2}$

c. $\aleph(G(p,r)) + \aleph(\overline{G(p,r)}) = \aleph(K_p)$.

Proof:

a) Case $\aleph(G(p,r)) = r\frac{p(p+1)}{2}$:Since $G(p,r)$ is a r-regular graph with p vertices, every vertex of $G(p,r)$ is adjacent to r vertices and from the proposition 2, any order labeling of the vertices of graph gives the *MVALN*, $\aleph$. Let vertices of $G(p,r), v_1, v_2, ...v_p$ are assigned the labels $1,2,3,...p$ respectively.

Then the contribution of vertex v_i to the $\aleph(G(p,r))$ is given by ri. ***(5)***

Thus $\aleph(G(p,r)) = \sum_{(u,v)\in E(G)} l(u)+l(v) = \sum_{i=1}^{p} ri$. ***(6)***

$\therefore \aleph(G(p,r)) = r\frac{p(p+1)}{2}$. ***(7)***

b) Case $\aleph(\overline{G(p,r)}) = (p-(1+r))\frac{p(p+1)}{2}$: As $G(p,r)$ is a r-regular graph, then $\overline{G(p,r)}$ is also a $p-1-r$ regular graph. Substituting $p-1-r$ for r in equation *(7)* $\aleph(\overline{G(p,r)}) = (p-(1+r))\frac{p(p+1)}{2}$.

(8)

c) Case $\aleph(G(p,r)) + \aleph(\overline{G(p,r)}) = \aleph(K_p)$: By adding (7),(8)

$\aleph(G(p,r)) + \aleph(\overline{G(p,r)}) = \aleph(K_p)$. □

Theorem 4. If $P_n(v_1, v_2, ...v_n)$ *is a path of length* n *,then* $\aleph(P_n) = n(n+1)-3$

Proof: Assign the first two natural numbers 1 and 2 to the end points of $P_n(v_1, v_2, ...v_n)$.

That is, $l(v_1)=1, l(v_n)=2$ and assign labels to the remaining vertices $\{v_2, v_3, \ldots, v_{n-1}\}$ from the set $\{3,4,\ldots,n\}$ in any order.

Then contribution of the end vertices v_1, v_n and any non end vertex v_i summation $\sum_{for\ each\ (u,v)\in E(G)} l(u)+l(v)$ is *1* ,2 and *2i* respectively. ***(9)***

$$\therefore \aleph(P_n) = 2\sum_{i=1}^{n} i - 1 - 2 . \qquad \textbf{(10)}$$

$$\therefore \aleph(P_n) = n(n+1) - 3 . \qquad \textbf{(11)}$$

□

Theorem 5. If $C_n(v_1, v_2, \ldots v_n, v_1)$ *is a path of length* n *,then* $\aleph(C_n) = n(n+1)$.

Proof: Assign labels to the vertices of the set $\{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ from the set $\{1,2,3,4,\ldots,n\}$ in any order. Then contribution of any vertex v_i to the summation $\sum_{for\ each\ (u,v)\in E(G)} l(u)+l(v)$ is $2l(v_i)$. Thus by adding contribution of all the vertices to the summation, $\aleph(C_n)$ is obtained.

$$\therefore \aleph(C_n) = 2\sum_{i=1}^{n} l(v_i) = 2\sum_{i=1}^{n} i . \qquad \textbf{(12)}$$

$$\therefore \aleph(C_n) = n(n+1). \qquad \textbf{(13)}$$

□

Theorem 6. If $T(p,h)$ *is a complete binary tree of* p *points with height* h***,*** *where* $p = 2^{h+1} - 1$***, then*** $\aleph(T) = \frac{5p^2 - 9}{4}$.

Proof: Complete tree of height h has $p = 2^{h+1} - 1$ vertices and among these vertices, 2^h vertices are leaves. So the vertices of the tree are assigned labels from the set $\{1,2,3\ldots 2^h, 2^h+1, \ldots 2^h + 2^{h-1}, \ldots\ldots, 2^{h+1}-1\}$. Assign the labels to the leaves from the set $\{1,2,3,\ldots 2^h\}$. Then assign the label 2^h+1 to the root node. Then all other non leaf and root nodes are from the set $\{2^h+2, \ldots, 2^{h+1}\}$.

