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PO-Γ-FILTERS IN PO-Γ-SEMIGROUPS

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ABSTRACT

The terms left po-Γ-filter, right po-Γ-filter, po-Γ-filter, are introduced. It is proved that a nonempty subset F of a po-Γ-semigroup S is a left po-Γ-filter if and only if S\F is a completely prime right po-Γ-ideal of S or empty. Further it is proved that S is a po-Γ-semigroup and F is a left po-Γ-filter, then S\F is a prime right po-Γ-ideal of S or empty and A nonempty subset F of a commutative po-Γ-semigroup S is a left po-Γ-filter if and only if S\F is a prime right po-Γ-ideal of S or empty. It is proved that a nonempty subset F of a po-Γ-semigroup S is a right po-Γ-filter if and only if S\F is a completely prime left po-Γ-ideal of S or empty. It is proved that every po-Γ-filter F of a po-Γ-semigroup S is a po-*c*-system. Further it is also proved that a nonempty subset F of a po-Γ-semigroup S is a po-Γ-filter if and only if S\F is a completely prime po-Γ-ideal of S or empty. It is proved that every po-Γ-filter F of a po-Γ-semigroup S is a po-*m*-system. It is proved that, if a nonempty subset F of a po-Γ-semigroup S is a po-Γ-filter, then F is a po-*d*-system of S or empty. Further it is proved that, every po-Γ-filter F of a po-Γ-semigroup S is a po-*n*-system of S. It is proved that the po-Γ-filter of a po-Γ-semigroup S generated by a nonempty subset A of S is the intersection of all po-Γ-filters of S containing A. It is proved that if $N(b) \subseteq N(a)$, then $N(a)\backslash N(b)$, if it is nonempty, is a completely prime po-Γ-ideal of $N(a)$.

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KEY WORDS: po-Γ-semigroup, po-Γ-ideal, prime po-Γ-ideal, po-Γ-filter.

1. INTRODUCTION :

Γ- semigroup was introduced by Sen and Saha [15] as a generalization of semigroup. Anjaneyulu. A [1], [2] and [3] initiated the study of ideals and radicals in semigroups. Many classical notions of semigroups have been extended to Γ-semigroups by Madhusudhana Rao, Anjaneyulu and Gangadhara Rao [11]. The concept of po-Γ-semigroup was introduced by Y. I. Kwon and S. K. Lee [10] in 1996, and it has been

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studied by several authors. In this paper we introduce the notions of po-Γ-filters, and characterize po-Γ-filters.

2. PRELIMINARIES :

DEFINITION 2.1 : Let S and Γ be two non-empty sets. Then S is called a ***Γ-semigroup*** if there exist a mapping from $S \times \Gamma \times S$ to S which maps $(a, \alpha, b) \rightarrow a\alpha b$ satisfying the condition : $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in S$ and $\gamma, \mu \in \Gamma$.

NOTE 2.2 : Let S be a Γ-semigroup. If A and B are two subsets of S, we shall denote the set $\{ a\gamma b : a \in A , b \in B \text{ and } \gamma \in \Gamma \}$ by AΓB.

DEFINITION 2.3: A Γ-semigroup S is said to a ***partially ordered Γ-semigroup*** if S is partially ordered set such that $a \le b \Rightarrow a\gamma c \le b\gamma c$ and $c\gamma a \le c\gamma b$ $\forall a,b,c \in S$ and $\gamma \in \Gamma$.

NOTE 2.4: A partially ordered Γ-semigroup simply called po-Γ-semigroup or ordered Γ-semigroup.

NOTATION 2.5 : Let S be a po- Γ-semigroup and T is a nonempty subset of S. If H is a nonempty subset of T, we denote the set $\{t \in T : t \le h$ for some $h \in H\}$ by $(H]_T$. The set $\{t \in T : h \le t$ for some $h \in H\}$ by $[H)_T$. $(H]_s$ and $[H)_s$ are simply denoted by (H] and [H) respectively.

DEFINITION 2.6 : Let S be a po- Γ-semigroup. A nonempty subset T of S is said to be a ***po-Γ-subsemigroup*** of S if $a\gamma b \in T$, for all $a, b \in T$ and $\gamma \in \Gamma$ and $t \in T$, $s \in S$, $s \le t \Rightarrow s \in T$.

THEOREM 2.7 : A nonempty subset T of a po-Γ-semigroup S is a po-Γ-subsemigroup of S iff (1) TΓT ⊆ T, (2) (T] ⊆ T.

THEOREM 2.8 : Let S be a po- Γ-semigroup and A is a subset of S. Then for all A, B ⊆ S (i) A ⊆ (A], (ii) ((A]] = (A], (iii) (A]Γ(B] ⊆ (AΓB] and for A ⊆ B (iv) A ⊆ (B], (v) (A] ⊆ (B] for A ⊆ B.

DEFINITION 2.9 : A nonempty subset A of a po-Γ-semigroup S is said to be a ***left* po-*Γ-ideal*** of S if

(1) $s \in S, a \in A, \alpha \in \Gamma$ implies $s\alpha a \in A$.
(2) $s \in S, a \in A$, $s \le a \Rightarrow s \in A$.

NOTE 2.10 : A nonempty subset A of a po-Γ-semigroup S is a left po-Γ-ideal of S iff (1) SΓA⊆A, and (2) (A] ⊆ A.

