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THRESHOLD THEOREMS OF TWO -SPECIES COMPETITIVE ECOLOGICAL MODEL WITH RESERVE FOR ONE SPECIES AND HARVESTING BOTH SPECIES AT RATES PROPORTIONAL TO THEIR RESPECTIVE SIZES

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ABSTRACT

In the present investigation we study a two Species competition model incorporating

i) A constant number of S_1 provided with reserve . and
ii) both the S_1 and the S_2 are harvested proportional to their population sizes.

The model is characterized by a couple of first order non-linear ordinary differential equations. All the four equilibrium points of the model are identified and stability criteria are outlined.Some threshold theorems have been derived for Normal steady state and results are illustrated.

Key words :Model equations , Stability ,Interactions,Equilibrium states, Threshold theorems
AMS Classification : 92 D 25, 92 D 40

1.1 Basic Equations:

The model equation for the present two Species competing system is given by the following system of non-linear ordinary differential equations

$$\frac{dN_1}{dt} = a_1(1-k_1)N_1 - a_{11}N_1^2 - a_{12}(1-k)N_1N_2. \quad (1.1.1)$$

$$\frac{dN_2}{dt} = a_2(1-k_2)N_2 - a_{22}N_2^2 - a_{21}(1-k)N_1N_2. \quad (1.1.2)$$

K is constant cover for Species S_1
K_1 is rate of decrease of the S_1 due to harvesting.
K_2 is rate of decrease of the S_2 due to harvesting.

1.2 Equilibrium states:

The system under investigation has four equilibrium states. They are

1. The fully washed out state. $\overline{N}_1=0, \overline{N}_2=0$ (1.2.1)
2. $\overline{N_1} = 0$; $\overline{N_2} = \frac{a_2(1-k_2)}{a_{22}}$ (1.2.2)

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In this state, only the S_2 survives and the S_1 are washed out

3. $\overline{N_1} = \dfrac{a_1(1-k_1)}{a_{11}} : \quad \overline{N_2} = 0$ (1.2.3)

In this state, only the S_1 survives and the S_2 are washed out.

4. $\overline{N_1} = \dfrac{a_1(1-k_1)a_{22} - a_2(1-k_2)a_{12}(1-k)}{a_{11}a_{22} - a_{12}a_{21}(1-k)^2}, \ \overline{N_2} = \dfrac{a_2(1-k_2)a_{11} - a_1(1-k_1)a_{21}(1-k)}{a_{11}a_{22} - a_{12}a_{21}(1-k)^2}$ (1.2.4)

The state in which both the S_1 and the S_2 co-exists and this state is possible only

When $k > \min.\{1 - \dfrac{a_1(1-k_1)a_{11}}{a_2(1-k_2)a_{12}}, \ 1 - \dfrac{a_2(1-k_2)a_{11}}{a_2(1-k_1)a_{12}}, \ 1 - \sqrt{\dfrac{a_{11}a_{22}}{a_{12}a_{21}}}\}$ (1.2.5)

Such that $a_{11}a_{22} < a_{12}a_{21}$.

The state 4 is called the "the normal steady state"

1.3 THERSHOLD THEOREMS:

The basic equations are:

$$\frac{dN_1}{dt} = \frac{a_1(1-k_1)N_1}{k_3}\{k_3 - N_1 - \beta_1 N_2\} \ \& \ \frac{dN_2}{dt} = \frac{a_2(1-k_2)N_2}{k_4}\{k_4 - N_2 - \beta_2 N_1\} \quad (1.3.1)$$

Where $k_3 = \dfrac{a_1(1-k_1)}{a_{11}}; k_4 = \dfrac{a_2(1-k_2)}{a_{22}}; \beta_1 = \dfrac{a_{12}(1-k)}{a_1(1-k_1)}; \beta_2 = \dfrac{a_{21}(1-k)}{a_2(1-k_2)};$

Theorem 3: *Principle of Competitive Exclusion for the Normal Steady State:*

$$\overline{N_1} = \frac{a_1(1-k_1)a_{22} - a_2(1-k_2)a_{12}(1-k)}{a_{11}a_{22} - a_{12}a_{21}(1-k)^2}, \overline{N_2} = \frac{a_2(1-k_2)a_{11} - a_1(1-k_1)a_{21}(1-k)}{a_{11}a_{22} - a_{12}a_{21}(1-k)^2}$$

Suppose that $\dfrac{k_3}{\beta_1} > k_4$ and $\dfrac{k_4}{\beta_2} > k_3$. Then every solution of $N_1(t)$, $N_2(t)$ of (1.3.1) approaches the equilibrium solution $N_1(t) = \overline{N_1}\ (\neq 0)$ and $N_2(t) = \overline{N_2}\ (\neq 0)$ as t approaches infinity . In other words, if Species 1 and 2 are nearly identical and the microcosm can support both the members of Species 1 and 2 depending up on the initial conditions.

