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NUMERICAL SOLUTION AND STABILITY ANALYSIS OF A CLASS OF THIRD ORDER DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we have discussed solution and stability analysis of a class of third order boundary value problem. We have solved a linear third order differential equation by using our method. It has been noticed that the solution obtained by our method is very nearer to the original solution

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Keywords: Numerical differentiation; Boundary Value Problem; Absolute stability, Multistep methods.

1. Introduction

Differential equations of the third order occur frequently in many fields of science, engineering and applied mathematics. The methods for the numerical solution of initial value problems in ordinary differential equations can be divided into two classes – single step methods and multistep methods. Single step methods are those which require the information about the solution at a single point $x = x_n$ to compute the value of y(x) at the next point $x = x_{n+1}$. Multistep methods are those which require information about the solution at more than one preceding point. Thus a k-step method requires information about the solution at k points $x_n, x_{n-1}, - - - - , x_{n-k+1}$ to compute the solution at the point x_{n+1}. An extensive study of the single step and multistep methods for the initial value problem of ordinary differential equations has been made by several researchers and a detailed treatment of the subject has been provided by many authors like Froberg [3], Gear [4], Gragg and Statter [5], P. Henrici [6] and J. D. Lambert [8]. Special multistep methods based on numerical integration and methods based on numerical differentiation for solving first-order differential equations have been derived in Peter Henrici [6] and Gear [4]. The methods based on numerical differentiation of first-order differential equations have been shown to be stiffly stable by Gear [4] and high order stiffly stable methods were considered by Jain [7]. Further information can be had from [1], [2], [10] and [11]. The motivation for the work carried out in this paper arises from the methods based on numerical differentiation for the special multistep methods based on numerical integration for the solution of the special second-order differential equations have been derived in Henrici [6] and special multistep methods based on numerical differentiation for solving the initial value problem have been derived in Rama Chandra Rao [11]. Special multistep methods have been derived by replacing y(x) on the left hand side of $F(x,\ y,\ y',\ y'',\ y''') = 0$ by an interpolating polynomial and differentiating it three times. We have investigated a class of method. It is found that the derived method has order (k - 2). Local truncation errors are provided. The regions of absolute stability of the methods are derived. We have solved a linear third order differential equation to show the efficiency of the method discussed in this paper. It has been noticed that the solution obtained by our method is very nearer to the original solution

2. General linear multistep methods for special third-order differential equations

The special third order differential equation $F(x,\ y,\ y',\ y'',\ y''')\ = 0$ (1)

frequently occurs in a number of applications of science and engineering.

A general linear multistep method of step number k for the numerical solution of equation (1) is given by $y_{n+1}\ =\ \sum_{j=1}^{k} a_j\, y_{n+1-j}\ +\ h^3 \sum_{j=0}^{k} b_j\, y_{n+1-j}$ (2)

where a_j, b_j are constants and 'h' is the step length.

Introducing the polynomials

$$\rho(\xi) =\ \xi^k - \sum_{j=1}^{k} a_j\, \xi^{k-1} \ \text{ and }\ \sigma(\xi) =\ \sum_{j=1}^{k} b_j\, \xi^{k-1} \quad (3)$$

Equation (2) can be written as

$$\rho(E)\, y_{n-k+1} - h^3\, \sigma(E) y''_{n-k+1} = 0, \text{ where } E(y_n) =\ y_{n+1} \quad (4)$$

Applying (4) to $y'' =\ \lambda y$, we get the characteristic equation

$$\rho(\xi) - \bar{h}\, \sigma(\xi) = 0, \qquad \text{where } \ \bar{h} = \lambda h^3 \quad (5)$$

The roots ξ_i of the characteristic equation (5) and $\bar{h}$ are in general, complex and the region of absolute stability is defined to be the region of the complex $\bar{h}$ - plane such that the roots of the characteristic equation (5) lie within the unit circle whenever $\bar{h}$ lies in the interior of the region. If we denote the region of absolute stability of R and its boundary by ∂R, then the locus of ∂R is given by $\bar{h}(\theta) =\ \rho(e^{i\theta})/\sigma(e^{i\theta}),\ \ 0\ \leq\ \theta\ \leq\ 2\pi$ (6)