DEFINITION 2.11 : A nonempty subset A of a po-Γ-semigroup S is said to be a ***right po- Γ-ideal*** of S if

(1) $s \in S, a \in A, \alpha \in \Gamma$ implies $a\alpha s \in A$.

(2) $s \in S, a \in A$, $s \leq a \Rightarrow s \in$ A.

NOTE 2.12 : A nonempty subset A of a po-Γ-semigroup S is a right po-Γ-ideal of S iff (1) AΓS⊆A and (2) (A] ⊆ A.

DEFINITION 2.13 : A nonempty subset A of a po-Γ-semigroup S is said to be a ***two sided* po-Γ-*ideal*** or simply a ***po*-Γ-*ideal*** of S if
(1) *s* ∈ S, *a* ∈ A, $\alpha \in \Gamma$ imply $s\alpha a$ ∈ A, $a\alpha s$ ∈ A.

(2) $s \in S, a \in A$, $s \leq a \Rightarrow s \in$ A.

NOTE 2.14 : A nonempty subset A of a po- Γ-semigroup S is a two sided po-Γ-ideal iff it is both a left po-Γ-ideal and a right po-Γ- ideal of S.

THEOREM 2.15 : The nonempty intersection of any family of po-Γ-ideals of a po-Γ-semigroup S is a po-Γ-ideal of S.

DEFINITION 2.16 : A (left, right) po-Γ-ideal P of a po-Γ-semigroup S is said to be ***completely prime* (*left, right*) *po*-Γ-*ideal*** provided *x*, *y* ∈ S and *x*Γ*y* ⊆ P implies either *x* ∈ P or *y* ∈ P.

DEFINITION 2.17: Let S be a po-Γ-semigroup. A nonempty subset A of S is said to be a ***po-c-system*** of S if for each *a*, *b* ∈ A and α ∈ Γ there exists an element *c* ∈ A such that $c \leq a\alpha b$.

NOTE 2.18 : A nonempty subset A of a po-Γ-semigroup S is said to be a po-*c*-system of S if for each *a*, *b* ∈ A there exists an element *c* ∈ A such that *c* ∈ (*a*Γ*b*].

THEOREM 2.19 : Every po-Γ-subsemigroup of a po-Γ-semigroup is a *po-c-system.*

THEOREM 2.20 : A po-Γ-ideal P of a po-Γ-semigroup S is completely prime if and only if S\P is either a *c*-system of S or empty.

DEFINITION 2.21 : A (left, right) po-Γ-ideal P of a po-Γ-semigroup S is said to be a ***prime* (*left, right*) *po*-Γ-*ideal*** provided A, B are two po- Γ-ideals of S and AΓB ⊆ P ⇒ either A ⊆ P or B ⊆ P.

THEOREM 2.22 : If P is a prime po-Γ-ideal of a po-Γ-semigroup S, then the following conditions are equivalent.

(1) If *A*, *B* are po-Γ- ideals of S and AΓB⊆P then either A⊆P or B⊆P.
(2) If *a, b* ∈ S such that $a\Gamma S^1 \Gamma b$ ⊆ P, then either *a* ∈ P or *b* ∈ P.

THEOREM 2.23 : Every completely prime (left, right) po-Γ-ideal of a po-Γ-semigroup S is a prime (left, right) po-Γ-ideal of S.

THEOREM 2.24 : Let S be a commutative po-Γ-semigroup. A (left, right) po-Γ-ideal P of S is prime (left, right) po-Γ-ideal if and only if P is a completely prime (left, right) po-Γ-ideal.

DEFINITION 2.25 : A nonempty subset A of a po-Γ-semigroup S is said to be an ***po-m-system*** provided for any *a*,*b* ∈ A and *α, β* ∈ Γ there exists an *c* ∈ A and *x* ∈ S such that $c \leq a\alpha x\beta b$.

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NOTE 2.26 : A nonempty subset A of a po-Γ-semigroup S is said to be an ***po-m-system*** provided for any $a,b \in$ A there exists an $c \in$ A and $x \in$ S such that $c \in (a\Gamma S\Gamma b]$.

THEOREM 2.27 : A po-Γ-ideal P of a po-Γ-semigroup S is a prime po-Γ-ideal of S if and only if S\P is an *m*-system of S or empty.

DEFINITION 2.28 : A po-Γ-ideal A of a po- Γ-semigroup S is said to be a ***completely semiprime po-Γ- ideal*** provided $x\Gamma x \subseteq A$; $x \in$S implies $x \in$A.

THEOREM 2.29 : Every completely prime (left, right) po-Γ-ideal of a po-Γ-semigroup S is a completely semiprime (left, right) po-Γ-ideal of S.

DEFINITION 2.30 : Let S be a po-Γ-semigroup. A nonempty subset A of S is said to be a ***po-d-system*** of S if for each $a \in$ A and $\alpha \in \Gamma$, there exists an element $c \in$ A such that $c \leq a\alpha a$.

NOTE 2.31 : A nonempty subset A of a po-Γ-semigroup S is said to be a po-*d*-system of S if for each $a \in$ A, there exists $c \in$ A such that $c \in (a\Gamma a]$.

THEOREM 2.32 : A po-Γ-ideal P of a po-Γ-semigroup S is a completely semiprime iff S\P is a po-*d*-system of S or empty.