The contribution of labels of any leaf vertex v_i ,root vertex and non leaf or root vertex u_j to the summation $\sum_{for\ each\ (u,v)\in E(G)} l(u)+l(v)$ is $i, 2(2^h+1)$ and $3(2^h+2+j)$ respectively.

(14)

$$\therefore \aleph(T) = \sum_{i=1}^{2^h} i + 3\sum_{j=1}^{2^h-2} 2^h + j + 2(2^h+1). \qquad \textbf{(15)}$$

$$= \frac{5(2^{h+1}-1)^2 - 9}{4}. \qquad \textbf{(16)}$$

Since $p = 2^{h+1} - 1$, substituting this in *16, we get*

$$\aleph(T) = \frac{5p^2 - 9}{4}. \qquad \textbf{(17)}$$

□

Theorem 7. Let $G(p,q)$ *is a graph, whose vertices* have degree either *m* or *n* with $m >= n$ .Let p_m *vertices have degree m and* p_n vertices have degree *n.* Then $\aleph(G(p,q)) = m\binom{p+1}{2} - (n-m)\binom{p_n+1}{2}$.

Proof: Without Loss of Generality, let the vertices of G $v_1, v_2, ... v_{p_n}, v_{p_n+1} ..., v_p$ are labeled with $1,2,3, ... p_n, p_{n+1}, ..., p$ respectively. $\Rightarrow$ vertices $v_1, v_2, ... v_{p_n}$ are having degree n and $v_{p_n+1} ..., v_{p=p_n+p_m}$ are having degree *m.* Then Contribution of labels of the vertices $v_1, v_2, ... v_{p_n}$ to the $\aleph(G(p,q))$ is $1n + 2n + ... p_n n = \sum_{i=1}^{p_n} in$ .Similarly the contribution of labels of the vertices $v_{p_n+1} ..., v_{p=p_n+p_m}$ to the $\aleph(G(p,q))$ is $(p_n+1)m + (p_n+2)m + ... (p_n + p_m)m = \sum_{i=p_n+1}^{p} (i)m$. By adding these two, $\aleph(G(p,q)) = \sum_{i=1}^{p_n} in + \sum_{j=p_n+1}^{p} (j)m$.

(18)

$$= n\binom{p+1}{2} + (m-n)\binom{p+1}{2} - (m-n)\binom{p_n+1}{2}. \qquad \textbf{(19)}$$

$$\therefore \aleph(G(p,q)) = m\binom{p+1}{2} - (n-m)\binom{p_n+1}{2}. \qquad \textbf{(20)} \square$$

Note1 : *By substituting* $m = p-1, n = 0, p_m = p, p_n = 0$ in equation (20), theorem (1) can be proved.

Note 2: *By substituting* $m = r, n = 0, p_m = p, p_n = 0$ in equation (20), theorem (2) can be proved.

Theorem 8. *If* $\zeta(G)$ is the number of *MAVLS* natural labeling in a graph G and let the vertex set is partitioned into *d* disjoint subsets $V_1, V_2, ... V_d$,where the vertices of graph has *d* distinct degrees and all the vertices of V_i has same degree d_i and $|V_i| = p_i$. Then $\zeta(G) = \prod_{i=1}^{d} \gamma(p_i + 1)$.

Proof: Arrange the degrees of the vertices of G in descending order. Without Loss of Generality, let the degree sequence be $d_1, d_2, ... d_d$ and corresponding vertex sets be $V_1, V_2, ..., V_d$ respectively. This implies every vertex of set V_k has degree d_k and $\bigcup_{i=1}^{d} V_i = V, \bigcap_{i=1}^{d} V_i = \Phi$. Assign labels to the vertices of the sets $V_1, V_2, ..., V_d$ from the sets $\{\{p, p-1, ... p-p_1\}, \{p-p_1-1, ..., p-p_1-p_2\}, ..., \{p-p_1-p_2-...-p_{d-1}-1, ..., 1\}\}$ respectively. From the proposition 3, this kind of graph labeling generates $\aleph(G)$. Note that, within the partition sets $V_1, V_2, ..., V_d$, by permuting labels assigned to vertices of the same set can also generate $\aleph(G)$. Thus there are $p_i!$ or $\gamma(p_i + 1)$ different ways of assigning labels to the vertices of a set V_i .

