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A Study on Discrete Model of Three Species Syn-Eco-System with Unlimited Resources for the First Species

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ABSTRACT: In this paper, the three species syn eco-system comprises of a commensal (S_1), two hosts S_2 and S_3 ie., S_2 and S_3 both benefit S_1, without getting themselves effected either positively or adversely. Further S_2 is a commensal of S_3, S_3 is a host of both S_1, S_2 and the first species has unlimited resources. The basic equations for this model constitute as three first order non-linear coupled ordinary difference equations. All possible equilibrium points are identified based on the model equations at two stages and criteria for their stability are discussed. Further the numerical solutions are computed for specific values of the various parameters and the initial conditions.

Keywords: Commensal, equilibrium state, host, stable, oscillatory.

2010 Mathematics Subject Classification: 92D25, 92D40

1. INTRODUCTION

Ecology, a branch evolutionary biology, deals with living species that coexist in a physical environment sustain themselves on common resources. It is a common observations that the species of same nature can not flourish is isolation without any interaction with species of different kinds. Syn-ecology is an ecosystem comprising of two or more distinct species. Species interact with each other in one way or other. The Ecological interactions can be broadly classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation, Parasitism and so on. Lotka[7], Svirezhev et al [22] and Volterra [25] pioneered theoretical ecology significantly and opened new eras in the field of life and biological sciences. The authors Rogers et al [18], Varma [23] and Veilleux [24] discussed prey-predator, competing ecological models. Colinvaux [3], Smith [20] and Wangersky [26] studied basic concepts of population models in ecology. Mathematical Modeling is a vital role in providing insight in to the mutual relationships (positive, negative) between the interacting species. The general concepts of modeling have been discussed by several authors Kapur [4], Kushing [6], Meyer [8] and Pieiou [15]. Srinivas [21] studied competitive ecosystem of two species and three species with limited and unlimited resources. Later, Laxminarayan et al [9] studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Stability analysis of competitive species was carried out by ([17], [19]), while Ravindra Reddy [16] investigated mutualism between two species. Acharyulu et al ([1], [2]) derived some productive results on various mathematical models of ecological Ammensalism with multifarious resources in the manifold directions. Further, Kumar [5] studied some mathematical models of ecological commensalism. The present author Prasad et al ([10]-[14]) investigated on the stability of a three and four species syn-ecosystems. The present investigation is a discrete model of three species (S_1, S_2, S_3) syn-eco system with unlimited resources for the first species. The system comprises of a commensal (S_1), two hosts S_2 and S_3. Further S_2 is a commensal of S_3, S_3 is a host of both S_1 and S_2.

Commensalism is a symbiotic interaction between two populations where one population (S_1) gets benefit from (S_2) while the other (S_2) is neither harmed nor benefited due to the interaction with (S_1). The benefited species (S_1) is called the commensal and the other, the helping one (S_2) is called the host species. A common example is an animal using a tree for shelter-tree (host) does not get any benefit from the animal (commensal).

2. BASIC EQUATIONS OF THE MODEL

2.1 Notation Adopted

$N_i(t)$	: The population strength of S_i at time t, $i = 1,2,3$
t	: Time instant
a_i	: Natural growth rate of S_i, $i = 1,2,3$
a_{ii}	: Self inhibition coefficients of S_i, $i = 2,3$
a_{12}, a_{13}	: Interaction coefficients of S_1 due to S_2 and S_1 due to S_3
a_{23}	: Interaction coefficient of S_2 due to S_3

Further the variables N_1, N_2, N_3 are non-negative and the model parameters a_1, a_2, a_3, a_{12}, $a_{22}, a_{33}, a_{13}, a_{23}$ are assumed to be non-negative constants.

2.2 Basic equations

Consider the growth of the species during the time interval $(t, t+1)$.

(i) Equation for the first species (N_1):

$$N_1(t+1) = N_1(t) + a_1 N_1(t) + a_{12} N_1(t)\, N_2(t) + a_{13} N_1(t)\, N_3(t) \tag{2.1}$$

(ii) Equation for the second species (N_2):

$$N_2(t+1) = N_2(t) + a_2 N_2(t) - a_{22} N_2^2(t) + a_{23} N_2(t) N_3(t) \tag{2.2}$$