3. Derivation of the methods

Let p(x) be the backward difference interpolating polynomial of y(x) at (k+1) abscissas x_{n+1}, x_n, ---------, x_{n-k+1}. Then p(x) is given by

$$p(x) = \sum_{m=0}^{k} (-1)^m \binom{-s}{m} \nabla^m y_{n+1}, \qquad s = \frac{(x - x_{n+1})}{h} \tag{7}$$

Differentiating (7) three times with respect to x, we get

$$p'''(x) = \left(\frac{1}{h^2}\right) \sum_{m=0}^{k} \frac{d^3}{ds^3} \left[(-1^m) \binom{-s}{m}\right] \nabla^m y_{n+1}.$$

Replacing $y'''(x)$ by $p'''(x)$ in equation (1) and putting x = x_{n+1-r} i.e. s = -r , we get,

$$\sum_{m=0}^{k} \delta_{r,\ m} \nabla^m y_{n+1} = h^3 f_{n+1-r} \tag{8}$$

where $$\delta_{r,m} = \frac{d^3}{ds^3}\left[(-1)^m \binom{-s}{m}\right] \tag{9}$$

4. Generating function for the coefficients $\boldsymbol{\delta_{r,m}}$

We define $$D_{r,t} = \sum_{m=0}^{\infty} \delta_{r,m}\, t^m \tag{10}$$

Substituting $\delta_{r,m}$ from (9) in (10) and simplifying, we get

$$D_{r,t} = \sum_{m=0}^{\infty} \delta_{r,m}\, t^m = -(1-t)^{-s} [\log(1-t)]^3$$

$$\therefore \quad \sum_{m=0}^{\infty} \delta_{r,m}\, t^m = (1-t)^r [\log(1-t)]^3 \ \ at\ \ s = -r \tag{11}$$

Taking r = $\frac{1}{2}$ in (8), a class of method can be attained which is given by

$$\sum_{m=0}^{k} \delta_{\frac{1}{2},\ m} \nabla^m y_{n+1} = h^3 f_{n+\frac{1}{2}} \tag{12}$$

From the above equation (12) it follows that $\delta_{\frac{1}{2},\ m}$ is the coefficient of t^m in the expansion of $-(1-t)^{\frac{1}{2}} [\log(1-t)]^3$ in powers of t. The coefficients $\delta_{\frac{1}{2},\ m}$ are shown in table 1.

Differences in (12) are expressed in terms of function values.

After simplification, the equation (12) will turn out into the form

$$\sum_{j=0}^{k} a_j y_{n+1-j} = h^3 f_{n+\frac{1}{2}} \tag{13}$$

The coefficients a_j are shown in table 2.

The local truncation error of the formula (13) is given by

$$LTE = \delta_{0,k+1} h^{k+1} y^{k+1}(\eta) \tag{14}$$

Table 1
Coefficients of $\delta_{\frac{1}{2},\ m}$; $m = 0\ (1)\ 9$

M	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
$\boldsymbol{\delta_{\frac{1}{2},\ m}}$	0	0	0	1	1	$\frac{7}{8}$	$\frac{3}{4}$	$\frac{1237}{1920}$	$\frac{357}{640}$	$\frac{471539}{967680}$