DEFINITION 2.33 : A po-Γ- ideal A of a po-Γ-semigroup S is said to be a ***semiprime po-Γ-ideal*** provided $x \in$ S, $x\Gamma S^1\Gamma x \subseteq A$ implies $x \in$ A.

THEOREM 2.34 : Every completely semiprime (left, right) po-Γ-ideal of a po-Γ-semigroup S is a semiprime (left, right) po-Γ-ideal of S.

THEOREM 2.35 : Let S be a commutative po-Γ-semigroup. A (left, right) po-Γ-ideal A of S is completely semiprime iff semiprime.

DEFINITION 2.36 : A nonempty subset A of a po-Γ-semigroup S is said to be a ***po-n-system*** provided for any $a \in$ A and some $\alpha, \beta \in \Gamma$ there exists an element $c \in$ A, $x \in$ S such that $c \leq a\alpha x\beta a$.

NOTE 2.37 : A nonempty subset A of a po-Γ-semigroup S is said to be an **po-*n-system*** provided for any $a \in$ A, $x \in$ S there exists an element $c \in$ A such that $c \in (a\Gamma S\Gamma a]$.

THEOREM 2.38 : A po-Γ- ideal Q of a po-Γ-semigroup S is a semiprime po-Γ-ideal iff S\Q is an po-*n*-system of S or empty.

3. PO-Γ-FILTERS IN PO- Γ-SEMIGROUPS :

DEFINITION 3.1 : A Γ-subsemigroup F of a po-Γ-semigroup S is said to be a ***left po-Γ-filter*** of S if

(1) $a, b \in$ S, $\alpha \in \Gamma$, $a\alpha b \in$ F implies $a \in$ F.
(2) $a \in$ F, $c \in$ S and $a \leq c$ implies $c \in$ F.

NOTE 3.2 : A Γ-subsemigroup F of a po-Γ-semigroup S is said to be a ***left po-Γ-filter*** of S if

(1) $a, b \in$ S, $a\Gamma b \subseteq$ F implies $a \in$ F.

(2) $[F) \subseteq$ F.

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THEOREM 3.3 : The nonempty intersection of two left po-Γ-filters of a po-Γ-semigroup S is also a left po-Γ-filter.

Proof **:** Let A, B be two left po-Γ-filters of S.
Let $a, b \in S$, $\alpha \in \Gamma$, $a\alpha b \in A \cap B$.

$a\alpha b \in A\cap B \Rightarrow a\alpha b \in A$ and $a\alpha b \in B$.

$a\alpha b \in A$, A is a left po-Γ-filter of S $\Rightarrow a \in A$.

$a\alpha b \in B$, B is a left po-Γ-filter of S $\Rightarrow a \in B$.

$a \in A$, $a \in B \Rightarrow a \in A \cap B$.

Let $a \in A\cap B$, $a \leq c$ for $c \in S$. Now $a \in A\cap B \Rightarrow a \in A$, $a \in B$.

$a \in A$, $a \leq c$ for $c \in S$, A is a left po-Γ-filter $\Rightarrow c \in A$.

$a \in B$, $a \leq c$ for $c \in S$, B is a left po-Γ-filter $\Rightarrow c \in B$.

$c \in A$, $c \in B \Rightarrow c \in A\cap B$. Thus $a \in A\cap B$, $c \in S$ and $a \leq c \Rightarrow c \in A\cap B$.

Therefore $A\cap B$ is a left po-Γ-filter of S.

THEOREM 3.4 : The nonempty intersection of a family of left po-Γ-filters of a po-Γ-semigroup S is also a left po-Γ-filter.

Proof **:** Let $\{F_\alpha\}_{\alpha\in\Delta}$ be a family of left po-Γ-filters of S and let $F = \bigcap_{\alpha\in\Delta} F_\alpha$.

Let $a, b \in S$, $\gamma \in \Gamma$, $a\alpha b \in F$. Now $a\alpha b \in F \Rightarrow a\alpha b \in \bigcap_{\alpha\in\Delta} F_\alpha \Rightarrow a\alpha b \in F_\alpha$ for each $\alpha \in \Delta$.

$a\alpha b \in F_\alpha$, $\gamma \in \Gamma$, F_α is a left po-Γ-filter of S

$\Rightarrow a \in F_\alpha$ for each $\alpha \in \Delta \Rightarrow a \in \bigcap_{\alpha\in\Delta} F_\alpha \Rightarrow a \in F$.

Let $a \in F$ and $a \leq c$ for $c \in S$. Now $a \in F \Rightarrow a \in \bigcap_{\alpha\in\Delta} F_\alpha \Rightarrow a \in F_\alpha$ for each $\alpha \in \Delta$

$a \in F_\alpha$, $a \leq c$ for $c \in S \Rightarrow c \in F_\alpha$ for all $\alpha \in \Delta$.
$\Rightarrow c \in \bigcap_{\alpha\in\Delta} F_\alpha \Rightarrow c \in F$. Therefore F is a left po-Γ-filter of S.

We now prove a necessary and sufficient condition for a nonempty subset to be a left po-Γ-filter in a po-Γ-semigroup.

THEOREM 3.5 : A nonempty subset F of a po-Γ-semigroup S is a left po-Γ-filter if and only if S\F is a completely prime right po-Γ-ideal of S or empty.