(iii) Equation for the third species (N_3):

$$N_3(t+1) = N_3(t) + a_3 N_3(t) - a_{33} N_3^2(t) \tag{2.3}$$

2.3 Species-growth equations in the discrete form

Consider the nonlinear autonomous system of discrete equations

$$N_1(t+1) = \alpha_1 N_1(t) + a_{12} N_1(t)\ N_2(t) + a_{13} N_1(t)\ N_3(t) \tag{2.4}$$

$$N_2(t+1) = \alpha_2 N_2(t) - a_{22} N_2^2(t) + a_{23} N_2(t) N_3(t) \tag{2.5}$$

$$N_3(t+1) = \alpha_3 N_3(t) - a_{33} N_3^2(t) \tag{2.6}$$

where $\alpha_i = a_i + 1, i = 1,2,3$ (2.7)

3. EQUILIBRIUM STATES

For a continuous model the equilibrium states are defined by $\frac{dN_i}{dt} = 0,\ i = 1,2,3$, the equilibrium states for a discrete model are defined in terms of the period of no growth.
i.e, $N_i(t+r) = N_i(t),\ r = 1,2,3,...$, where r is the period of the equilibrium state.
Kapur J.N, Mathematical Modeling through Discrete Mathematics-Fascinating World of Mathematical Sciences, 1989, Volume-II, pp.71-80, Mathematical Science Trust Society, New Delhi.

3.1 One period equilibrium states (Stage-I)

$$N_i(t+1) = N_i(t),\ i = 1,2,3 \tag{3.1}$$

The system under investigation has **five** equilibrium states given by

(i) Fully washed out state

$E_1 : \bar{N}_1 = 0,\ \bar{N}_2 = 0,\ \bar{N}_3 = 0$

(ii) The state in which only the first species survives

$E_2 : \bar{N}_1 = 0,\ \bar{N}_2 = 0,\ \bar{N}_3 = \frac{\alpha_3 - 1}{a_{33}}$, when $\alpha_3 > 1$

$E_3 : \bar{N}_1 = 0,\ \bar{N}_2 = \frac{\alpha_2 - 1}{a_{22}},\ \bar{N}_3 = 0$, when $\alpha_2 > 1$

$E_4 : \bar{N}_1 = 0,\ \bar{N}_2 = \frac{1}{a_{22}}\left[(\alpha_2 - 1) + a_{23}\left(\frac{\alpha_3 - 1}{a_{33}}\right)\right],\ \bar{N}_3 = \frac{\alpha_3 - 1}{a_{33}}$, when $\alpha_2 > 1$ and $\alpha_3 > 1$

$E_5 : \bar{N}_1 = 0,\ \bar{N}_2 = a_{23}\left(\frac{\alpha_3 - 1}{a_{22} a_{33}}\right),\ \bar{N}_3 = \frac{\alpha_3 - 1}{a_{33}}$, when $\alpha_2 = 1$ and $\alpha_3 > 1$

3.1.1 Stability of equilibrium states

Stability of $E_1(0,0,0)$:

$$N_1(t)=N_1(t+1)=N_1(t+2)=...=0;\ N_2(t)=N_2(t+1)=N_2(t+2)=...=0$$

$$N_3(t)=N_3(t+1)=N_3(t+2)=...=0$$

i.e, $N_i(t+r)=0$, where r is an integer and $i=1,2,3$

Hence, $E_1(0,0,0)$ is **stable**.

Stability of E_2:

$$N_1(t)=N_1(t+1)=N_1(t+2)=...=0;\ N_2(t)=N_2(t+1)=N_2(t+2)=...=0$$

$$N_3(t)=N_3(t+1)=N_3(t+2)=...=\frac{\alpha_3-1}{a_{33}}$$

i.e, $N_i(t+r)=0,\ N_3(t+r)=\frac{\alpha_3-1}{a_{33}}$, where r is an integer and $i=1,2$

Hence, E_2 is **stable**.

Stability of E_3:

$$N_1(t)=N_1(t+1)=N_1(t+2)=...=0;\ N_3(t)=N_3(t+1)=N_3(t+2)=...=0$$

$$N_2(t)=N_2(t+1)=N_2(t+2)=...=\frac{\alpha_2-1}{a_{22}}$$

i.e, $N_i(t+r)=0,\ N_2(t+r)=\frac{\alpha_2-1}{a_{22}}$, where r is an integer and $i=1,3$

Hence, E_3 is **stable**.

Stability of E_4:

$$N_1(t)=N_1(t+1)=N_1(t+2)=...=0;\ N_3(t)=N_3(t+1)=N_3(t+2)=...=\frac{\alpha_3-1}{a_{33}}$$

$$N_2(t)=N_2(t+1)=N_2(t+2)=...=\frac{1}{a_{22}}\left[\left(\alpha_2-1\right)+a_{23}\left(\frac{\alpha_3-1}{a_{33}}\right)\right]$$

i.e, $N_1(t+r)=0,\ N_2(t+r)=\frac{1}{a_{22}}\left[\left(\alpha_2-1\right)+a_{23}\left(\frac{\alpha_3-1}{a_{33}}\right)\right],\ N_3(t+r)=\frac{\alpha_3-1}{a_{33}}$ where r is an integer

Hence, E_4 is **stable**.

