INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES, TECHNOLOGY AND HUMANITIES

www.internationalejournals.com

Available online at www.internationaleJournals.com

International eJournals

International Journal of Mathematical Sciences, Technology and Humanities 6 (2011) 61 – 68

Author's personal copy

ON $(\overline{N}, p_n)(E, q)$ PRODUCT SUMMABILITY OF FOURIER SERIES

[1]Mahendra Misra, [2]U.K.Misra, [3]B.P.Padhy and [4]M.K.Muduli

[1]Government Science College
Malkangiri, Orissa
e-mail:mahendramisra@2007.gmail.com

[2]Department of Mathematics
Berhampur University
Berhampur – 760007,Orissa, India
e-mail: umakanta_misra@yahoo.com

[3]Roland Institute of Technology
Golanthara-760008, Odisha, India.
e-mail:iraady@gmail.com

[4]Roland Institute of Technology
Golanthara-760008, Odisha, India.
malayamudulimath@gmail.com

Abstract: In this paper, a theorem on $(\overline{N}, p_n)(E, q)$ product summability of Fourier series is proved.

Keywords: $(\overline{N}, p_n)$ mean**,** $(\overline{N}, p_n)(E, q)$ product mean and Fourier series**.**

Mathematics subjection classification: 42B05, 42B08.

1. Introduction:

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$**.**Let $p = \{p_n\}$ be a sequence of non-negative, non-increasing real constants such that

(1.1) $$P_n = \sum_{\upsilon=0}^{n} p_\upsilon \to \infty \text{ as } n \to \infty \ , \quad P_{-i} = p_{-i} = 0 \quad \forall \ i \geq 1$$

The sequence –to-sequence transformation

(1.2)
$$t_n = \frac{1}{P_n}\sum_{\upsilon=0}^{n} p_\upsilon \, s_\upsilon$$

defines the sequence $\{t_n\}$ of the $(\overline{N}, p_n)$ - mean of the sequence $\{s_n\}$,[**1**].

Clearly, $(\overline{N}, p_n)$ is regular if

(1.3)
$$\begin{cases} (i)\ P_n \to \infty, as\ n \to \infty. \\ (ii) \sum_{i=0}^{n} p_i \le C|P_n|, as\ n \to \infty. \end{cases}$$

If $t_n \to s$, as $n \to \infty$, then the series $\sum a_n$ is said to be $(\overline{N}, p_n)$ summable to s**.**

If

(1.4)
$$(E,q) = E_n^q = \frac{1}{(1+q)^n}\sum_{\upsilon=0}^{n}\binom{n}{\nu} q^{n-\upsilon} s_\upsilon \to s \quad , \text{as} \quad n \to \infty.$$

Then the series $\sum a_n$ is said to be summable (E,q) to a definite number s.

The product of $(\overline{N}, p_n)$ summability with (E,q) summability defines $(\overline{N}, p_n)(E,q)$ summability and we denote it by $\overline{N}_p E_n^q$ then if

(1.5)
$$\overline{N}_p E_n^q = \frac{1}{P_n}\sum_{k=0}^{n}\frac{p_k}{(1+q)^k}\sum_{\upsilon=0}^{k}\binom{k}{\nu} q^{k-\upsilon} s_\upsilon \to s \quad \text{,as} \quad n \to \infty.$$

Then the series $\sum a_n$ is said to be summable to s by the $(\overline{N}, p_n)(E,q)$ method.

It is known that (E,q) is regular. It is supposed through out this paper that the method $(\overline{N}, p_n)(E,q)$ is regular**.**

Let $f(t)$ be a periodic function with period 2π, integrable in the sense of Lebesgue over $(-\pi,\pi)$, then

(1.6)
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t)$$

is the Fourier series associated with f.

We use the following notation through out this paper

Author's personal copy

Author's personal copy

$$\phi(t) = f(x+t) + f(x-t) - 2f(x),$$

$$K_n(t) = \frac{1}{2\pi P_n} \sum_{k=0}^{n} \frac{p_k}{(1+q)^k} \sum_{\nu=0}^{k} \binom{k}{\nu} q^{k-\upsilon} \frac{\sin\left(\upsilon + \frac{1}{2}\right)t}{\sin\frac{t}{2}}.$$

2. Known Theorem:

Dealing with $(N, p_n)(E, q)$ method of a Fourier series, Nigam,et.al[**2**] proved the following theorem:

Theorem-2.1:

Let $\{p_n\}$ be a positive, monotonic, non-increasing sequence of real constants such that

$$P_n = \sum_{\upsilon=0}^{n} p_\upsilon \to \infty \quad \text{, as} \quad n \to \infty.$$

If

(2.1.1) $$\Phi(t) = \int_0^t |\phi(u)|\, du = O\left\{\frac{t}{\alpha\left(\frac{1}{t}\right)}\right\} \text{, as} \quad t \to +0$$

and

(2.1.2) $$\alpha(n) \to \infty \quad \text{as} \quad n \to \infty,$$

where $\alpha(t)$ be a positive ,non-increasing function of t , then the Fourier series $\sum_{n=0}^{\infty} A_n(t)$ is summable $(N, p_n)(E, q)$ to $f(x)$ at the point $t = x$.