Proof **:** Assume that $S\backslash F \neq \emptyset$. Let $x \in S\backslash F$ and $y \in S$, $\alpha \in \Gamma$.
Suppose that $x\alpha y \notin S\backslash F$, then $x\alpha y \in F$. Since F is a left po-Γ-filter, $x \in F$.

It is a contradiction. Thus $x\alpha y \in S\backslash F$, and so $(S\backslash F)\Gamma S \subseteq S\backslash F$.

Let $x \in S\backslash F$ and $y \leq x$ for $y \in S$. If $y \notin S\backslash F$, then $y \in F$.

Since F is a left po-Γ-filter, $x \in F$. It is a contradiction.

Thus $y \in S\backslash F$. Therefore S\F is a right po-Γ-ideal.

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Next we shall prove that S\F is completely prime.

Let $x\alpha y \in$ S\F for $x, y \in$ S and $\alpha \in \Gamma$. Suppose that $x \notin$ S\F and $y \notin$ S\F.

Then $x \in$ F and $y \in$ F. Since F is a Γ-subsemigroup of S, $x\alpha y \in$ F.

It is a contradiction. Thus $x \in$ S\F or $y \in$ S\F.

Hence S\F is completely prime.

Therefore S\F is a completely prime right po-Γ-ideal of S.

Conversely suppose that S\F is a completely prime right po-Γ-ideal of S or empty.

If S\F = ∅, then F = S. Thus F is a left po-Γ-filter of S.

Assume that S\F is a completely prime right po-Γ-ideal of S.

Let $x, y \in$ F, for $\alpha \in \Gamma$. Suppose if possible $x\alpha y \notin$ F.

Then $x\alpha y \in$ S\F. Since S\F is completely prime, $x \in$ S\F or $y \in$ S\F. It is a contradiction.

Thus $x\alpha y \in$ F and hence F is a Γ-subsemigrupup of S.

Let $x, y \in$ S, $\alpha \in \Gamma$, $x\alpha y \in$ F. If $x \notin$ F, then $x \in$ S\F.

Since S\F is a completely prime right po-Γ-ideal of S, $x\alpha y \in$ (S\F)ΓS ⊆ S\F.

It is a contradiction. Thus $x \in$ F.

Let $x \in$ F, $y \in$ S and $x \leq y$. If $y \notin$ F, then $y \in$ S\F.

Since S\F is a right po-Γ-ideal of S, $x \in$ S\F. It is a contradiction.

Thus $y \in$ F. Therefore F is a left po-Γ-filter of S.

COROLLARY 3.6 : Let S is a po-Γ-semigroup and F is a left po-Γ-filter. Then S\F is a prime right po-Γ-ideal of S or empty.

Proof **:** Since F is a left po-Γ-filter. By theorem 3.5, S\F is a completely prime right po-Γ-ideal of S or empty. By theorem 2.23, S\F is a prime right po-Γ-ideal of S or empty.

COROLLARY 3.7 : A nonempty subset F of a commutative po-Γ-semigroup S is a left po-Γ-filter if and only if S\F is a prime right po-Γ-ideal of S or empty.

Proof **:** Suppose that F is a left po-Γ-filter of po-Γ-semigroup S. Then by Corollary 3.6, S\F is a prime right po-Γ-ideal of S or empty.

Conversely suppose that S\F is a prime right po-Γ-ideal of S or empty. By theorem 2.24, S\F is completely prime right po-Γ-ideal of S or empty. By theorem 3.5, F is a left po-Γ-filter of S.

DEFINITION 3.8 : A Γ-subsemigroup F of a po- Γ-semigroup S is said to be ***right po- Γ-filter*** of S if

(1) $a, b \in$ S, $\alpha \in \Gamma$, $a\alpha b \in$ F implies $b \in$ F.

(2) a ∈ F, $c \in$ S and $a \leq c$ implies $c \in$ F.

NOTE 3.9: A Γ-subsemigroup F of a po-Γ-semigroup S is said to be a ***right po- Γ-filter*** of S if

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(1) $a, b \in S$, $a\Gamma b \subseteq F$ implies $b \in F$.

(2) $[F) \subseteq F$.

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THEOREM 3.10 : The nonempty intersection of two right po-Γ-filters of a po-Γ-semigroup S is also a right po-Γ-filter.

Proof **:** Let A, B be two right po-Γ-filters of S.

Let $a, b \in S$, $\alpha \in \Gamma$, $a\alpha b \in A \cap B$.
$a\alpha b \in A \cap B \Rightarrow a\alpha b \in A$ and $a\alpha b \in B$.
$a\alpha b \in A$, A is a right po-Γ-filter of S $\Rightarrow b \in A$.
$a\alpha b \in B$, B is a right po-Γ-filter of S $\Rightarrow b \in B$.
$b \in A, b \in B \Rightarrow b \in A \cap B$.
Let $b \in A \cap B$, $b \leq c$ for $c \in S$. Now $b \in A \cap B \Rightarrow b \in A, b \in B$.
$b \in A$, $b \leq c$ for $c \in S$, A is a right po-Γ-filter $\Rightarrow c \in A$.
$b \in B$, $a \leq c$ for $c \in S$, B is a right po-Γ-filter $\Rightarrow c \in B$.
$c \in A, c \in B \Rightarrow c \in A \cap B$. Thus $b \in A \cap B$, $b \leq c$ for $c \in S \Rightarrow c \in A \cap B$.
Therefore $A \cap B$ is a right po-Γ-filter of S.

