DUO Chained $\Gamma$-Semigroups

A. Gangadhara Rao$^1$, A. Anjaneyulu$^2$, D. Madhusudhana Rao$^3$.

Dept. of Mathematics, V S R & N V R College, Tenali, A.P. India.

$^1$raog1967@gmail.com, $^2$anjaneyulu.addala@gmail.com, $^3$dmrmaths@gmail.com

ABSTRACT

In this paper we introduce the terms left $\alpha$-cancellative, right $\alpha$-cancellative, $\alpha$-cancellative, left $\Gamma$-cancellative, right $\Gamma$-cancellative, $\Gamma$-cancellative, strongly left cancellative, strongly right cancellative, strongly cancellative elements, $\alpha$-inverse, $\Gamma$-inverse, complete inverse of an element, unit in a $\Gamma$-semigroup and a duo chained $\Gamma$-semigroup. It is proved that if $P$ is a prime $\Gamma$-ideal of a duo chained $\Gamma$-semigroup $S$ and $x \not\in P$ then $P = \bigcap_{n=1}^{\infty} (x\Gamma)^n P$. It is also proved that every duo chained $\Gamma$-semigroup is a semiprimary $\Gamma$-semigroup. It is proved that (1) if $a \in S$ is a semisimple elements of a duo chained $\Gamma$-semigroup $S$, then $<a>^w \neq \emptyset$. (2) if a duo chained $\Gamma$-semigroup $S$ has no $\Gamma$-idempotent elements, then for any $a \in S$, $<a>^w = \emptyset$ or $<a>^w$ is a prime $\Gamma$-ideal. In a duo chained $\Gamma$-semigroup $S$ if $S \neq S \Gamma S$ then $S \Gamma S = \{x\}$ for some $x \in S$. Further it is proved that in a duo chained $\Gamma$-semigroup $S$, if $S \neq S \Gamma S$ such that $S \Gamma S = \{x\}$ for some $x \in S$, then (1) $S = x \Gamma S^1 = S^1 \Gamma x$ and $S \Gamma S = x \Gamma S = S \Gamma x$ is the unique maximal $\Gamma$-ideal of $S$. (2) If $a \in S$ and $a \not\in <x>^w$ then $a \in (x\Gamma)^n x$ for some natural number $n > 1$. (3) If $S$ contains strongly cancelable elements then $x$ is a strongly cancelable element and $<x>^w$ is either empty or a prime $\Gamma$-ideal of $S$. It is proved that, if $S$ is a duo chained $\Gamma$-semigroup, then $S$ is an archimedean $\Gamma$-semigroup without $\Gamma$-idempotents if and only if $<a>^w = \emptyset$ for every $a \in S$. It is proved that if $S$ be a strongly cancellable archimedean duo chained $\Gamma$-semigroup with $<a>^w \neq \emptyset$ for some $a \in S$, then $S$ is a $\Gamma$-group. Also it is proved that, let $S$ be a duo chained $\Gamma$-semigroup containing strongly cancellative elements and $<a>^w = \emptyset$ for every $a \in S$, then $S$ is a strongly cancellative $\Gamma$-semigroup.


KEY WORDS : chained $\Gamma$-semigroup, duo chained $\Gamma$-semigroup, $\alpha$-cancellative element, $\Gamma$-cancellative element, strongly cancellative $\Gamma$-semigroup, $\alpha$-inverse, $\Gamma$-inverse, complete inverse of an element, unit in a $\Gamma$-semigroup,
1. INTRODUCTION:

Γ-semigroup was introduced by SEN and SAHA [12] as a generalization of semigroup. ANJANEYULU [1, 2] and [3] initiated the study of pseudo symmetric ideals, radicals, semipseudo symmetric ideals in semigroups and N(A)-semigroups and primary and semiprimary ideals in semigroups. GIRI and WAZALWAR [6] initiated the study of prime radicals in semigroups. MADHUSUDHANA RAO, ANJANEYULU and GANGADHARA RAO [7], [8], [9], [10], [11] and [12] initiated the study of pseudo symmetric Γ-ideals, prime Γ-radicals and semipseudo symmetric Γ-ideals in Γ-semigroups and N(A)-semigroups pseudo integral Γ-semigroups, Primary and semi primary Γ-ideals in Γ-semigroups. In this paper we introduce the notions of duo chained Γ-semigroup and characterize duo chained Γ-semigroup.

2. PRELIMINARIES:

DEFINITION 2.1: Let S and Γ be any two non-empty sets. Then S is said to be a Γ-semigroup if there exist a mapping from S × Γ × S to S which maps (a, γ, b) → aγb satisfying the condition: (aαb)βγc = aα(bβc) for all a, b, c ∈ S and α, β ∈ Γ.

NOTE 2.2: Let S be a Γ-semigroup. If A and B are two subsets of S, we shall denote the set \{ aγb : a ∈ A, b ∈ B and γ ∈ Γ \} by AΓB.

DEFINITION 2.3: A nonempty subset A of a Γ-semigroup S is said to be a left Γ-ideal of S if s ∈ S, a ∈ A, α ∈ Γ implies saα ∈ A.

NOTE 2.4: A nonempty subset A of a Γ-semigroup S is a left Γ-ideal of S iff SΓA ⊆ A.

DEFINITION 2.5: A nonempty subset A of a Γ-semigroup S is said to be a right Γ-ideal of S if s ∈ S, a ∈ A, α ∈ Γ implies aaα ∈ A.

NOTE 2.6: A nonempty subset A of a Γ-semigroup S is a right Γ-ideal of S iff AΓS ⊆ A.

DEFINITION 2.7: A nonempty subset A of a Γ-semigroup S is said to be a two sided Γ-ideal or simply a Γ-ideal of S if s ∈ S, a ∈ A, α ∈ Γ imply sαa ∈ A, aaa ∈ A.

NOTE 2.8: A nonempty subset A of a Γ-semigroup S is a two sided Γ-ideal iff it is both a left Γ-ideal and a right Γ-ideal of S.

THEOREM 2.9: The nonempty intersection of any two (left or right) Γ-ideals of a Γ-semigroup S is a (left or right) Γ-ideal of S.

THEOREM 2.10: The nonempty intersection of any family of (left or right) Γ-ideals of a Γ-semigroup S is a (left or right) Γ-ideal of S.

THEOREM 2.11: The union of any two (left or right) Γ-ideals of a Γ-semigroup S is a (left or right) Γ-ideal of S.