Proof: The first step in our proof is to show that $N_1(t)$ and $N_2(t)$ can never become negative. The this end observe that

$$N_1(t) = \overline{N_1} = \frac{a_1(1-k_1)a_{22} - a_2(1-k_2)a_{12}(1-k)}{a_{11}a_{22} - a_{12}a_{21}(1-k)^2} \quad \text{and}$$

$$N_2(t) = \overline{N_2} = \frac{a_2(1-k_2)a_{11} - a_1(1-k_1)a_{21}(1-k)}{a_{11}a_{22} - a_{12}a_{21}(1-k)^2}$$

Is a solution of (1.3.1) for any choice of $N_1(0)$. The orbit of this solution in the N_1 - N_2 plane is the point (0,0) for $N_1(0) = 0$; the line $0 < N_1 < k_3$, $N_2 = 0$ for $N_1(0) > k_3$ .Thus the N_1axis, for $N_1 \geq \infty$ is the union of four distinct orbits of (1.3.1) . Similarly the N_2 axis , for $N_2 \geq 0$, is the union of four distinct orbits of (1.3.1) . This implies that all solution $N_1(t)$, $N_2(t)$ of (1.3.1) which start in the first quadrant ($N_1(t) > 0$, $N_2 > 0$) of the N_1 - N_2 plane must remain there for all future time.

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The second step in our proof is to split the first quadrant into regions in which both $\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$ have fixed signs. This is accomplished in the following manner.

Let l_1 and l_2 be the lines $k_3 - N_1 - \beta_1 N_2 = 0$ and $k_4 - N_2 - \beta_2 N_1 = 0$ respectively and the point of their intersection, is $(\overline{N_1}, \overline{N_2})$. Observe that $\frac{dN_1}{dt}$ is negative if (N_1, N_2) lies above the line l_1 and positive if (N_1, N_2) lies below . Thus the two lines l_1 and l_2 split the first quadrant of the $N_1 - N_2$ plane into four regions in which both $\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$ have fixed signs.

$N_1(t)$, $N_2(t)$ both increase with time (along any solution of (1.3.1) in region I :
$N_1(t)$ increases and $N_2(t)$ decreases with time in region II:
$N_1(t)$ decreases and $N_2(t)$ increases with time in region III:
And both $N_1(t)$ and $N_2(t)$ decrease with time in region IV in this region both the S_1&S_2compete with each other but do not flourish and at the same tine do not get extinct. This is shown in Fig.1.1.

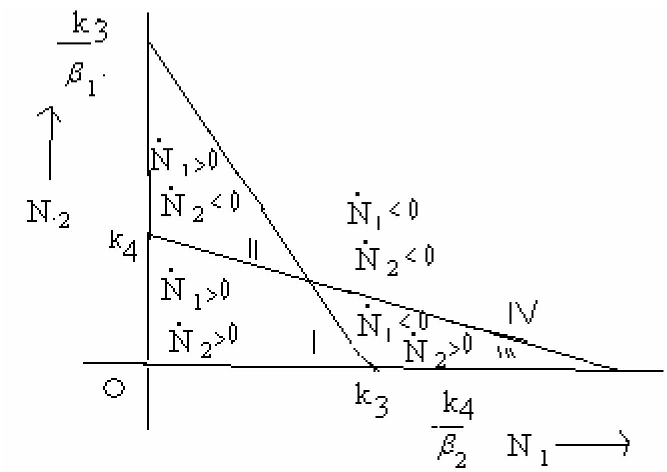

Fig 1.1

Finally we require the following four lemmas.