THEOREM 3.11 : The nonempty intersection of a family of right po-Γ-filters of a po-Γ-semigroup S is also a right po-Γ-filter.

Proof **:** Let $\{F_\alpha\}_{\alpha \in \Delta}$ be a family of right po-Γ-filters of S and let $F = \bigcap_{\alpha \in \Delta} F_\alpha$.

Let $a, b \in S$, $\gamma \in \Gamma$, $a\alpha b \in F$. Now $a\alpha b \in F \Rightarrow a\alpha b \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow a\alpha b \in F_\alpha$ for each $\alpha \in \Delta$.

$a\alpha b \in F_\alpha$, $\gamma \in \Gamma$, F_α is a right po-Γ-filter of S $\Rightarrow b \in F_\alpha$.
Let $b \in F$ and $b \leq c$ for $c \in S$. Now $b \in F \Rightarrow b \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow b \in F_\alpha$ for each $\alpha \in \Delta$

$b \in F_\alpha$, $b \leq c$ for $c \in S \Rightarrow c \in F_\alpha$ for all $\alpha \in \Delta$
$\Rightarrow b \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow b \in F$ and $b \in F_\alpha \Rightarrow c \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow c \in F$.
Therefore F is a right po-Γ-filter of S.

We now prove a necessary and sufficient condition for a nonempty subset to be a right po-Γ-filter in a po-Γ-semigroup.

THEOREM 3.12 : A nonempty subset F of a po-Γ-semigroup S is a right po-Γ-filter if and only if S\F is a completely prime left po-Γ-ideal of S or empty.

Proof **:** Assume that $S \setminus F \neq \emptyset$. Let $x \in S \setminus F$ and $y \in S$, $\alpha \in \Gamma$.
Suppose that $y\alpha x \notin S \setminus F$, then $y\alpha x \in F$.
Since F is a right po-Γ-filter, $x \in F$. It is a contradiction.
Thus $y\alpha x \in S \setminus F$, and so $S\Gamma(S \setminus F) \subseteq S \setminus F$.
Let $x \in S \setminus F$ and $y \leq x$ for $y \in S$.
If $y \notin S \setminus F$, then $y \in F$. Since F is a right po-Γ-filter, $x \in F$. It is a contradiction.
Thus $y \in S \setminus F$. Therefore S\F is a left po-Γ-ideal.
Next we shall prove that S\F is completely prime.

Let $x\alpha y \in S \backslash F$ for $x, y \in S$ and $\alpha \in \Gamma$. Suppose that $x \notin S\backslash F$ and $y \notin S\backslash F$.

Then $x \in F$ and $y \in F$. Since F is a Γ-subsemigroup of S, $x\alpha y \in F$.

It is a contradiction. Thus $x \in S\backslash F$ or $y \in S\backslash F$.

Hence S\F is completely prime and hence S\F is a completely prime left po-Γ-ideal of S.

Conversely suppose that S\F is a completely prime left po-Γ-ideal of S or empty.

If $S\backslash F = \emptyset$, then F = S. Thus F is a right po-Γ-filter of S.

Assume that S\F is a completely prime left po-Γ-ideal of S.

Let $x, y \in F$, $\alpha \in \Gamma$. Suppose if possible $x\alpha y \notin F$. Then $x\alpha y \in S\backslash F$.

Since S\F is completely prime, $x \in S\backslash F$ or $y \in S\backslash F$. It is a contradiction.

Thus $x\alpha y \in F$ and hence F is a Γ-subsemigroup of S.

Let $x, y \in S$, $\alpha \in \Gamma$, $x\alpha y \in F$. If $y \notin F$, then $y \in S\backslash F$.

Since S\F is a completely prime left po-Γ-ideal of S, $x\alpha y \in S\Gamma(S\backslash F) \subseteq S\backslash F$.

It is a contradiction. Thus $y \in F$.

Let $x \in F$ and $x \leq y$ for $y \in S$. If $y \notin F$, then $y \in S\backslash F$.

Since S\F is a left po-Γ-ideal of S, $y \in S\backslash F$. It is a contradiction. Thus $y \in F$.

Therefore F is a right po-Γ-filter of S.

COROLLARY 3.13 : Let S is a po-Γ-semigroup and F is a right po-Γ-filter. Then S\F is a prime left po-Γ-ideal of S or empty.

Proof **:** Since F is a right po-Γ-filter. By theorem 3.12, S\F is a completely prime left po-Γ-ideal of S or empty. By theorem 2.23, S\F is a prime left po-Γ-ideal of S or empty.

COROLLARY 3.14 : A nonempty subset F of a commutative po-Γ-semigroup S is a right po-Γ-filter if and only if S\F is a prime left po-Γ-ideal of S or empty.

Proof **:** Suppose that F is a right po-Γ-filter of po-Γ-semigroup S. Then by Corollary 3.13, S\F is a prime left po-Γ-ideal of S or empty.

Conversely suppose that S\F is a prime left po-Γ-ideal of S or empty. By theorem 2.24, S\F is completely prime left po-Γ-ideal of S or empty. By theorem 3.12, F is a right po-Γ-filter of S.

DEFINITION 3.15 : A Γ-subsemigroup F of a po-Γ-semigroup S is said to be ***po-Γ-filter*** of S if

(1) $a, b \in S$, $\alpha \in \Gamma$, $a\alpha b \in F$ implies $a, b \in F$.