THEOREM 2.12: The union of any family of (left or right) Γ-ideals of a Γ-semigroup S is a (left or right) Γ-ideal of S.
DEFINITION 2.13: A \( \Gamma \)-semigroup \( S \) is said to be a **left duo \( \Gamma \)-semigroup** provided every left \( \Gamma \)-ideal of \( S \) is a two sided \( \Gamma \)-ideal of \( S \).

DEFINITION 2.14: A \( \Gamma \)-semigroup \( S \) is said to be a **right duo \( \Gamma \)-semigroup** provided every right \( \Gamma \)-ideal of \( S \) is a two sided \( \Gamma \)-ideal of \( S \).

DEFINITION 2.15: A \( \Gamma \)-semigroup \( S \) is said to be a **duo \( \Gamma \)-semigroup** provided it is both a left duo \( \Gamma \)-semigroup and a right duo \( \Gamma \)-semigroup.

THEOREM 2.16: A \( \Gamma \)-semigroup \( S \) is a duo \( \Gamma \)-semigroup if and only if \( xS^1 = S^1x \) for all \( x \in S \).

THEOREM 2.17: Let \( A \) be a \( \Gamma \)-ideal in a duo \( \Gamma \)-semigroup \( S \) and \( a, b \in S \). Then \( a\Gamma b \subseteq A \) if and only if \( <a> \Gamma <b> \subseteq A \).

DEFINITION 2.18: A \( \Gamma \)-ideal \( P \) of a \( \Gamma \)-semigroup \( S \) is said to be a **completely prime \( \Gamma \)-ideal** provided \( x, y \in S \) and \( xy \Gamma \subseteq P \) implies either \( x \in P \) or \( y \in P \).

DEFINITION 2.19: A \( \Gamma \)-ideal \( P \) of a \( \Gamma \)-semigroup \( S \) is said to be a **prime \( \Gamma \)-ideal** provided \( A, B \) are two \( \Gamma \)-ideals of \( S \) and \( AB \subseteq P \Rightarrow \) either \( A \subseteq P \) or \( B \subseteq P \).

COROLLARY 2.20: A \( \Gamma \)-ideal \( P \) of a \( \Gamma \)-semigroup \( S \) is a prime \( \Gamma \)-ideal iff \( a, b \in S \) such that \( a\Gamma S^1b \subseteq P \), then either \( a \in P \) or \( b \in P \).

THEOREM 2.21: Every completely prime \( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( S \) is a prime \( \Gamma \)-ideal of \( S \).

THEOREM 2.22: Let \( S \) be a commutative \( \Gamma \)-semigroup. A \( \Gamma \)-ideal \( P \) of \( S \) is prime \( \Gamma \)-ideal if and only if \( P \) is a completely prime \( \Gamma \)-ideal.

DEFINITION 2.23: A \( \Gamma \)-ideal \( A \) of a \( \Gamma \)-semigroup \( S \) is said to be a **completely semiprime \( \Gamma \)-ideal** provided \( x\Gamma X \subseteq A \); \( x \in S \) implies \( x \in A \).

THEOREM 2.24: Every completely prime \( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( S \) is a completely semiprime \( \Gamma \)-ideal of \( S \).

THEOREM 2.25: The nonempty intersection of any family completely prime \( \Gamma \)-ideals of a \( \Gamma \)-semigroup \( S \) is a completely semiprime \( \Gamma \)-ideal of \( S \).

DEFINITION 2.26: A \( \Gamma \)-ideal \( A \) of a \( \Gamma \)-semigroup \( S \) is said to be a **semiprime \( \Gamma \)-ideal** provided \( x \in S \), \( x\Gamma S^1 \Gamma x \subseteq A \) implies \( x \in A \).

THEOREM 2.27: Every completely semiprime \( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( S \) is a semiprime \( \Gamma \)-ideal of \( S \).

THEOREM 2.28: Let \( S \) be a commutative \( \Gamma \)-semigroup. A \( \Gamma \)-ideal \( A \) of \( S \) is completely semiprime iff semiprime.

THEOREM 2.29: Every prime \( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( S \) is a semiprime \( \Gamma \)-ideal of \( S \).
THEOREM 2.30: The nonempty intersection of any family of prime \( \Gamma \)-ideals of a \( \Gamma \)-semigroup \( S \) is a semiprime \( \Gamma \)-ideal of \( S \).

NOTATION 2.31: If \( A \) is a \( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( S \), then we associate the following four types of sets.

\( A_1 = \) The intersection of all completely prime \( \Gamma \)-ideals of \( S \) containing \( A \).

\( A_2 = \{ x \in S : (\lambda\Gamma)^n \subseteq A \text{ for some natural number } n \} \)

\( A_3 = \) The intersection of all prime ideals of \( S \) containing \( A \).

\( A_4 = \{ x \in S : (\langle x \rangle \Gamma)^n \subseteq A \text{ for some natural number } n \} \)

THEOREM 2.32: If \( A \) is a \( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( S \), then \( A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1 \).

THEOREM 2.33: If \( A \) is a \( \Gamma \)-ideal in a duo \( \Gamma \)-semigroup \( S \) then \( A_1 = A_2 = A_3 = A_4 \).

DEFINITION 2.34: If \( A \) is a \( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( S \), then the intersection of all prime \( \Gamma \)-ideals of \( S \) containing \( A \) is called \( \text{prime } \Gamma \)-radical or simply \( \Gamma \)-radical of \( A \) and it is denoted by \( \sqrt{A} \) or \( \text{rad } A \).

DEFINITION 2.35: If \( A \) is a \( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( S \), then the intersection of all completely prime \( \Gamma \)-ideals of \( S \) containing \( A \) is called \( \text{complete prime } \Gamma \)-radical or simply \( \text{complete } \Gamma \)-radical of \( A \) and it is denoted by \( \text{c. rad } A \).

NOTE 2.36: If \( A \) is a \( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( S \) then \( \text{rad } A = A_3 \) and \( \text{c. rad } A = A_4 \).

THEOREM 2.37: If \( A \) is a \( \Gamma \)-ideal of a duo \( \Gamma \)-semigroup \( S \), then \( \text{rad } A = \text{c. rad } A \).

DEFINITION 2.38: A \( \Gamma \)-ideal \( A \) of a \( \Gamma \)-semigroup \( S \) is said to be a \text{left primary } \Gamma \)-ideal provided

i) If \( X, Y \) are two \( \Gamma \)-ideals of \( S \) such that \( X\Gamma Y \subseteq A \) and \( Y \not\subseteq A \) then \( X \subseteq \sqrt{A} \).

ii) \( \sqrt{A} \) is a prime \( \Gamma \)-ideal of \( S \).