Lemma 1: Any solution of $N_1(t)$, $N_2(t)$ of (1.3.1) which starts in region I at time $t = t_0$ will remain in this region for all future time $t \geq t_0$,and ultimately approach the equilibrium solution $N_1(t) = \overline{N_1}$, $N_2(t) = \overline{N_2}$ (Fig.1.1)

Proof: Suppose that a solution $N_1(t)$, $N_2(t)$ of (1.3.1) leaves region I at time t = t* .Then either $\frac{dN_1}{dt}(t^*)$ or $\frac{dN_2}{dt}(t^*)$ is zero. Since the only way a solution of (1.3.1) can leave region *I* is by crossing l_1 or l_2 . Assume that $\frac{dN_1}{dt}(t^*) = 0$, Differentiation both sides of the first equation of (1.3.1) with respect to t and setting t = t* gives

$$\frac{d^2 N_1(t^*)}{dt} = \frac{-a_1 \beta_1 N_1(t^*)}{k_3} \frac{dN_2(t^*)}{dt} < 0 \tag{1.3.2}$$

Hence $N_1(t)$ is monotonic increasing and it has maximum whenever a solution of $N_1(t)$, $N_2(t)$ of (1.3.1) is in region I.

Similarly, if $\frac{dN_2}{dt}(t^*) = 0$, then

$$\frac{d^2N_2(t^*)}{dt} = \frac{-a_2\beta_2N_2(t^*)}{k_4}\frac{dN_1dt(t^*)}{dt} < 0 \quad (1.3.3)$$

Implies that $N_2(t)$ is monotonic increasing and it has maximum whenever a solution $N_1(t)$, $N_2(t)$ of (1.3.1) is in region I

If a solution $N_1(t)$, $N_2(t)$ of (1.3.1) remains in region I for $t \geq t_0$, then both $N_1(t)$ and $N_2(t)$ are monotonic increasing functions of time for $t \geq t_n$, with $N_1(t) > k_3$ and $N_2(t) < k4$, consequently, both $N_1(t)$ and $N_2(t)$ have limits ξ, η respectively. As t approaches infinity. This in turn implies that (ξ, η) is an equilibrium point of (1.3.1). Now, (ξ, η) obviously cannot equal (0,0): $(k_3,0)$ or $(0,k_4)$. Consequently $(\xi,\eta) = (\overline{N_1}, \overline{N_2})$.

Lemma2: *Any solution of $N_1(t)$, $N_2(t)$ of (1.3.1) which starts in region II at time t= t_0 will remain in this region for all future time $t \geq t_0$, and ultimately approach the equilibrium solution $N_1(t) = \overline{N_2}$, $N_2(t) = \overline{N_2}$ (Fig.1.1).*

Proof: Suppose that a solution $N_1(t)$, $N_2(t)$ *of (1.3.1)* leaves region II at time $t = t^*$. Then either $\frac{dN_1}{dt}(t^*)$ or $\frac{dN_2}{dt}(t^*)$ is zero. Since the only way a solution of *(1.3.1)* can leave region II is by crossing l_1 or l_2. Assume that $\frac{dN_1(t^*)}{dt} = 0$. Differentiation both sides of the first equation of *(9.5.1)* with respect to t and setting $t = t^*$ gives

$$\frac{d^2N_1(t^*)}{dt} = \frac{-a_1\beta_1N_1(t^*)}{k_3}\frac{dN_2(t^*)}{dt} \quad (1.3.4)$$

This quantity is positive. Hence $N_1(t)$ has a minimum at $t = t^*$. However, this is impossible, since $N_1(t)$ is increasing whenever a solution of $N_1(t)$, $N_2(t)$ *of (1.3.1)* is in region II.

Similarly, if $\frac{dN_2}{dt}(t^*) = 0$,

Then

$$\frac{d^2N_2(t^*)}{dt} = \frac{-a_2\beta_2N_2(t^*)}{k_4}\frac{dN_1(t^*)}{dt} \quad (1.3.5)$$

This quantity is negative, implying that $N_1(t)$ has a maximum at $t = t^*$, but this is impossible, since is decreasing whenever a solution $N_1(t)$, $N_2(t)$ *of (1.3.1)* is in region II.

The previous argument shows that any solution $N_1(t)$, $N_2(t)$ *of (1.3.1)* which starts in region II at time $t = t_0$ will remain in region II for all future time $t \geq t_0$. This implies that $N_1(t)$ is monotonic increasing and $N_2(t)$ is monotonic decreasing for $t \geq t_0$: with $N_1(t) > k_3$ and $N_2(t) < k_4$. Consequently, both $N_1(t)$ and $N_2(t)$ have limits ξ, η respectively, as approaches infinity. This in turn implies that (ξ, η) is an equilibrium point of *(1.3.1)*.