(2) $a \in F$, $c \in S$ and $a \leq c$ implies $c \in F$.

NOTE 3.16 : A Γ-subsemigroup F of a po-Γ-semigroup S is said to be ***po-Γ-filter*** of S if

(1) for $a, b \in S$, $a\Gamma b \in F$ implies $a, b \in F$.

(2) $[F) \subseteq F$

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NOTE 3.17 : A Γ-subsemigroup F of a po- Γ-semigroup S is said to be ***po*-Γ-*filter*** of S iff F is a left po-Γ-filter and a right po-Γ-filter of S.

EXAMPLE 3.18 : Let S = { a, b, c} and Γ = {γ **}** with the multiplication defined by

$$x\gamma y = \begin{cases} b & \text{if } x = y = b \\ c & \text{if } x = y = c \\ a & \text{otherwise} \end{cases}$$

Define a relation ≤ on S as ≤: 1_S ∪ {(a, b), (a, c)}. Then S is a po-Γ-semigroup and {a, b, c}, {b}, {c} are all the po-Γ-filters of S.

DEFINITION 3.19 : A po-Γ-filter F of a po-Γ-semigroup S is said to be a ***proper po*-Γ-*filter*** if F ≠ S.

THEOREM 3.20 : The nonempty intersection of two po-□-filters of a po-□-semigroup is also a po- Γ -filter.

Proof **:** Let A, B be two po-Γ-filters of S.

Let a, b ∈ S, □ ∈ Γ, a□b ∈ A∩B.

a□b ∈ A∩B ⇒ a□b ∈ A and a□b ∈ B.

a□b ∈ A, A is a po-Γ-filter of S ⇒ a, b ∈ A.

a□b ∈ B, B is a po-Γ-filter of S ⇒ a, b ∈ B.

a, b ∈ A, a, b ∈ B ⇒ a, b ∈ A ∩B.

Let a ∈ A∩B, c ∈ S and $a \leq c$. Now a ∈ A∩B ⇒ a ∈ A, a ∈ B.

a ∈ A, $a \leq c$ for c ∈ S, A is po-Γ-filter ⇒ c ∈ A.

a ∈ B, $a \leq c$ for c ∈ S, B is po-Γ-filter ⇒ c ∈ B.

a ∈ A∩B, $a \leq c$ for c ∈ S ⇒ c ∈ A∩B.

Therefore A∩B is a Γ-filter of S.

THEOREM 3.21 : The nonempty intersection of a family of po-□-filters of a po-□-semigroup is also a po-□-filter.

Proof **:** Let $\{F_\alpha\}_{\alpha \in \Delta}$ be a family of po-Γ-filters of S and let $F = \bigcap_{\alpha \in \Delta} F_\alpha$.

Let a, b ∈ S, γ ∈ Γ, a□b ∈ F. Now a□b ∈ F ⇒ a□b ∈ $\bigcap_{\alpha \in \Delta} F_\alpha$ ⇒ $a\alpha b \in F_\alpha$ for each α ∈ Δ.

$a\alpha b \in F_\alpha$, γ ∈ Γ, F_α is a po-Γ-filter of S ⇒ a, b ∈ F_α .

a ∈ F , c ∈ S, $a \leq c$. $a \leq c$ for c ∈ S ⇒ c ∈ F_α for all α ∈ Δ ⇒ a, b ∈ $\bigcap_{\alpha \in \Delta} F_\alpha$ ⇒ a, b ∈ F and $a \in F_\alpha$, $a \leq c$ for $c \in$ S $\Rightarrow c \in \bigcap_{\alpha \in \Delta} F_\alpha \Rightarrow c \in$ F. Therefore F is a po-Γ-filter of S.

NOTE 3.22: In general, the union of two po-Γ-filters is not a po-Γ-filter.

EXAMPLE 3.23 : As in the example 3.18, S is a po-Γ-semigroup and $\{b\}$,$\{c\}$ are po-Γ-filters, but $\{b\} \cup \{c\}$ is not a po-Γ-filter of S because b□$c = a$ is not in $\{b\} \cup \{c\}$.

We now prove a necessary and sufficient condition for a nonempty subset to be a po-Γ-filter in a po-Γ-semigroup.

THEOREM 3.24 : A nonempty subset F of a po-□-semigroup S is a po-□-filter if and only if S\F is a completely prime po-□-ideal of S or empty.

***Proof* :** Assume that S\F $\neq \emptyset$. Let $x, y \in$ S\F, □ $\in$ Γ.

Suppose that x□$y \notin$ S\F, then x□$y \in$ F. Since F is a po-Γ-filter and hence $x, y \in$ F.

It is a contradiction. Thus $x\alpha y \in$ S\F, and so (S\F)ΓSΓ(S\F) ⊆ S\F.

Let $x \in$ S\F and $y \leq x$ for $y \in$ S. If $y \notin$ S\F, then $y \in$ F.

Since F is a po-Γ-filter, $x \in$ F.

It is a contradiction. Thus $y \in$ S\F. Therefore S\F is a po-Γ-ideal.

Next we shall prove that S\F is completely prime.

Let x□$y \in$ S\F for $x, y \in$ S and □ $\in$ Γ. Suppose that $x \notin$ S\F and $y \notin$ S\F.