DEFINITION 2.39: A \( \Gamma \)-ideal \( A \) of a \( \Gamma \)-semigroup \( S \) is said to be a \text{right primary } \Gamma \)-ideal provided

i) If \( X, Y \) are two \( \Gamma \)-ideals of \( S \) such that \( X\Gamma Y \subseteq A \) and \( X \not\subseteq A \) then \( Y \subseteq \sqrt{A} \).

ii) \( \sqrt{A} \) is a prime \( \Gamma \)-ideal of \( S \).

DEFINITION 2.40: A \( \Gamma \)-ideal \( A \) of a \( \Gamma \)-semigroup \( S \) is said to be a \text{primary } \Gamma \)-ideal provided \( A \) is both a left primary \( \Gamma \)-ideal and a right primary \( \Gamma \)-ideal.

THEOREM 2.41: Let \( A \) be a \( \Gamma \)-ideal of a \( \Gamma \)-semigroup \( S \). Then \( X, Y \) are two \( \Gamma \)-ideals of \( S \) such that \( X\Gamma Y \subseteq A \) and \( Y \not\subseteq A \Rightarrow X \subseteq \sqrt{A} \) if and only if \( x, y \in S \), \( \langle x \rangle \Gamma \langle y \rangle \subseteq A \) and \( y \not\in A \Rightarrow x \in \sqrt{A} \).
THEOREM 2.42: Let $A$ be a $\Gamma$-ideal of a $\Gamma$-semigroup $S$. Then $X$, $Y$ are two $\Gamma$-ideals of $S$ such that $X \Gamma Y \subseteq A$ and $X \not\subseteq A \Rightarrow Y \subseteq \sqrt{A}$ if and only if $x, y \in S$, $< x > \Gamma < y > \subseteq A$ and $x \not\in A \Rightarrow y \in \sqrt{A}$.

DEFINITION 2.43: A $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$ is said to be semiprimary provided $\sqrt{A}$ is a prime $\Gamma$-ideal of $S$.

DEFINITION 2.44: A $\Gamma$-semigroup $S$ is said to be a semiprimary $\Gamma$-semigroup provided every $\Gamma$-ideal of $S$ is a semiprimary $\Gamma$-ideal.

THEOREM 2.45: Every left primary or right primary $\Gamma$-ideal of a $\Gamma$-semigroup is a semiprimary $\Gamma$-ideal.

DEFINITION 2.46: An element $a$ of $\Gamma$-semigroup $S$ is said to be semisimple provided $a \in < a > \Gamma < a >$, that is, $< a > \Gamma < a > = < a >$.

DEFINITION 2.47: A $\Gamma$-semigroup $S$ is said to be semisimple $\Gamma$-semigroup provided every element is a semisimple.

DEFINITION 2.48: An element $a$ of $\Gamma$-semigroup $S$ is said to be an $\alpha$-idempotent if $a \alpha a = a$ for all $\alpha \in \Gamma$.

DEFINITION 2.49: A $\Gamma$-semigroup $S$ is said to be an idempotent $\Gamma$-semigroup provided every element is an idempotent.

DEFINITION 2.50: An element $a$ of a $\Gamma$-semigroup $S$ is said to be regular provided $a = a \alpha x \beta a$, for some $x \in S$, $\alpha, \beta \in \Gamma$. i.e., $a \in a \Gamma \Gamma a$.

DEFINITION 2.51: A $\Gamma$-semigroup $S$ is said to be a strongly idempotent $\Gamma$-semigroup provided every element in $S$ is an idempotent.

DEFINITION 2.52: An element $a$ of a $\Gamma$-semigroup $S$ is called an idempotent, then $a \Gamma a = a$.

DEFINITION 2.53: A $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$ is said to be a maximal $\Gamma$-ideal provided $A$ is a proper $\Gamma$-ideal of $S$ and $A$ is not properly contained in any proper $\Gamma$-ideal of $S$.

THEOREM 2.58: If $S$ is a duo $\Gamma$-semigroup, then the following are equivalent for any element $a \in S$.

1) $a$ is completely regular.
2) $a$ is regular. 
3) $a$ is left regular. 
4) $a$ is right regular. 
5) $a$ is intra regular. 
6) $a$ is semisimple.

**THEOREM 2.59:** If a $\Gamma$-semigroup $S$ contains regular elements then $S$ contains idempotents.

**DEFINITION 2.60:** A $\Gamma$- semigroup $S$ is said to be an archimedean $\Gamma$- semigroup provided for any $a, b \in S$, there exists a natural number $n$ such that $(a\Gamma)^{n-1}a \subseteq <b>$.

**DEFINITION 2.61:** A $\Gamma$-semigroup $S$ is said to be a strongly archimedean $\Gamma$-semigroup provided for any $a, b \in S$, there exists a natural number $n$ such that $(a\Gamma)^{n-1}a \subseteq <b>$.

**THEOREM 2.62:** If $S$ is a duo $\Gamma$-semigroup, then the conditions (1) $S$ is strongly Archimedean, (2) $S$ is Archimedean and (3) $S$ has no proper prime $\Gamma$-ideals are equivalent.

3. DUO CHAINED $\Gamma$-SEMIGROUP :

**DEFINITION 3.1:** A $\Gamma$-semigroup $S$ is said to be a chained $\Gamma$-semigroup if the $\Gamma$-ideals in $S$ are linearly ordered by set inclusion.

**THEOREM 3.2:** Let $S$ be a duo chained $\Gamma$-semigroup and $x \in S$. If $P$ is a prime $\Gamma$-ideal of $S$ and $x \not\in P$ then $P = \bigcap_{n=1}^{\infty} (x\Gamma)^{n}P$.