Table 2
Coefficients of α_j ; $j = 0\,(1)\,k,\ k = 3\,(1)\,9$

K	J									
	0	1	2	3	4	5	6	7	8	
3	1	−3	3	−1						
4	2	−7	9	−5	1					
5	$\frac{23}{8}$	$\frac{-91}{8}$	$\frac{142}{8}$	$\frac{-110}{8}$	$\frac{43}{8}$	$\frac{-7}{8}$				
6	$\frac{29}{8}$	$\frac{-127}{8}$	$\frac{232}{8}$	$\frac{-230}{8}$	$\frac{133}{8}$	$\frac{-43}{8}$	$\frac{6}{8}$			
7	$\frac{8197}{1920}$	$\frac{-39139}{1920}$	$\frac{81657}{1920}$	$\frac{-98495}{1920}$	$\frac{75215}{1920}$	$\frac{-36297}{1920}$	$\frac{10099}{1920}$	$\frac{-1237}{1920}$		
8	$\frac{9268}{1920}$	$\frac{-47707}{1920}$	$\frac{111645}{1920}$	$\frac{-158471}{1920}$	$\frac{150185}{1920}$	$\frac{-96273}{1920}$	$\frac{40087}{1920}$	$\frac{-9805}{1920}$	$\frac{1071}{1920}$	
9	$\frac{5142611}{967680}$	$\frac{-28288179}{967680}$	$\frac{73244484}{967680}$	$\frac{-119478660}{967680}$	$\frac{135107154}{967680}$	$\frac{-107935506}{967680}$	$\frac{59813124}{967680}$	$\frac{-21917124}{967680}$	$\frac{4783635}{967680}$	$\frac{-471539}{967680}$

It follows that the k-step method (14) has the order k-2, which is absolutely stable

For the method (13), we have

$$\rho(\xi) = \sum_{j=0}^{k} a_j \xi^{k-j} \ \ and\ \sigma(\xi) = \ \xi^k. \tag{15}$$

The regions of absolute stability of the method for k = 3 (1) 8 are shown in figures 1 to 7 (Taking real part on x-axis and imaginary part on y-axis). The region of absolute stability is the region lying outside the boundary.