Now (ξ, η) obviously cannot equal (0, 0); $(k_3,0)$ or $(0,k_4)$. Consequently, $(\xi, \eta) = (\overline{N}_1, \overline{N}_2)$ and this proves Lemma 2

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Lemma 3: Any solution of $N_1(t)$, $N_2(t)$ of (1.3.1) which starts in region III at time $t= t_0$ will remain in this region for all future time $t \geq t_0$, and ultimately approach the equilibrium solution $N_1(t) = \overline{N}_1$, $N_2(t) = \overline{N}_2$, (Fig.1.1)

Proof: Suppose that a solution $N_1(t)$, $N_2(t)$ *of* (1.3.1) leaves region III at time $t = t^*$. Then either $\frac{dN_1}{dt}(t^*)$ or $\frac{dN_2}{dt}(t^*)$ is zero. Since the only way a solution of (1.3.1) can leave region II is by crossing l_1 or l_2. Assume that $\frac{dN_1}{dt}(t^*) = 0$, Differentiation both sides of the first equation of (1.3.1) with respect to t and setting $t = t^*$ gives

$$\frac{d^2N_1(t^*)}{dt} = \frac{-a_1\beta_1 N_1(t^*)}{k_3}\frac{dN_2(t^*)}{dt} \tag{1.3.6}$$

since $N_1(t)$ is decreasing whenever a solution of $N_1(t)$, $N_2(t)$ *of* (1.3.1) is in region II . This quantity is negative. Hence $N_1(t)$ has a maximum at $t = t^*$. However, this is impossible

Similarly, if $\frac{dN_2}{dt}(t^*) = 0$, then

$$\frac{d^2N_2(t^*)}{dt} = \frac{-a_2\beta_2 N_2(t^*)}{k_4}\frac{dN_1(t^*)}{dt} \tag{1.3.7}$$

This quantity is positive, implying that $N_2(t)$ has a minimum at $t = t^*$, but this is impossible, since $N_2(t)$ is increasing whenever a solution $N_1(t)$, $N_2(t)$ *of* (1.3.1) is in region III.

The previous argument shows that any solution $N_1(t)$, $N_2(t)$ *of* (1.3.1) which starts in region III at time $t= t_0$ will remain in region III for all future time $t \geq t_0$ .This implies that is monotonic increasing and $N_2(t)$ is monotonic decreasing for $t \geq t_0$; with . Consequently, both $N_1(t)$ *and* $N_2(t)$ have limits ξ, η . Now (ξ, η) obviously cannot equal (0,0): $(k_3,0)$ or $(0,k_4)$. Consequently,$(\xi, \eta) = (\overline{N}_1 , \overline{N}_2)$ and this proves Lemma3

Lemma 4: *Any solution of* $N_1(t)$, $N_2(t)$ of (9.5.1) which starts in region IV at time $t= t_0$ will remain in this region for all future time $t \geq t_0$, and ultimately approach the equilibrium solution $N_1(t) = \overline{N}_1$,$N_2=\overline{N}_2$ (Fig.1.1)

Proof: Suppose that a solution $N_1(t)$, $N_2(t)$ *of* (*1.3.1*) leaves region VI at tome $t = t^*$. Then either $\frac{dN_1}{dt}(t^*)$ or $\frac{dN_2}{dt}(t^*)$ is zero. Since the only way a solution of (*1.3.1*) can leave region IV is by crossing l_1 or l_2 . Assume that $\frac{dN_1}{dt}(t^*) =0$. Differentiation both sides of the first equation of (*1.3.1*) with respect to t and setting $t = t^*$ gives

$$\frac{d^2N_1(t^*)}{dt} = \frac{-a_1\beta_1 N_1(t^*)}{k_3}\frac{dN_2(t^*)}{dt} \tag{1.3.8}$$

This quantity is positive. Hence $N_1(t)$ is monotonic decreasing and it has minimum whenever a solution of $N_1(t)$, $N_2(t)$ *of* (*1.3.1*) is in region VI.

Similarly, if $\frac{dN_2}{dt}(t^*) = 0$, then

$$\frac{d^2N_2(t^*)}{dt} = \frac{-a_2\beta_2 N_2(t^*)}{k_4}\frac{dN_1(t^*)}{dt} \tag{1.3.9}$$

This quantity is positive .implying that $N_2(t)$ is

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monotonic decreasing and it has minimum whenever a solution $N_1(t)$, $N_2(t)$ *of* (*1.3.1*) is in region IV.

If a solution $N_1(t)$, $N_2(t)$ *of* (*1.3.1*) remain in region VI for $t \geq t_0$, then both $N_1(t)$ *and* $N_2(t)$ are monotonic decreasing functions of time for $t \geq t_0$, with $N_1(t) > k_3$ *and* $N_2(t) > k_4$. consequently , both $N_1(t)$ *and* $N_2(t)$ have limits ξ, η respectively ,as t approaches infinity . This, in turn implies that (ξ, η) is an equilibrium point of (*1.3.1*). Now, (ξ, η) obviously cannot equal (0,0): (k_3,0) or (0,k_4) .Consequently (ξ, η) = ($\overline{N_1}$, $\overline{N_2}$).