**Proof:** Since $x \not\in P$ and $P$ is prime, $(x\Gamma)^{n-1}x \not\subseteq P$ for all natural numbers $n$. Since $(x\Gamma)^{n-1}x \subseteq S$ and $P$ is a $\Gamma$-ideal of $S$, it follows that $(x\Gamma)^{n-1}xP \subseteq P$ for all natural numbers $n$ and hence $(x\Gamma)^{n}P \subseteq P$ for all natural numbers $n$. Therefore $\bigcap_{n=1}^{\infty} (x\Gamma)^{n}P \subseteq P$. Since $S$ is a duo chained $\Gamma$-semigroup, $(x\Gamma)^{n}S^{1} \subseteq P$ and since $S$ is a chained $\Gamma$-semigroup, $P \subseteq (x\Gamma)^{n}S^{1}$ for all natural numbers $n$. Let $y \in P$. Then $y \in (x\Gamma)^{n}S^{1}$. Therefore $y \in (x\Gamma)^{n}z$ for some $z \in S^{1}$. Therefore $y = x\alpha_{1}x\alpha_{2}...x\alpha_{n}z$ for some $z \in S^{1}$ and $\alpha_{1}, \alpha_{2}, ..., \alpha_{n} \in \Gamma$. Since $P$ is prime, $y = x\alpha_{1}x\alpha_{2}...x\alpha_{n}z \in P$, $x \not\in P$, we get $z \in P$. Therefore $y \in (x\Gamma)^{n}P$ for all natural numbers $n$. Hence $P \subseteq \bigcap_{n=1}^{\infty} (x\Gamma)^{n}P$. Therefore $P = \bigcap_{n=1}^{\infty} (x\Gamma)^{n}P$.

**THEOREM 3.3:** If $S$ is a duo chained $\Gamma$-semigroup, then $S$ is a semiprimary $\Gamma$-semigroup.

**Proof:** Let $A$ be a $\Gamma$-ideal of $S$. We have $\sqrt{A} = \bigcap_{\alpha \in \Delta} P_{\alpha}$ = Intersection of all prime $\Gamma$-ideals of $S$ containing $A$. Since $S$ is a duo chained $\Gamma$-semigroup, we have $\{P_{\alpha} : \alpha \in \Delta\}$
forms a chain. By Zorn’s lemma \( \{ P_{\alpha} : \alpha \in \Delta \} \) has a minimal element say \( P_\beta \). Therefore \( \sqrt{A} = P_\beta \) and \( P_\beta \) is a prime \( \Gamma \)-ideal of \( S \) and hence \( \sqrt{A} \) is prime. Therefore \( A \) is a semiprimary \( \Gamma \)-ideal of \( S \) and hence \( S \) is a semiprimary \( \Gamma \)-semigroup.

**NOTE 3.4:** If \( S \) is a \( \Gamma \)-semigroup and \( a \in S \) then we denote \( < a >^w = \bigcap_{n=1}^{\infty} (a \Gamma)^{n-1} a < a >. \)

**NOTE 3.5:** If \( S \) is a duo \( \Gamma \)-semigroup then \( < a >^w = \bigcap_{n=1}^{\infty} (a \Gamma)^{n-1} a = \bigcap_{n=1}^{\infty} (a \Gamma)^{n-1} a \Gamma S^1 \)

**THEOREM 3.6:** Let \( S \) be duo chained \( \Gamma \)-semigroup. If \( a \in S \) is a semisimple element of \( S \), then \( < a >^w \neq \emptyset \).

**Proof:** Suppose that \( a \) is a semisimple element of \( S \). Therefore \( a \in < a > \Gamma < a > \), implies that \( < a > = < a > \Gamma < a > \). Therefore \( a \in < a > = (a \Gamma)^{n-1} a \) for all natural numbers \( n \).

Hence \( a \in \bigcap_{n=1}^{\infty} (a \Gamma)^{n-1} a = a < a >^w \) and hence \( a < a >^w \neq \emptyset \).

**THEOREM 3.7:** Let \( S \) be a duo chained \( \Gamma \)-semigroup. If \( < a >^w = \emptyset \) for all \( a \in S \), then \( S \) has no semisimple elements.

**Proof:** Suppose that \( < a >^w = \emptyset \) for all \( a \in S \). Suppose if possible \( S \) has a semisimple element \( a \). By theorem 3.6, \( a < a >^w \neq \emptyset \). It is a contradiction. Therefore \( S \) has no semisimple elements.

**THEOREM 3.8:** Let \( S \) be a duo chained \( \Gamma \)-semigroup. If \( S \) has no \( \Gamma \)-idempotents elements, then for any \( a \in S \), \( < a >^w \neq \emptyset \) or \( < a >^w \) is a prime \( \Gamma \)-ideal of \( S \).

**Proof:** Suppose that \( S \) has no \( \Gamma \)-idempotent elements and \( a \in S \). We have \( < a >^w = \bigcap_{n=1}^{\infty} (a \Gamma)^{n-1} a < a > \). Assume that \( < a >^w \neq \emptyset \). If possible suppose that \( < a >^w \) is not prime. Then there exists \( x, y \in S \) such that \( x \Gamma y \subseteq < a >^w \), \( x \not\in < a >^w \) and \( y \not\in < a >^w \). By theorem 2.17, \( x \Gamma y = < x \Gamma y > \subseteq < a >^w \).

Now \( x, y \not\in < a >^w \), implies that there exists natural numbers \( n, m \) such that

\[
x \not\in < a >^n \subseteq < a >, \ y \not\in < a >^m \subseteq < a >.
\]

Consider \( k = \min \{ n, m \} \).

Then \( x, y \not\in < a >^k \subseteq < a > \). Since \( S \) is a duo chained \( \Gamma \)-semigroup, we have \( < a >^k \subseteq < a > \subseteq < x > \) and \( < a >^k \subseteq < a > \subseteq < y > \).

Therefore \( < a >^k \subseteq < a > \subseteq < a >^k \subseteq < a > \subseteq < x > \Gamma < y > \subseteq < x > \Gamma y > \subseteq < a >^k \subseteq < a >^k \subseteq < a > \Gamma < a >^k \subseteq < a >^k < a > \). Therefore \( < a >^k < a > \subseteq < a >^k < a > \subseteq < a >^k < a > \Gamma < a >^k < a > \).

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Therefore $a^{2k}$ is a semisimple element of $S$. By theorem 2.58, $a^{2k}$ is a regular element of $S$. Therefore $a^{2k} = a^{2k} \Gamma x a^{2k}$ for some $x \in S$, implies that $(a^{2k} \Gamma x) (a^{2k} \Gamma x) = a^{2k} \Gamma x$ and hence $a^{2k} \Gamma x$ is a $\Gamma$-idempotent of $S$. So $S$ has $\Gamma$-idempotent elements.

It is a contradiction. Hence $< a > ^w$ is a prime $\Gamma$-ideal of $S$.

**DEFINITION 3.9**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be left $\alpha$-cancellative provided for $\alpha \in \Gamma$, $aab = aac$ implies $b = c$.