Then $x \in$ F and $y \in$ F. Since F is a Γ-subsemigroup of S, x□$y \in$ F.

It is a contradiction. Thus $x \in$ S\F or $y \in$ S\F.

Hence S\F is completely prime and hence S\F is a completely prime right po-Γ-ideal of S.

Conversely suppose that S\F is a completely prime po-Γ-ideal of S or empty.

If S\F = $\emptyset$, then F = S. Thus F is a po-Γ-filter of S.

Assume that S\F is a completely prime po-Γ-ideal of S.

Suppose that for $\alpha \in$ Γ, $x, y \in$ F, x□$y \notin$ F. Then x□$y \in$ S\F for $x, y \in$ F, $\alpha \in$ Γ.

Since S\F is completely prime, $x \in$ S\F or $y \in$ S\F. It is a contradiction.

Thus x□$y \in$ F and hence F is a Γ-subsemigroup of S.

Let $x, y \in$ S, □ $\in$ Γ, x□$y \in$ F. If $x, y \notin$ F, then $x, y \in$ S\F.

Since S\F is a completely prime po-Γ-ideal of S, x□$y \in$ (S\F)ΓSΓ(S\F) ⊆ S\F.

It is a contradiction. Thus $x, y \in$ F.

Let $x \in$ F and $x \leq y$ for $y \in$ S. If $y \notin$ F, then $y \in$ S\F.

Since S\F is a po-Γ-ideal of S, $x \in$ S\F. It is a contradiction. Thus $y \in$ F.

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Therefore F is a po-Γ-filter of S.

COROLLARY 3.25 : Let S is a po-□-semigroup and F is a po-□-filter. Then S\F is a prime po-□-ideal of S or empty.

***Proof* :** Since F is a po-Γ-filter. By theorem 3.24, S\F is a completely prime po-Γ-ideal of S or empty. By theorem 2.23, S\F is prime po-Γ-ideal of S or empty.

COROLLARY 3.26 : A nonempty subset F of a commutative po-□-semigroup S is a po-□-filter if and only if S\F is a prime po-□-ideal of S or empty.

***Proof* :** Suppose that S\F is po-Γ-filter of commutative po-Γ-semigroup S. By Corollary 3.25, S\F is prime po-Γ-ideal of S or empty.
Conversely suppose that S\F is a prime po-Γ-ideal of S or empty. If S\F = $\emptyset$, then F = S. Thus F is a po-Γ-filter of S. Assume that S\F is a prime po-Γ-ideal of S. By theorem 2.23, S\F is a completely po-Γ-ideal of S or empty. By theorem 3.24, F is a po-Γ-filter of S.

THEOREM 3.27 : Every po-□-filter F of a po-□-semigroup S is a po-*c*-system of S.

***Proof* :** Suppose that F is a po-Γ-filter. By theorem 3.24, S\F is completely prime po-Γ-ideal of S. By the theorem 2.20, F is po-*c*-system of S.

THEOREM 3.28 : A po-Γ-semigroup S does not contain proper po-Γ-filters if and only if S does not contain proper completely prime po-Γ-ideals.

***Proof* :** Suppose that po-Γ-semigroup S does not contain proper po-Γ-filters. Let A be a completely prime po-Γ-ideal of S, $A \subset S$. Then $\emptyset \neq S\backslash A \subseteq S$ and S\(S\A) (= A) is a completely prime po-Γ-ideal of S. Since S\A is the complement of A to S, by theorem 3.24, S\A is a po-Γ-filter of S. Then S\A = S and A = $\emptyset$. It is a contradiction and hence S does not contain proper completely prime po-Γ-ideals.

Conversely suppose that S does not contain proper completely prime po-Γ-ideals. Let F is a po-Γ-filter of S, $F \subset S$. Since $S\backslash F \neq \emptyset$, by theorem 3.24, S\F is a completely prime po-Γ-ideal of S. Then S\F = S and F = $\emptyset$. It is a contradiction and hence S does not contain proper po-Γ-filters.

THEOREM 3.29 : Every po-□-filter F of a po-□-semigroup S is a po-*m*-system of S.

***Proof* :** Suppose that F is a po-Γ-filter of a po-Γ-semigroup S. By corollary 3.25, S\F is a prime po-Γ-ideal of S. By theorem 2.27, S\(S\F) = F is a po-*m*-system of S or empty.

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THEOREM 3.30 : Let S is a po-Γ-semigroup and F is a po-Γ-filter. Then S\F is a completely semiprime po-Γ-ideal of S.

***Proof* :** Since F is a po-Γ-filter of a po-Γ-semigroup S. By theorem 3.24, S\F is a completely prime po-Γ-ideal of S. By theorem 2.29, S\F is a completely semiprime po-Γ-ideal of S.

THEOREM 3.31: Every po-Γ-filter F of a po-Γ-semigroup S is a po-*d*-system of S.

***Proof* :** Since F is a po-Γ-filter of a po-Γ-semigroup S. By theorem 3.30, S\F is a completely semiprime po-Γ-ideal of S. By theorm 2.32, S\(S\F) = F is a po-*d*-system of S or empty.

THEOREM 3.32 : Let S is a po-Γ-semigroup and F is a po-Γ-filter. Then S\F is a semiprime po-Γ-ideal of S.