Proof of theorem: Lemmas 1,2,3 and 4 state that every solution $N_1(t)$, $N_2(t)$ *of* (*1.3.1*) which starts in region I,II , III or IV at time $t= t_0$ and remains there for all future time must also approach equilibrium solution $N_1(t) = \overline{N_2}$, $N_2(t) = \overline{N_2}$, as t approaches infinity. Next , observe that any solution $N_1(t)$, $N_2(t)$ of (1) which starts on l_1 or l_2 must immediately afterwards enter region I,II,III or IV. Finally the solution approaches the equilibrium solution $N_1(t) = \overline{N_2}$, $N_2(t) = \overline{N_2}$. This is illustrated in Fig.1.2

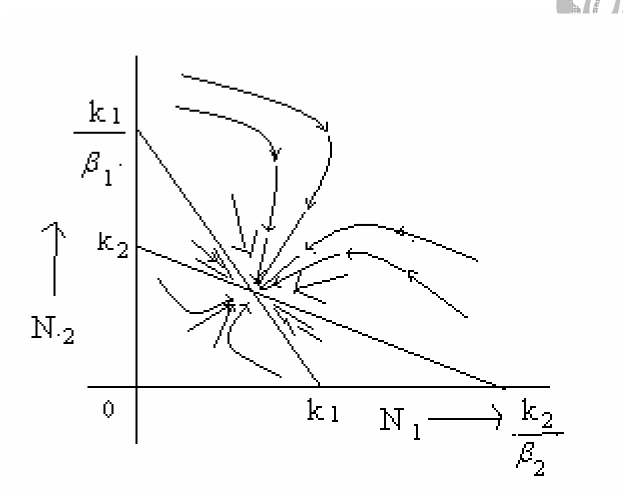

Fig 1.2

REFERENCES

1. Bhaskara Rama Sarma .B & N. Ch. Pattabhiramacharyulu: *'Stability Analysis of a two-species competitive eco-system-'*, **International Journal of Logic Based Intelligent Systems,** Volume-2, No.1, 2008,PP.79-86.

2 .Bhaskara Rama Sarma .B & N. Ch. Pattabhiramacharyulu:*'On two species competitive eco-system with decay and replenishment for one of the species'*, **Proceedings of National Seminar on Uniform Hypergraphs and their applications** held during January 30th to Feb.5th at Eluru,A.P,pp1-16.

3. Bhaskara Rama Sarma .B & N. Ch. Pattabhiramacharyulu: **'***Local stability analysis of two-species competitive eco-system with deathrate and emigration for one of the species'*, **International Journal of Applied Mathematical Analysis andApplications'**, NewDelhi, Vol.3 No.2,2008,Pp153-159.

4. Bhaskara Rama Sarma .B & N. Ch. Pattabhiramacharyulu: **'***On Global stability analysis of two-species competing eco-system with decay and replenishment for one of the species*', **International Journal of Scientific Computing'**, NewDelhi, Vol.2 No.2,2008,Pp177- 181

5. Bhaskara Rama Sarma .B & N. Ch. Pattabhiramacharyulu: *'A Two Species Competing Ecological Model with Reserve and Unlimited Resources for one of the Species',* Accepted for publication in **International review of Pure and Applied Mathematics Journal,**NewDelhi
[6].Bhaskara Rama Sarma .B & N. Ch.Pattabhiramacharyulu: *'A Recursive Procedure for two-species competing eco-system with decay and replenishment for one species' ,* Published in **Acta Ciencia Indica**,Meerut,Vol.35,2009
[7] Lakshmi Narayan. K: "A Mathematical study of a prey- predator ecological model with a partial cover for the prey and alternate food for the predator",Ph.D Thesis, J.N.T.U(2005)
[8] Srinivas .N.C :"Some Mathematical Aspects of Modelling in Bio-Medical Sciences"- Ph .D Thesis Submitted to Kakatiya University (1991)
[7] Lotka . A.J : Elements of Physical Biology , Williams & Wilkins, Baltimore (1925)
[8] Meyer. W.J : Concepts of Mathematical modeling, Mc. Grawhill, 1985
[9] Paul Colinvaux . A : Ecology, John Wiley, New York, 1986
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