**DEFINITION 3.10**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be right $\alpha$-cancellative provided for $\alpha \in \Gamma$, $baa = caa$ implies $b = c$.

**DEFINITION 3.11**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be $\alpha$-cancellative provided $a$ is both a left $\alpha$-cancellative element and a right $\alpha$-cancellative element.

**DEFINITION 3.12**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be left $\Gamma$-cancellative provided $a$ is left $\alpha$-cancellative for all $\alpha \in \Gamma$.

**DEFINITION 3.13**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be right $\Gamma$-cancellative provided $a$ is right $\alpha$-cancellative for all $\alpha \in \Gamma$.

**DEFINITION 3.14**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be $\Gamma$-cancellative provided $a$ is a both left $\Gamma$-cancellative and $\Gamma$-cancellative.

**DEFINITION 3.15**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be strongly left $\Gamma$-cancellative provided $a \Gamma b = a \Gamma c$ implies $b = c$.

**NOTE 3.16**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be strongly left $\Gamma$-cancellative provided $aab = a\beta c$, $\alpha, \beta \in \Gamma \Rightarrow b = c$.

**DEFINITION 3.17**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be strongly right $\Gamma$-cancellative provided $b \Gamma a = c \Gamma a$ implies $b = c$.

**NOTE 3.18**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be strongly right $\Gamma$-cancellative provided $a ba = c \beta a$, $\alpha, \beta \in \Gamma \Rightarrow b = c$.

**DEFINITION 3.19**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be strongly $\Gamma$-cancellative provided $a$ is a both strongly left $\Gamma$-cancellative and strongly right $\Gamma$-cancellative.

**DEFINITION 3.20**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be a left identity of $S$ provided $aas = s$ for all $s \in S$ and $a \in \Gamma$.

**DEFINITION 3.21**: An element ‘$a$’ of a $\Gamma$-semigroup $S$ is said to be a right identity of $S$ provided $saa = s$ for all $s \in S$ and $a \in \Gamma$.

**DEFINITION 3.22**: An element ‘$a$’ of a $\Gamma$-semigroup $S$ is said to be a two sided identity or an identity provided it is both a left identity and a right identity of $S$.

**THEOREM 3.23**: If $a$ is a left identity and $b$ is a right identity of a $\Gamma$-semigroup $S$, then $a = b$. 

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THEOREM 3.24: Any \( \Gamma \)-semigroup \( S \) has at most one identity.

Proof: Let \( a, b \) be two identity elements of the \( \Gamma \)-semigroup \( S \). Now \( a \) can be considered as a left identity and \( b \) can be considered as a right identity of \( S \). By theorem 3.23, \( a = b \). Then \( S \) has at most one identity.

NOTE 3.25: The identity (if exists) of a \( \Gamma \)-semigroup is usually denoted by \( e \) or \( 1 \).

DEFINITION 3.26: An element \( b \) of a \( \Gamma \)-semigroup \( S \) is said to be a left \( \alpha \)-inverse of \( a \) of a \( \Gamma \)-semigroup \( S \) provided for \( \alpha \in \Gamma \), \( b\alpha a = e \).

DEFINITION 3.27: An element \( b \) of a \( \Gamma \)-semigroup \( S \) is said to be a right \( \alpha \)-inverse of \( a \) of a \( \Gamma \)-semigroup \( S \) provided for \( \alpha \in \Gamma \), \( a\alpha b = e \).

THEOREM 3.28: If \( b \) is a left \( \alpha \)-inverse and \( c \) is a right \( \alpha \)-inverse of an element \( a \) of a \( \Gamma \)-semigroup \( S \), then \( b = c \).

Proof: Since \( b \) is a left \( \alpha \)-inverse of an element \( a \) in \( S \), \( b\alpha a = e \) and \( c \) is a right \( \alpha \)-inverse of an element \( a \) in \( S \), \( a\alpha c = e \) for \( \alpha \in \Gamma \).

Now \( b = b\alpha e = b\alpha (a\alpha c) = (b\alpha a)c = eac = c \).

DEFINITION 3.29: An element \( b \) of a \( \Gamma \)-semigroup \( S \) is said to be a \( \alpha \)-inverse of \( a \) of a \( \Gamma \)-semigroup \( S \) provided for \( \alpha \in \Gamma \), \( a\alpha b = b\alpha a = e \).

THEOREM 3.30: The \( \alpha \)-inverse of an element \( a \) in a \( \Gamma \)-semigroup \( S \) (if exists) is unique.

Proof: Let \( b, c \) be two \( \alpha \)-inverse elements of an element \( a \) in a \( \Gamma \)-semigroup \( S \). If \( b \) is a \( \alpha \)-inverse of \( a \) then \( a\alpha b = b\alpha a = e \) and if \( c \) is a \( \alpha \)-inverse of \( a \) then \( a\alpha c = c\alpha a = e \).

Now \( b = b\alpha e = b\alpha (a\alpha c) = (b\alpha a)c = eac = c \).

DEFINITION 3.31: An element \( b \) of a \( \Gamma \)-semigroup \( S \) is said to be a left \( \Gamma \)-inverse of \( a \) of a \( \Gamma \)-semigroup \( S \) provided \( b\Gamma a = e \) for all \( \alpha \in \Gamma \).

NOTE 3.32: An element \( b \) of a \( \Gamma \)-semigroup \( S \) is said to be a left \( \Gamma \)-inverse of \( a \) of a \( \Gamma \)-semigroup \( S \) provided \( b\Gamma a = e \).

DEFINITION 3.33: An element \( b \) of a \( \Gamma \)-semigroup \( S \) is said to be a right \( \Gamma \)-inverse of \( a \) of a \( \Gamma \)-semigroup \( S \) provided \( a\Gamma b = e \) for all \( \alpha \in \Gamma \).

NOTE 3.34: An element \( b \) of a \( \Gamma \)-semigroup \( S \) is said to be a right \( \Gamma \)-inverse of \( a \) of a \( \Gamma \)-semigroup \( S \) provided \( a\Gamma b = e \).

THEOREM 3.35: If \( b \) is a left \( \Gamma \)-inverse and \( c \) is a right \( \Gamma \)-inverse of an element \( a \) of a \( \Gamma \)-semigroup \( S \), then \( b = c \).
**Theorem 3.38**: The $\Gamma$-inverse of an element $a$ in a $\Gamma$-semigroup $S$ (if exists) is unique.