***Proof* :** Since F is a po-Γ-filter of a po-Γ-semigroup S. By theorem 3.24, S\F is a completely prime po-Γ-ideal of S. By theorem 2.29, S\F is a completely semiprime po-Γ-ideal of S. By theorem 2.34, S\F is a semiprime po-Γ-ideal of S.

THEOREM 3.33 : Every po-Γ-filter F of a po-Γ-semigroup S is a po-*n*-system of S.

***Proof* :** Since F is a po-Γ-filter of a po-Γ-semigroup S. By theorem 3.32, S\F is a semiprime po-Γ-ideal of S. By theorem 2.38, S\(S\F) = F is a po-*n*-system of S.

DEFINITION 3.34 : Let S be a po- Γ-semigroup and A be a nonempty subset of S. The smallest left po-Γ-filter of S containing A is called ***left po-Γ-filter of* S *generated by A*** and it is denoted by $F_l(A)$.

THEOREM 3.35 : The left po-Γ-filter of a po-Γ-semigroup S generated by a nonempty subset A of S is the intersection of all left po-Γ-filters of S containing A.

***Proof* :** Let Δ be the set of all left po-Γ-filters of S containing A.

Since S itself is a left po-Γ-filter of S containing A, S ∈ Δ. So $\Delta \neq \emptyset$.

Let $F^* = \bigcap_{T\in\Delta} F$. Since $A \subseteq F$ for all F ∈ Δ, $A \subseteq F^*$.

By theorem 3.4, F^* is a left po-Γ-filter of S.

Let K is a left po-Γ-filter of S containing A.

Clearly A ⊆ K and K is a left po-Γ-filter of S.

Therefore K ∈ Δ ⇒F^* ⊆ K. Therefore F^* is the left po-Γ-filter of S generated by A.

DEFINITION 3.36 : Let S be a po- Γ-semigroup and A be a nonempty subset of S. The smallest right po-Γ-filter of S containing A is called ***right po-Γ-ideal of*** **S** ***generated by A*** and it is denoted by $F_r(A)$.

THEOREM 3.37 : The right po-Γ-filter of a po-Γ-semigroup S generated by a nonempty subset A is the intersection of all right po-Γ-filters of S containing A.

Proof **:** Let Δ be the set of all right po-Γ-filters of S containing A.

Since S itself is a right po-Γ-filter of S containing A, S ∈ Δ. So $\Delta \neq \emptyset$.

Let $F^* = \bigcap_{T \in \Delta} F$. Since $A \subseteq F$ for all F ∈ Δ, $A \subseteq F^*$.

By theorem 3.11, T^* is a right po-Γ-filter of S.

Let K is a right po-Γ-filter of S containing A.

Clearly A ⊆ K and K is a right po-Γ-filter of S.

Therefore K ∈ Δ ⇒F^* ⊆ K. Therefore F^* is the right po-Γ-filter of S generated by A.

DEFINITION 3.38 : Let S be a po- Γ-semigroup and A be a nonempty subset of S. The smallest po-Γ-filter of S containing A is called ***po*-Γ*-filter of*** **S** ***generated by A*** and it is denoted by N(A).

THEOREM 3.39 : The po-Γ-filter of a Γ-semigroup S generated by a nonempty subset A is the intersection of all po-Γ-filters of S containing A.

Proof **:** Let Δ be the set of all po-Γ-filters of S containing A.

Since S itself is a po-Γ-filter of S containing A, S ∈ Δ. So $\Delta \neq \emptyset$.

Let $F^* = \bigcap_{T \in \Delta} F$. Since $A \subseteq F$ for all F ∈ Δ, $A \subseteq F^*$.

By theorem 3.21, F^* is a po-Γ-filter of S.

Let K is a po-Γ-filter of S containing A.

Clearly A ⊆ K and K is a po-Γ-filter of S.

Therefore K ∈ Δ ⇒F^* ⊆ K.

Therefore F^* is the po-Γ-filter of S generated by A.

DEFINITION 3.40 : A po-Γ-filter F of a po-Γ-semigroup S is said to be a ***principal po-□-filter*** provided F is a po-Γ-filter generated by {*a*} for some *a* ∈ S. It is denoted by N(*a*).

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EXAMPLE 3.41 : As in the example 3.18, S is a po-Γ-semigroup and N(a) = {a, b, c}, N(b) = {b} and N(c) = {c} are all the principal po-Γ-filters of the po-Γ-semigroup S.

COROLLARY 3.42: Let S is a po-Γ-semigroup and $a \in$ S. Then N(a) is the least filter of S containing {a}.

NOTE 3.43 : For every $a \in$ S, the intersection of all po-Γ-filters containing a is again a po-Γ-filter and thus the least po-Γ-filter containing a.

THEOREM 3.44 : If N(b) ⊆ N(a), then N(a)\N(b), if it is nonempty, is a completely prime po-Γ-ideal of N(a).

***Proof* :** By the theorem 3.24, N(a)\N(b) is a completely prime po-Γ-ideal of N(a).

LEMMA 3.45 : Let $a, b \in$ S and $b \in$ N(a), then N(b) ⊆ N(a).

***Proof* :** From the definition of the principal po-Γ-filter it is clear.

COROLLARY 3.46 : Let $a, b \in$ S and $a \leq b$ then N(b) ⊆ N(a).

***Proof* :** Since $a \leq b$ then it is clear that $b \in$ N(a). By lemma 3.45, we have N(b) ⊆ N(a).

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