Thus $\zeta(G)=\gamma(p_1+1)\gamma(p_2+1).....\gamma(p_d+1)$. ***(21)***

$$\zeta(G)=\prod_{i=1}^{d}\gamma(p_i+1). \qquad \textbf{(22)}$$

□

4. Corollary

1. $\zeta(Kp)=\gamma(p+1)$.

Proof: All the vertices in K_p has degree $p-1$. By theorem 6 and substituting $d=1$, $V_1=V$, $p_1=|V_1|=p$ in equation 20, $\zeta(K_p)=\gamma(p+1)$.

2. If $G(p,q)$ is a *r*-regular graph, then $\zeta(G)=\gamma(p+1)$.

Proof: All the vertices in $G(p,q)$ has degree r By theorem 6 and substituting $d=1$, $V_1=V$, $p_1=|V_1|=p$ in equation 20, $\zeta(K_p)=\gamma(p+1)$.

3. $\zeta(C_p)=\gamma(p+1)$.

Proof: By immediate consequence of corollary *2*.

4. $\zeta(P_n(v_1,v_2,.....v_n))=2\gamma(n-1)$.

Proof: The vertex set is partitioned into two sets with $V_1=\{v_2,.....,v_{n-1}\}, V_2=\{v_1,v_n\}. \Rightarrow p_1=|V_1|=n-2, p_2=|V_2|=2$ and $d=2$. By theorem 6 and substituting these values in equation *22*, $\zeta(P_n(v_1,v_2,.....v_n))=\gamma(n-2+1)\gamma(2+1)=\gamma(n-1)\gamma(3)$.
$\therefore \zeta(P_n(v_1,v_2,.....v_n))=2\gamma(n-1)$.

5. $\zeta(K_{m,n})=\begin{cases}\gamma(2n+1) & \text{for } m=n\\ \beta(m+1,n+1)\gamma(m+n+2) & \text{for } m\neq n\end{cases}$

Proof: Case 1 : $m=n$, Here, all the vertices have same degree n. By theorem 6 and substituting $d=1$, $V_1=V$, $p_1=|V_1|=2n$ in equation *22*, $\zeta(K_{n,n})=\gamma(2n+1)$.

Case 2: $m\neq n$, the vertex set is portioned into two sets with V_1,V_2 such that $d_1=m, d_2=n$, Without Loss of Generality, let the vertices of sets V_1 and V_2 has degree m and n respectively. Then $p_1=|V_1|=n, p_2=|V_2|=m$ and $d=2$. By theorem 6 and substituting these values in equation *22*, $\zeta(K_{m,n})=\gamma(m+1)\gamma(n+1)$, by using $\beta(x,y)\gamma(x+y)=\gamma(x)\gamma(y), \zeta(K_{m,n})=\beta(m+1,n+1)\gamma(m+n+2)$.

6. $\zeta(T(p,h))=\beta\left(\frac{(p+3)}{2},\frac{(p-3)}{2}\right)\gamma(p)$.

Proof: The vertex set V of T is partitioned into 3 sets $V_1=\{v \mid d(v)=3, v\ is\ not\ root\ or\ leaf\ node\}, V_2=\{v \mid d(v)=2, v\ is\ root\ node\}$ and $V_3=\{v \mid d(v)=1, v\ is\ leaf\ node\}$. Cardinality of these sets p_1,p_2,p_3 respectively needs to be determined. Total number of nodes, leaf nodes in a complete tree T with height h is given by $p=2^{h+1}-1$ and $p_3=2^h$ respectively and $p_2=1, \because only\ one\ root\ node\ exists\ in\ T$.

Hence, $p_1 = 2^h - 2$. By theorem *6* and substituting these values along with $d = 3$ in equation *(22)*, $\zeta(T(p,h)) = \gamma(2^h - 1)\gamma(2)\gamma(2^h + 1)$. Expressing h in terms of p, $\zeta(T(p,h)) = \gamma\left(\frac{p-1}{2}\right)\gamma\left(\frac{p+3}{2}\right)$.

$$\therefore \zeta(T(p,h)) = \beta\left((p-1)/2, (p+3)/2\right)\gamma(p+1).$$

7. *Let* $G(p,q)$ *is a graph, whose vertices* have degree either *m* or *n*. Let p_m *vertices have degree m and* p_n vertices have degree *n*. Then

$$\zeta(G(p,q)) = \begin{cases} \gamma(p+1) & \text{for } m = n \\ \beta(p_m + 1, p_n + 1)\gamma(p_m + p_n + 2) & \text{for } m \neq n \end{cases}$$

Proof: Let V_1, V_2 be the set of vertices having degree *m* and *n* respectively. By substituting $p_1 = |V_1| = p_m$, $p_2 = |V_2| = p_n$ in equation *22*, proof is obtained.