**Proof**: Let $b$, $c$ be two $\Gamma$-inverse elements of an element $a$ in a $\Gamma$-semigroup $S$. If $b$ is a $\Gamma$-inverse of $a$ then $a\Gamma b = b\Gamma a = e$ and if $c$ is a $\Gamma$-inverse of $a$ then $a\Gamma c = c\Gamma a = e$.

Now $b = b\Gamma e = b\Gamma(a\Gamma c) = (b\Gamma a)c\Gamma = e\Gamma c = c$.

**Definition 3.39**: An element $a$ of a $\Gamma$-semigroup $S$ is said to be a *unit* if it has $\Gamma$-inverse.

**Theorem 3.40**: If $S$ is a duo chained strongly cancellative $\Gamma$-semigroup with an identity then for every nonunit $a$, $<a>^w$ is either empty or a prime $\Gamma$-ideal of $S$.

**Proof**: Suppose that $a$ is a nonunit in $S$. If $<a>^w = \emptyset$ then the proof is trivial.

Let $<a>^w \neq \emptyset$. If possible suppose that $<a>^w$ is not a prime $\Gamma$-ideal of $S$.

Then there exists $x, y \in S$ such that $x\Gamma y \subseteq <a>^w$ and $x, y \notin <a>^w$.

By theorem 2.17, $<x>\Gamma <y> = <x\Gamma y > \subseteq <a>^w$. Now $x, y \notin <a>^w$, implies that there exists natural numbers $n, m$ such that $x \notin (a \Gamma)^{k-1} <a>$ and $y \notin (a > \Gamma)^{m-1} <a>$.

Consider $k = \min\{n, m\}$. Then $x, y \notin (a > \Gamma)^{k-1} <a>$.

Since $S$ is duo chained $\Gamma$-semigroup, we have $(a \Gamma)^{k-1} <a> \subseteq <x >$ and $(<a> \Gamma)^{k-1} <a> \subseteq <y >$.

Therefore $(a > \Gamma)^{2k-1} <a> = (a > \Gamma)^{k-1} <a> > (a > \Gamma)^{k-1} <a> \subseteq <x > \Gamma <y >$ and hence $a^{2k} \in (a > \Gamma)^{2k-1} <a>$.

Therefore $a^{2k}$ is a semisimple element of $S$.

By theorem 2.58, $a^{2k}$ is a regular element of $S$.

Therefore $a^{2k} = a^{2k} \Gamma x \Gamma a^{2k}$ for some $x \in S$ implies that $(a^{2k} \Gamma x) \Gamma (a^{2k} \Gamma x) = a^{2k} \Gamma x$ and hence $S$ has $\Gamma$-idempotent elements. Since $S$ is strongly cancellative and $a^{2k} \Gamma x \Gamma e = (a^{2k} \Gamma x) \Gamma (a^{2k} \Gamma x)$ implies that $a^{2k} \Gamma x = e$ and hence $a\Gamma(a^{2k-1} \Gamma x) = e$.

Hence $a$ is a unit in $S$. It is a contradiction. Thus $<a>^w$ is a prime $\Gamma$-ideal of $S$. Hence $<a>^w = \emptyset$ or $<a>^w$ is a prime $\Gamma$-ideal of $S$.

**Theorem 3.41**: Let $S$ be a duo chained $\Gamma$-semigroup. If $S \neq S \Gamma S$ then $S\Gamma S = \{x\}$ for some $x \in S$. 

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Proof : Suppose if possible $x, y \in S \subseteq S$ and $x \neq y$. Since $S$ is a chained $\Gamma$-semigroup, $\langle x \rangle \leq \langle y \rangle$ or $\langle y \rangle \leq \langle x \rangle$. If $\langle x \rangle \leq \langle y \rangle$ then $x \in \langle y \rangle$ and hence $x \in \gamma\Gamma S$ for some $s \in S$. Therefore $x \in S \Gamma S$, which is not true. If $\langle y \rangle \leq \langle x \rangle$, then $y \in \langle x \rangle$ and hence $y \in \gamma\Gamma S$ for some $s \in S$. Therefore $y \in S \Gamma S$, which is not true. It is not a contradiction. Therefore $x = y$. So there exists unique $x \in S$ such that $x \not\in S \Gamma S$. Therefore $S \setminus S \Gamma S = \{ x \}$ for some $x \in S$.

**THEOREM 3.42** : Let $S$ be a duo chained $\Gamma$-semigroup with $S \setminus S \Gamma S = \{ x \}$ for some $x \in S$. Then $S \setminus \{ x \}$ is a $\Gamma$-ideal of $S$.

Proof : Let $a \in S \setminus \{ x \}$ and $s \in S$. Since $\{x\} \not\subset S \Gamma S$ we have $\alpha \Gamma s \neq \{x\}$ and hence $a \Gamma s \subseteq S \setminus \{ x \}$. Therefore $S \setminus \{ x \}$ is a right $\Gamma$-ideal of $S$. Since $S$ is a duo $\Gamma$-semigroup, $S \setminus \{ x \}$ is a $\Gamma$-ideal of $S$.

**THEOREM 3.43** : Let $S$ be a duo chained $\Gamma$-semigroup. If $S \neq S \Gamma S$ such that $S \setminus S \Gamma S = \{ x \}$ for some $x \in S$ then $S = x \Gamma S^1 = S^1 \Gamma x$ and $S \Gamma S = x \Gamma S = S \Gamma x$ is the unique maximal $\Gamma$-ideal of $S$.

Proof : Since $S \setminus S \Gamma S = \{ x \}$, $S \Gamma S = S \setminus \{ x \}$. Now $x \Gamma S^1$ is a $\Gamma$-ideal of $S$ and $S \Gamma S$ is a $\Gamma$-ideal of $S$. Since $\{x\} \not\subset S \Gamma S$ and since $S$ is a chained $\Gamma$-semigroup, $S \Gamma S \subseteq x \Gamma S^1$. So $x \Gamma S \subseteq S \Gamma S \subseteq \langle x \rangle \Gamma S \cup \{ x \}$ and $\{x\} \not\subset S \Gamma S$. Thus $S \Gamma S = x \Gamma S$. So $S = x \Gamma S^1 = S^1 \Gamma x$ and $S \Gamma S = x \Gamma S = S \Gamma x$. Since $S \Gamma S$ is trivial, $S \Gamma S = x \Gamma S = S \Gamma x$ is a maximal $\Gamma$-ideal.

Since $S$ is a chained $\Gamma$-semigroup, $S \Gamma S = x \Gamma S = S \Gamma x$ is the unique maximal $\Gamma$-ideal of $S$.