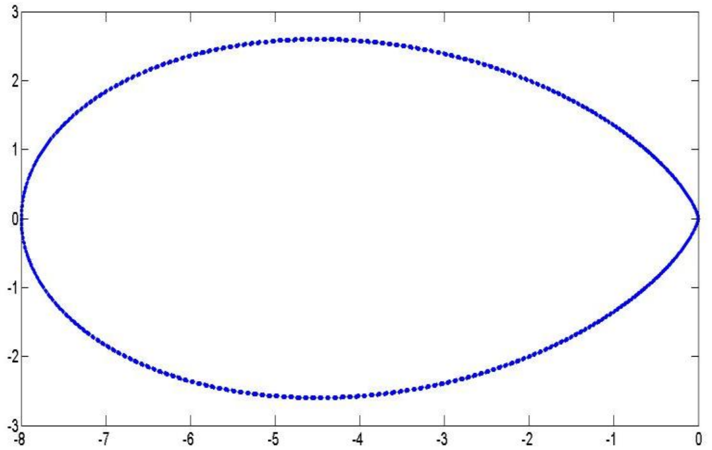

Figure 1

Figure 1: The region of absolute stability of the method (13) for k = 3

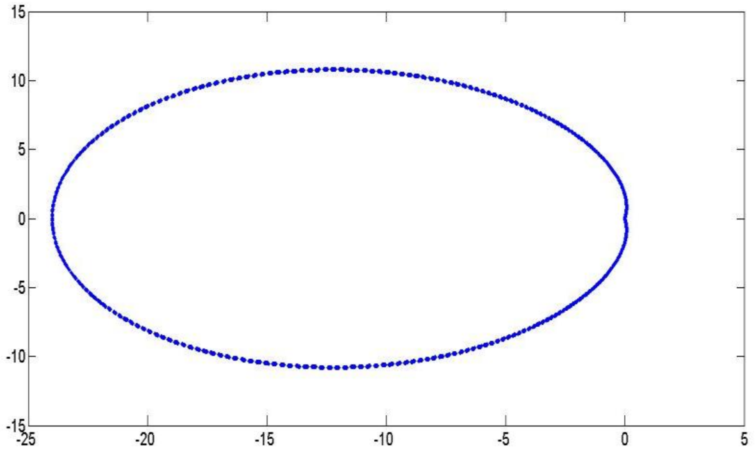

Figure 2

Figure 2: The region of absolute stability of the method (13) for k = 4

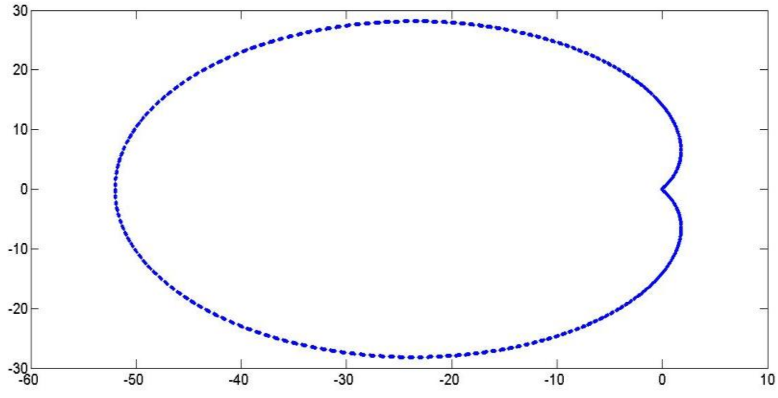

Figure 3

Figure 3: The region of absolute stability of the method (13) for k = 5

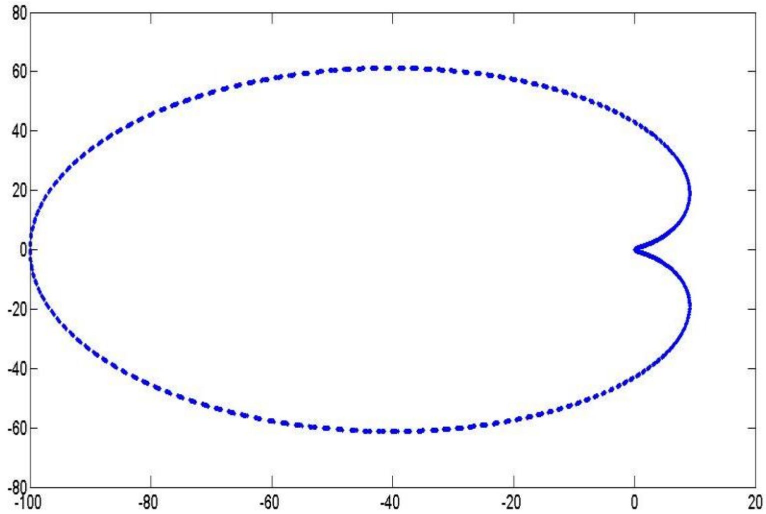

Figure 4

Figure 4: The region of absolute stability of the method (13) for k = 6

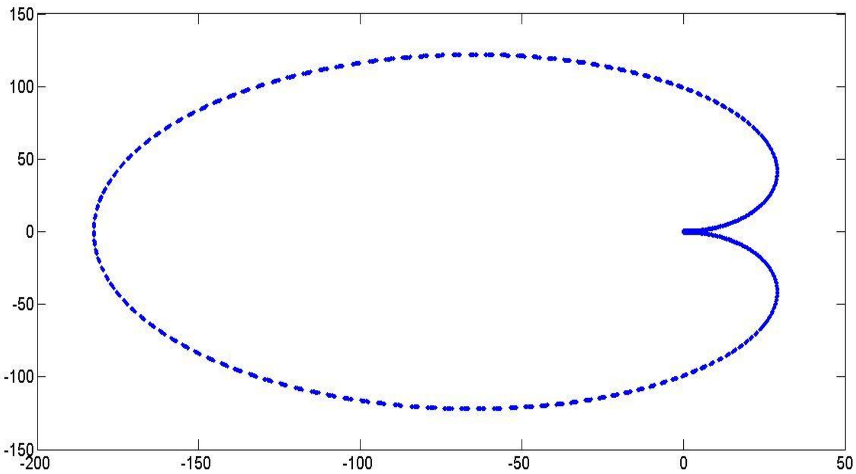

Figure 5

Figure 5: The region of absolute stability of the method (13) for k = 7

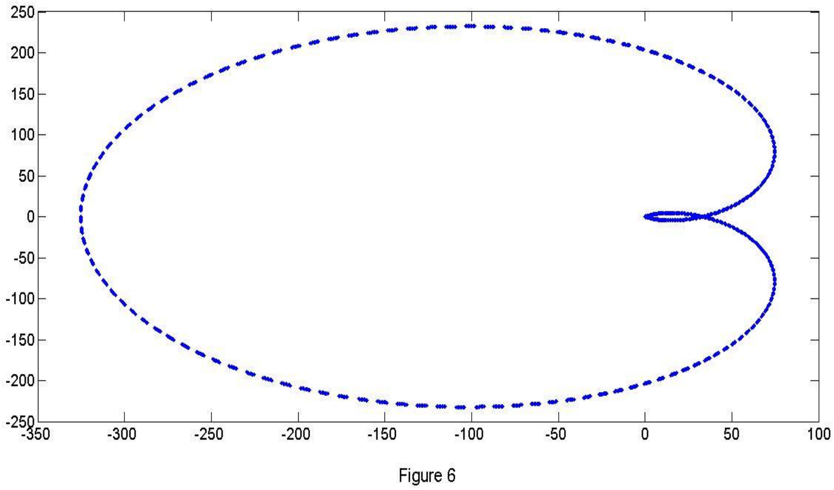

Figure 6: The region of absolute stability of the method (13) for k = 8

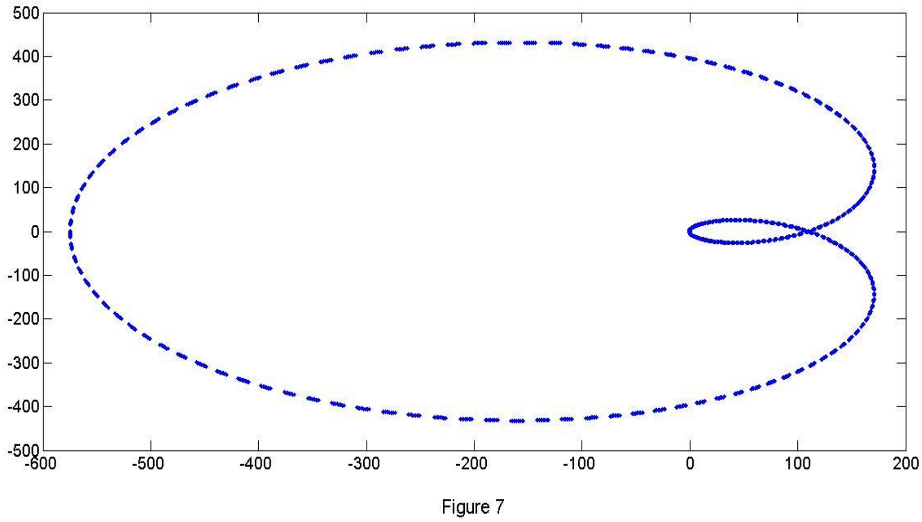

Figure 7: The region of absolute stability of the method (13) for k = 9

5. Numerical example

In this section, we have considered the differential equation:

$$y''' = 2e^x - y\,,\quad y(0) = 0,\ y'(0) = 2,\ y''(0) = 0 \tag{16}$$