In this paper , we have generalized it to $(\overline{N}, p_n)(E, q)$ summability of Fourier series .

3. Main theorem:

Theorem -3.1:

Let $\{p_n\}$ a positive, monotonic, non-increasing sequence of real constants satisfying (1.1) and

(3.1.1) $$\Phi(t) = \int_0^t |\phi(u)|\, du = O\left\{\frac{t}{\alpha\left(\frac{1}{t}\right)}\right\} \text{, as} \quad t \to +0$$

and

(3.1.2) $$\alpha(n) \to \infty \quad \text{as} \quad n \to \infty$$

where $\alpha(t)$ is positive ,non-increasing function of t , then the Fourier series $\sum_{n=0}^{\infty} A_n(t)$ is summable $(\overline{N}, p_n)(E,q)$ at the point t .

4. Required Lemmas:

We require the following Lemmas to prove the theorem

Lemma -4.1:

$$|K_n(t)| = O(n) \quad ,0 \le t \le \frac{1}{n+1}.$$

Proof:

For $0 \le t \le \frac{1}{n+1}$, we have $\sin nt \le n \sin t$ .Then

$$|K_n(t)| = \frac{1}{2\pi P_n}\left|\sum_{k=0}^{n}\frac{p_k}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu} q^{k-\upsilon} \quad \frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|$$

$$\le \frac{1}{2\pi P_n}\left|\sum_{k=0}^{n}\frac{p_k}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu} q^{k-\upsilon} \quad \frac{(2\nu+1)\sin\frac{t}{2}}{\sin\frac{t}{2}}\right|$$

$$\le \frac{1}{2\pi P_n}\left|\sum_{k=0}^{n}\frac{p_k(2k+1)}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu} q^{k-\upsilon}\right|$$

$$= \frac{1}{2\pi P_n}\left|\sum_{k=0}^{n}\frac{p_k(2k+1)}{(1+q)^k}(1+q)^k\right| , as \sum_{\nu=0}^{k}\binom{k}{\nu} q^{k-\upsilon} = (1+q)^k$$

$$= \frac{(2n+1)}{2\pi P_n}\left|\sum_{k=0}^{n} p_k\right|.$$

$$= O(n)$$

Lemma-4.2:

$$|K_n(t)| = O\left(\frac{1}{t}\right) \quad , for \ \frac{1}{n} \le t \le \pi .$$

Proof:

Author's Personal copy

For $\frac{1}{n} \le t \le \pi$, we have by Jorden's lemma, $\sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}$, $\sin nt \le 1$.

Then

$$\left| K_n(t) \right| \le \frac{1}{2\pi P_n} \left| \sum_{k=0}^{n} \frac{p_k}{(1+q)^k} \sum_{\upsilon=0}^{k} \binom{k}{\nu} q^{k-\upsilon} \frac{\pi}{t} \right|$$

$$= \frac{1}{2t P_n} \sum_{k=0}^{n} \frac{p_k}{(1+q)^k} (1+q)^k$$

$$= \frac{1}{2t P_n} \sum_{k=0}^{n} p_k$$

$$= O\left(\frac{1}{t}\right).$$

5. Proof of the theorem- 3.1:

If $s_n(f;x)$ is the n-th partial sum of the Fourier series $\sum_{n=0}^{\infty} A_n(t)$ of $f(t)$, then by using Riemann-Lebesgue theorem , following Titechmarch [**3**] we have

$$s_n(f;x) - f(x) = \frac{1}{2\pi} \int_0^{\pi} \phi(t) \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} dt$$

Thus , the (E,q) transform E_n^q of $s_n(f;x)$ is given by

$$E_n^q - f(x) = \frac{1}{2\pi(1+q)^n} \int_0^{\pi} \frac{\phi(t)}{\sin\left(\frac{t}{2}\right)} \left\{ \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin\left(k+\frac{1}{2}\right)t \right\} dt.$$

If $\overline{N}_p E_n^q$ denote the $(\overline{N}, p_n)(E,q)$ transform of $s_n(f;x)$, we then prove