**THEOREM 3.44** : Let $S$ be a duo chained $\Gamma$-semigroup with $S \neq S \Gamma S$ such that $S \setminus S \Gamma S = \{ x \}$ for some $x \in S$. If $a \in S$ and $a \notin \langle x \rangle$ then $a \in (x\Gamma)^{n-1} x$ for some natural number $n > 1$.

Proof : Since $S$ is a duo chained $\Gamma$-semigroup with $S \neq S \Gamma S$ such that $S \setminus S \Gamma S = \{ x \}$ for some $x \in S$, by theorem 3.28, $S \Gamma S = x \Gamma S = S \Gamma x = S \setminus \{ x \}$.

Since $a \notin \langle x \rangle^\omega \subseteq \bigcup_{n=1}^{\infty} \langle x\Gamma \rangle^{n-1} x$, there exists a natural number $k$ such that $a \notin \langle x\Gamma \rangle^{k-1} x$.

Let $n$ be the least positive integer such that $a \notin \langle x\Gamma \rangle^{n-1} x$ and $a \in \langle x\Gamma \rangle^{n-2} x$. Now $a \in (x\Gamma)^{n-2} \Gamma S^1$ and $a \notin (x\Gamma)^{n-1} \Gamma S^1$.

Now $a \in (x\Gamma)^{n-2} \Gamma S^1 \Rightarrow a \in (x\Gamma)^{n-2} \Gamma S$ for some $s \in S$. $s \in S$, $s \neq x \Rightarrow s \in x \Gamma S$ Therefore $a \in (x\Gamma)^{n-2} \Gamma x \Gamma S = (x\Gamma)^{n-1} x \Gamma S$. It is a contradiction and hence $s = x$. Therefore $a \in (x\Gamma)^{n-2} \Gamma x = (x\Gamma)^{n-1} x$.

**THEOREM 3.45** : Let $S$ be a duo chained $\Gamma$-semigroup with $S \neq S \Gamma S$ such that $S \setminus S \Gamma S = \{ x \}$ for some $x \in S$. If $a \in S$ and $a \notin \langle x \rangle^\omega$ then $a \in (x\Gamma)^{r-1} x$ for some natural number $r$ or $a \in (x\Gamma)^{n-1} x \Gamma S_n$, $s_n \in \langle x \rangle^\omega$ for all natural numbers $n$.
Proof: Since $S$ is a duo chained $\Gamma$-semigroup with $S \neq S \Gamma S$ such that $x \in S \Gamma S \Gamma S$, by theorem 3.44, $S \Gamma S = x \Gamma S = S \Gamma x = S \{x\}$. Let $a \in S$. Suppose that $a \in <x>^\omega$. Then $a \in \bigcap_{n=1}^\infty (x \Gamma)^{n-1} x \Gamma S^1$. Therefore $a \in (x \Gamma)^{n-1} x \Gamma S^1$ for all natural numbers $n$. Hence $a \in (x \Gamma)^{n-1} x$ or $a \in (x \Gamma)^{n-1} x S_n$ for some $s_n \in S$. If $s_n \notin <x>^\omega$ then by theorem 3.44, $s_n \in (x \Gamma)^{r-1} x$ for some natural number $r$ and hence $a \in (x \Gamma)^{n-1} x \Gamma (x \Gamma)^{r-1} x = (x \Gamma)^{n+r-1} x$. If $s_n \notin <x>^\omega$ then $a \in (x \Gamma)^{n-1} x \Gamma S_n$.

**THEOREM 3.46**: Let $S$ be a duo chained $\Gamma$-semigroup with $S \Gamma S = \{x\}$ for some $x \in S$. If $S$ contains strongly cancelable elements then $x$ is a strongly cancellable element and $<x>^\omega$ is either empty or a prime $\Gamma$-ideal of $S$.

Proof: Suppose if possible $x$ is not strongly cancellable element in $S$. Let $Z$ be the set of all non strongly cancellable elements of $S$. Clearly $x \in Z$. So $Z$ is nonempty subset of $S$. Let $a \in Z$ and $s \in S$. Since $a \in Z$, $a$ is not strongly cancellable in $S$. So there exists $b, c \in S$ such that $a \Gamma b = a \Gamma c$ and $b \neq c$. Now $a \Gamma b = a \Gamma c$ implies $s \Gamma (a \Gamma b) = s \Gamma (a \Gamma c)$ and hence $(s \Gamma a) \Gamma b = (s \Gamma a) \Gamma c$ and $b \neq c$.

Therefore $s \Gamma a$ is a set of nonstrongly cancellable elements of $S$. Therefore $s \Gamma a \subseteq Z$ and hence $Z$ is left $\Gamma$-ideal of $S$. Since $S$ is a duo $\Gamma$-semigroup, $Z$ is a $\Gamma$-ideal of $S$. Since $S \Gamma S \Gamma S = \{x\}$, by theorem 3.27, $S = x \Gamma S^1$. Since $x \in Z$ and $Z$ is a $\Gamma$-ideal of $S$, $Z = S$. It is a contradiction. Therefore $x$ is a strongly cancellable element in $S$.

Suppose that $<x>^\omega = \emptyset$. Let $a, b \in S$ and $a \Gamma b \subseteq <x>^\omega$. Suppose if possible $a \notin <x>^\omega$ and $b \notin <x>^\omega$. Now $a, b \notin <x>^\omega$, by theorem 3.28, $a \in (x \Gamma)^{n-1} x, b \in (x \Gamma)^{m-1} x$ for some natural numbers $n, m$.

Therefore $(x \Gamma)^{n+m-1} x = [(x \Gamma)^{n-1} x] \Gamma [(x \Gamma)^{m-1} x] = a \Gamma b \subseteq <x>^\omega \subseteq (x \Gamma)^{n+m} S$.

Therefore $x \in x \Gamma S \subseteq S \Gamma S$. It is a contradiction.

Therefore either $a \in <x>^\omega$ or $b \in <x>^\omega$ and hence $<x>^\omega$ is a prime $\Gamma$-ideal of $S$.

**THEOREM 3.47**: Let $S$ be a duo chained $\Gamma$-semigroup. Then $S$ is an archimedean $\Gamma$-semigroup without $\Gamma$-idempotents if and only if $<a>^\omega = \emptyset$ for every $a \in S$.