5. Recurrence Relation for $\aleph$

1. $\aleph(K_{p+1}) = \aleph(K_p)\frac{p+2}{p-1}$.
2. $\aleph(P_{n+1}) = \frac{(n+2)\aleph(P_n) + 3}{n}$, *for* $n \succ 1$.
3. $\aleph(C_{n+1}) = \left(1 + \frac{2}{n}\right)\aleph(C_{n+1})$.
4. $\aleph(G(p, r+1)) = (1 + 1/r)\aleph(G(p,r))$.

Recurrence relation for complete bi-graph

a. $\aleph(K_{n,n+1}) = \aleph(K_{n+1,n}) = \aleph(K_{n,n}) + \frac{n(7n+5)}{2}$.

b. $\aleph(K_{n+1,n+1}) = \aleph(K_{n},_{n}) + 6n^2 + 8n + 3$.

c. $\aleph(K_{m+1,n}) = \aleph(K_{m,n}) + (m+n+1)\frac{3m+n+2}{2} - (m+1)\frac{3m-2n+2}{2}$, if $m+1 \geq n$.

d. $\aleph(K_{m,n+1}) = \aleph(K_{m,n}) + (m+n+1)\frac{3n+m+2}{2} - (n+1)\frac{3n-2m+2}{2}$,if $n+1 \geq m$.

6. Bounds for $\aleph(G)$

1. For any graph $G(p,q)$, $0 \leq \aleph(G) \leq \frac{p(p^2-1)}{2}$.
2. For any connected graph $G(p,q)$, $\frac{p(p+1)}{2} \leq \aleph(G) \leq \frac{p(p^2-1)}{2}$.
3. For any graph $G(p,q)$, $\aleph(G) \leq \aleph(K_p)$and
4. For any graph $G(p,q)$, $\aleph(L(G)) \leq (p-1)\frac{\binom{p}{2}\left(\binom{p}{2}+1\right)}{2}$.

7. Algorithm to compute $\aleph(G)$

Input: Adjacency Matrix $A(G)$, order of graph *G, p*.

1. Compute the degree of the each vertex of *G*.
2. Arrange the vertices of *G* by the decreasing order of their degrees.
3. Let $v_{i_1}, v_{i_2}, \ldots\ldots\ldots, v_{i_p}$ be the arrangement of the vertices and corresponding degree sequence be $d_{i_1}, d_{i_2}, \ldots\ldots\ldots, d_{i_p}$
4. $\aleph(G) = 0$
5. *For* $k = p$ to *1*
6. *Begin*
7. $\aleph(G) = \aleph(G) + k * d_{i_{p-k+1}}$
8. $k = k - 1$
9. *End*
10. *Print* $\aleph(G)$

8. $\aleph(G)$ for special graphs

1. If W_p is a wheel graph with *p* vertices, then $\aleph(W_p) = 5\binom{p}{2}, \zeta(W_p) = \gamma(p)$.
2. If $G = \overline{L(K_5)}$ is a Peterson graph with 10 vertices, then $\aleph(G) = 165$, $\zeta(G) = \gamma(11)$.
3. If $G = L(K_p)$ is a line graph of K_p, then $\aleph(L(K_p)) = (p-1)\dfrac{\binom{p}{2}\left(\binom{p}{2}+1\right)}{2}$, $\zeta(L(K_p)) = \gamma(p+1)$.
4. $\aleph(kuratowaski's\ graph1) = 60$, $\zeta(kuratowaski's\ graph1) = \gamma(6)$. $\aleph(kuratowaski's\ graph2) = 144$, $\zeta(kuratowaski's\ graph2) = \gamma(13)$.

9. Conclusion

In this paper, a novel vertex and edge natural labeling of a graph is introduced. Since labeling of graphs find applications in resource allocation, channel allocation in computer networks and communication. Exploring the application of this new labeling of graphs is under way. A useful algorithm to find $\aleph$ of any graph is described along with some properties and relations of graphs in terms of $\aleph$. As part of continue to this work, edge labeling of graphs and labelings on graph operations are studied in a separate paper.

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