in the interval [0, 2] with h = 0.01 and h = 0.02.

The exact solution of the above problem is $y = e^x - e^{-x}$

We have applied the ND methods to solve the differential equation and the results are shown in tables 3 and 4.

(1) The third order numerical differentiation method (13) derived in this paper with k = 4 is

$$y_{n+1} = \frac{7}{2}y_n - \frac{9}{2}y_{n-1} + \frac{5}{2}y_{n-2} - \frac{1}{2}y_{n-3} + \frac{1}{2}h^3 f_{n+\frac{1}{2}} \quad (17)$$

(2) The fourth order numerical differentiation method (13) derived in this paper with k = 5 is

$$y_{n+1} = \frac{91}{23}y_n - \frac{142}{23}y_{n-1} + \frac{110}{23}y_{n-2} - \frac{43}{23}y_{n-3} + \frac{7}{23}y_{n-4} + \frac{8}{23}h^3 f_{n+\frac{1}{2}} \quad (18)$$

(3) The fifth order numerical differentiation method (13) derived in this paper with k = 6 is

$$y_{n+1} = \frac{127}{23}y_n - \frac{232}{29}y_{n-1} + \frac{230}{29}y_{n-2} - \frac{133}{29}y_{n-3} + \frac{43}{29}y_{n-4} - \frac{6}{29}y_{n-5} + \frac{8}{29}h^3 f_{n+\frac{1}{2}} \quad (19)$$