Proof: Suppose that $S$ is an archimedean $\Gamma$-semigroup without $\Gamma$-idempotents. If possible suppose that $<a>^\omega \neq \emptyset$ for some $a \in S$. By theorem 3.8, $<a>^\omega$ is a prime $\Gamma$-ideal of $S$. Since $S$ is an archimedean duo $\Gamma$-semigroup, by theorem 2.57, $S$ has no proper prime $\Gamma$-ideals. Therefore $<a>^\omega = S$. Now $a \in <a>^\omega \subseteq <a> \Gamma <a>$. Thus $a$ is semisimple element. By theorem 2.55, $a$ is regular element. By theorem 2.56, $S$ has $\Gamma$-idempotent elements. It is a contradiction. Hence $<a>^\omega = \emptyset$ for every $a \in S$. Conversely suppose that $<a>^\omega = \emptyset$ for every $a \in S$. Since $<a>^\omega = \emptyset$ for every $a \in S$, by corollary 3.6, $S$ has no semi simple elements. By theorem 2.55, $S$ has no regular elements. By theorem 2.59, $S$ has no $\Gamma$-idempotent elements. If possible, suppose that $P$ is proper prime $\Gamma$-ideal of $S$. Let $x \in S$ such that $x \notin P$. Since $x \notin P$ by theorem 3.2, $P = \bigcap_{n=1}^\infty (x \Gamma)^{n} P$. Therefore $P \subseteq <a>^\omega = \emptyset$. It is a contradiction. Hence $S$ has no proper prime $\Gamma$-ideals. By theorem 2.62, $S$ is an archimedean $\Gamma$-semigroup.
DEFINITION 3.48: A $\Gamma$-semigroup $S$ is said to be a $\Gamma$-group if
(1) \( \exists \, e \in S \Rightarrow a^\Gamma e = e^\Gamma a = a \) for all \( a \in S \).
(2) every element \( a \in S \) has an \( \alpha \)-inverse in \( S \) for some \( \alpha \in \Gamma \).

THEOREM 3.49: Let \( S \) be a strongly cancellative archimedean duo chained $\Gamma$-semigroup with \( < a >^\wedge \neq \emptyset \) for some \( a \in S \), then \( S \) is a $\Gamma$-group.

Proof: Suppose that \( S \) is a strongly cancellative archimedean duo chained $\Gamma$-semigroup with \( < a >^\wedge \neq \emptyset \) for some \( a \in S \). Suppose if possible \( S \) has no $\Gamma$-idempotent elements. Since \( < a >^\wedge \neq \emptyset \), by theorem 3.7, \( < a >^\wedge \) is a prime $\Gamma$-ideal of \( S \). Since \( S \) is an archimedean duo $\Gamma$-semigroup, by theorem 2.64, \( S \) has no proper prime $\Gamma$-ideals. It is a contradiction. Hence \( S \) has $\Gamma$-idempotent elements. Let \( e \) be a $\Gamma$-idempotent element in \( S \). Then \( x\alpha(e^\beta e) = xae \) for every \( x \in S \) and \( \alpha, \beta \in \Gamma \). Since \( S \) is strongly cancellative, we have \( xae = x \) for every \( x \in S, \alpha \in \Gamma \). Similarly \( eax = x \) for every \( x \in S, \alpha \in \Gamma \). Therefore \( e\Gamma x = x\Gamma e = x \). Hence \( e \) is the identity element in \( S \). Let \( a \in S \). Since \( e, a \in S \) and \( S \) is archimedean $\Gamma$-semigroup, \( (e\Gamma)^{-1} e \subseteq S\Gamma a\Gamma S \). Therefore \( e \in S\Gamma a\Gamma S \).

Since \( S \) is duo $\Gamma$-semigroup \( S\Gamma a\Gamma S = (S\Gamma S)\Gamma a = a\Gamma(S\Gamma S) \). Therefore \( e \in (S\Gamma S)\Gamma a \) and hence \( e = x\alpha a \) for some \( x \in S, \alpha \in \Gamma \) and \( S = S\Gamma S \subseteq S \). Now \( e = eae = (x\alpha a)\alpha (x\alpha a) \) implies that \( x\alpha (aax)aa = e = x\alpha a = xa(eaa) \). Since \( S \) is strongly cancellative, we have \( aax = e \).

Similarly \( x\alpha a = e \). Therefore \( x\alpha a = aax = e \) and hence \( x \) is the $\alpha$-inverse of \( a \) in \( S \). Therefore \( S \) is a $\Gamma$-group.

THEOREM 3.50: Let \( S \) be a duo chained $\Gamma$-semigroup containing strongly cancellative elements and \( < a >^\wedge = \emptyset \) for every \( a \in S \), then \( S \) is a strongly cancellative $\Gamma$-semigroup.

Proof: Let \( S \) be a duo chained $\Gamma$-semigroup containing strongly cancellable elements. Suppose that \( < a >^\wedge = \emptyset \) for every \( a \in S \). Let \( Z \) be the set of all non strongly $\Gamma$-cancellable elements in \( S \). Suppose if possible \( Z \) is a nonempty subset of \( S \). If \( x \in Z \), then there exists \( y, z \in S, \alpha, \beta \in \Gamma \) such that \( x\alpha y = x\beta z \) and \( y \neq z \). Therefore for any \( s \in S, \gamma \in \Gamma \), \( s\gamma(x\alpha y) = s\gamma(x\beta z) \Rightarrow (s\gamma\alpha y)\beta z = (s\gamma\beta z)\alpha y \) and \( y \neq z \). Hence \( s\gamma y \in Z \). Therefore \( Z \) is a left $\Gamma$-ideal of \( S \) and hence \( Z \) is a $\Gamma$-ideal of \( S \). If possible, suppose that \( Z \) is not prime. Then there exists \( a, b \in S \) such that \( a\gamma b \in Z \) and \( a, b \notin Z \). Since \( a \notin Z \), \( b\gamma c = b\beta d \) for some \( c, d \in S, \alpha, \beta \in \Gamma \). Since \( b \notin Z, c = d \). It is a contradiction. Therefore \( Z \) is a prime $\Gamma$-ideal of \( S \). Since \( < a >^\wedge = \emptyset \) for every \( a \in S \), by theorem 3.47, we have \( S \) is an archimedean $\Gamma$-semigroup without $\Gamma$-idempotents. Therefore by theorem 2.59, \( S \) has no prime $\Gamma$-ideals and hence \( Z = S \). It is a contradiction to \( S \) contains strongly cancellable elements. Hence \( Z = \emptyset \). Thus \( S \) is a strongly cancellative $\Gamma$-semigroup.
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