Author's personal copy

$$\overline{N}_p E_n^q - f(x) = \frac{1}{2\pi\ P_n}\sum_{k=0}^{n}\frac{p_k}{(1+q)^k}\int_0^{\pi}\frac{\phi(t)}{\sin\left(\frac{t}{2}\right)}\left\{\sum_{\upsilon=0}^{k}\binom{k}{\nu}q^{k-\upsilon}\ \sin\left(\upsilon+\frac{1}{2}\right)t\right\}dt$$

$$= \int_0^{\pi}\phi(t)\ K_n(t)dt$$

In order to prove the theorem, under our assumption, it is sufficient to show that

$$\int_0^{\pi}\phi(t)\ K_n(t)\ dt = O(1) \quad \text{as} \quad n\to\infty$$

For $0<\delta<\pi$, we have $\overline{N}_p E_n^q - f(x) = \int_0^{\pi}\phi(t)K_n(t)\ dt$

$$= \left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi}\right)\phi(t)\ K_n(t)\ dt.$$

$$= I_1 + I_2 + I_3 \quad ,Say.$$

Now $|I_1| = \left|\int_0^{1/n}\phi(t)\ K_n(t)\ dt\right| \le \int_0^{1/n}|\phi(t)|\ |K_n(t)|dt.$

$\le O(n)\int_0^{1/n}|\phi(t)|\ dt$, using Lemma -1

$=O(n)\left\{O\left(\frac{1}{n\alpha(n)}\right)\right\}$, using (3.1.1).

$=O\left(\frac{1}{\alpha(n)}\right)$, as $n\to\infty$.

$=O(1)$, as $n\to\infty$, using (3.1.2).

Next

$$|I_2| \le \left|\int_{1/n}^{\delta}|\phi(t)||k_n(t)|\ dt\right|$$

$= O\left\{\int_{1/n}^{\delta}\frac{|\phi(t)|}{t}\ dt\right\}$, using lemma -2

Author's personal copy

$$= O\left\{\left[\frac{\Phi(t)}{t}\right]_{1/n}^{\delta} + \int_{1/n}^{\delta} \frac{\Phi(t)}{t^2}\, dt\right\}.$$

$$= O\left\{ o\left[\frac{1}{\alpha\left(\frac{1}{t}\right)}\right]_{1/n}^{\delta} + \int_{1/\delta}^{n} o\left(\frac{1}{u\alpha(u)}\right) du \right\}$$, where u=1/t

and $0 < \delta < 1$.

$$= o\left(\frac{1}{\alpha(n)}\right) + o\left(\frac{1}{n\alpha(n)}\right)\int_{1/\delta}^{n} du$$, using second mean-value theorem for the integral in the 2nd term as $\alpha(n)$ is monotonic.

$$= O(1) + O(1) \quad as \quad n \to \infty$$, using (3.1.2)

$$= O(1), \text{ as } \quad n \to \infty .$$

Finally

$$|I_3| \le \int_{\delta}^{\pi} |\phi(t)||K_n(t)|\, dt \quad = O(1) \quad \text{as} \quad n \to \infty .$$

by using Riemann –Lebesgue theorem and the regularity condition of the method of summability.

Then, $\overline{N}_p E_n^q - f(x) = O(1)$, as $n \to \infty$.

This completes the proof of the theorem.

6. Corollaries:

The following corollaries can be derived from our main theorem,

Corollary-6.1: If $\Phi(t) = \int_0^t |\phi(u)|\, du = O\left\{\frac{t}{\alpha\left(\frac{1}{t}\right)}\right\}$, $as\ t \to +0$ and $\alpha(n) \to \infty\ as\ n \to \infty$ where $\alpha(t)$ be a positive , non- increasing function of t , then the series $\sum_{n=0}^{\infty} A_n(x)$ is summable $(C,1)(E,q)$ to $f(x)$ at the point $t = x$.

Author's personal copy

Proof:

If we take $p_n = 1, \forall n$, the result follows from the main theorem.

Corollary - 6.2:

If $\Phi(t) \equiv 0\left\{\dfrac{t}{\log\left(\dfrac{1}{t}\right)}\right\}$, as $t \to +0$ then the Fourier series (1.6) is $(\overline{N}, p_n)(E,q)$ summable to $f(x)$.

Under the hypothesis, condition (3.1.2) is obviously satisfied on $\log n \to \infty$ as $n \to \infty$.

Corollary - 6.3:

If $\Phi(t) \equiv 0(t)$, as $t \to +0$, then the Fourier series $(\overline{N}, p_n)(E,q)$ summable to $f(x)$ without employing (3.1.2).

REFERENCES

1. Borwein, D: On product of sequences, Journal of London Math. Society.33 (1958), 352-357.
2. Nigam,H.K and Kusum sharma : A study on $(N, p_n)(E,q)$ product summability of fourier series , ultra scientist ,vol. 22(3) ,m 927-932, (2010).
3. Titechmalch , E. C :The theory of functions ,oxford university press , 402- 403(1939).

Author's personal copy