Table 3
Solution by the fourth order numerical differentiation (ND) method with k = 5 and h = 0.01

X	Exact Solution	Numerical Solution by fourth order ND	Absolute Error
0	0.0000000000E+00	-1.0939058455E-14	1.0939058455E-14
0.1	2.0033350004E-01	2.0033350004E-01	4.0550895974E-14
0.2	4.0267200508E-01	4.0267200508E-01	9.2925667161E-14
0.3	6.0904058689E-01	6.0904058689E-01	1.4677148386E-13
0.4	8.2150465161E-01	8.2150465161E-01	2.0350388041E-13
0.5	1.0421906110E+00	1.0421906110E+00	2.5734969711E-13
0.6	1.2733071643E+00	1.2733071643E+00	3.1841196346E-13
0.7	1.5171674037E+00	1.5171674037E+00	3.7925218521E-13
0.8	1.7762119644E+00	1.7762119644E+00	4.4919623576E-13
0.9	2.0530334514E+00	2.0530334514E+00	5.1869619710E-13
1	2.3504023873E+00	2.3504023873E+00	5.9641180883E-13
1.1	2.6712949402E+00	2.6712949402E+00	6.7901240186E-13
1.2	3.0189227108E+00	3.0189227108E+00	7.6383344094E-13
1.3	3.3967648746E+00	3.3967648746E+00	8.6020079948E-13
1.4	3.8086030029E+00	3.8086030029E+00	9.7077901273E-13
1.5	4.2585589102E+00	4.2585589102E+00	1.0800249584E-12
1.6	4.7511359064E+00	4.7511359064E+00	1.2132517213E-12
1.7	5.2912638677E+00	5.2912638677E+00	1.3473666627E-12
1.8	5.8843485762E+00	5.8843485762E+00	1.5010215293E-12
1.9	6.5363258231E+00	6.5363258231E+00	1.6635581801E-12
2	7.2537208157E+00	7.2537208157E+00	1.8456347561E-12

Table 4
Solution by the fourth order numerical differentiation (ND) method with k = 5 and h = 0.02

X	Exact Solution	Numerical Solution by fourth order ND	Absolute Error
0	0.0000000000E+00	-1.4301991335E-12	1.4301991335E-12
0.1	2.0033350004E-01	2.0033350004E-01	1.9108048477E-12
0.2	4.0267200508E-01	4.0267200509E-01	5.2701731867E-12
0.3	6.0904058689E-01	6.0904058690E-01	8.6829432533E-12
0.4	8.2150465161E-01	8.2150465162E-01	1.2182810316E-11
0.5	1.0421906110E+00	1.0421906110E+00	1.5803802711E-11
0.6	1.2733071643E+00	1.2733071643E+00	1.9582779842E-11
0.7	1.5171674037E+00	1.5171674037E+00	2.3558932583E-11
0.8	1.7762119644E+00	1.7762119644E+00	2.7771118738E-11
0.9	2.0530334514E+00	2.0530334514E+00	3.2261748828E-11
1	2.3504023873E+00	2.3504023873E+00	3.7070346792E-11
1.1	2.6712949402E+00	2.6712949403E+00	4.2258196942E-11
1.2	3.0189227108E+00	3.0189227109E+00	4.7860826413E-11
1.3	3.3967648746E+00	3.3967648746E+00	5.3943516320E-11
1.4	3.8086030029E+00	3.8086030030E+00	6.0571547777E-11
1.5	4.2585589102E+00	4.2585589103E+00	6.7799987846E-11
1.6	4.7511359064E+00	4.7511359065E+00	7.5712769387E-11
1.7	5.2912638677E+00	5.2912638678E+00	8.4384055299E-11
1.8	5.8843485762E+00	5.8843485763E+00	9.3894669817E-11
1.9	6.5363258231E+00	6.5363258232E+00	1.0434320075E-10
2	7.2537208157E+00	7.2537208158E+00	1.1585044035E-10

6. Discussion and Conclusion

It is noticed that the numerical solution obtained by the Numerical Differentiation method derived in this paper is more accurate and very near to the original solution. The absolute errors are very very small. The region of absolute stability is the region lying out side the boundary.

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