Stability of E_5:

$$N_1(t) = N_1(t+1) = N_1(t+2) = ... = 0\,;\; N_2(t) = N_2(t+1) = N_2(t+2) = ... = a_{23}\left(\frac{\alpha_3 - 1}{a_{22}a_{33}}\right)$$

$$N_3(t) = N_3(t+1) = N_3(t+2) = ... = \frac{\alpha_3 - 1}{a_{33}}$$

i.e, $N_1(t+r) = 0,\ N_2(t+r) = a_{23}\left(\frac{\alpha_3 - 1}{a_{22}a_{33}}\right),\ N_3(t+r) = \frac{\alpha_3 - 1}{a_{33}}$, where r is an integer

Hence, E_5 is **stable**.

At this stage all the five equilibrium states are **stable**.

3.2 Two period equilibrium states (Stage-II)

$$N_i(t+2) = N_i(t),\ i = 1,2,3 \tag{3.2}$$

The system under investigation has **twenty five** equilibrium states given by

(i) Fully washed out state

$E_1 : \bar{N}_1 = 0,\ \bar{N}_2 = 0,\ \bar{N}_3 = 0$.

(ii) States in which only the third species survives

$E_2 : \bar{N}_1 = 0,\ \bar{N}_2 = 0,\ \bar{N}_3 = \frac{\alpha_3 - 1}{a_{33}}$, when $\alpha_3 > 1$

$E_3 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = \frac{(\alpha_3 + 1) + \sqrt{(\alpha_3 + 1)(\alpha_3 - 3)}}{2a_{33}}$, when $\alpha_3 > 3$

$E_4 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = \frac{(\alpha_3 + 1) - \sqrt{(\alpha_3 + 1)(\alpha_3 - 3)}}{2a_{33}}$, when $\alpha_3 > 3$

$E_5 : \bar{N}_1 = 0, \bar{N}_2 = 0,\ \bar{N}_3 = \frac{2}{a_{33}}$, when $\alpha_3 = 3$

The states E_3 and E_4 coincide when $\alpha_3 = 3$ and do not exist when $\alpha_3 < 3$.

(iii) States in which only the second species survives

$E_6 : \bar{N}_1 = 0,\ \bar{N}_2 = \frac{\alpha_2 - 1}{a_{22}},\ \bar{N}_3 = 0$, when $\alpha_2 > 1$

$E_7 : \bar{N}_1 = 0, \bar{N}_2 = \frac{(\alpha_2 + 1) + \sqrt{(\alpha_2 + 1)(\alpha_2 - 3)}}{2a_{22}}, \bar{N}_3 = 0$, when $\alpha_2 > 3$

$E_8 : \bar{N}_1 = 0, \bar{N}_2 = \frac{(\alpha_2 + 1) - \sqrt{(\alpha_2 + 1)(\alpha_2 - 3)}}{2a_{22}}, \bar{N}_3 = 0$, when $\alpha_2 > 3$

$E_9 : \bar{N}_1 = 0, \bar{N}_2 = \frac{2}{a_{22}}, \ \bar{N}_3 = 0$, when $\alpha_2 = 3$

The states E_7 and E_8 coincide when $\alpha_2 = 3$ and do not exist when $\alpha_2 < 3$.

(iv) States in which only the second and third species survives

$E_{10} : \bar{N}_1 = 0, \ \bar{N}_2 = \frac{\beta_2 - 1}{a_{22}}, \ \bar{N}_3 = \frac{\alpha_3 - 1}{a_{33}}$, when $\beta_2, \alpha_3 > 1$

where $\beta_2 = \alpha_2 + a_{23}\left(\frac{\alpha_3 - 1}{a_{33}}\right)$

$E_{11} : \bar{N}_1 = 0, \ \bar{N}_2 = \frac{(\beta_2+1)+\sqrt{(\beta_2+1)(\beta_2-3)}}{2a_{22}}, \ \bar{N}_3 = \frac{\alpha_3 - 1}{a_{33}}$, when $\beta_2 > 3, \alpha_3 > 1$

$E_{12} : \bar{N}_1 = 0, \ \bar{N}_2 = \frac{(\beta_2+1)-\sqrt{(\beta_2+1)(\beta_2-3)}}{2a_{22}}, \ \bar{N}_3 = \frac{\alpha_3 - 1}{a_{33}}$, when $\beta_2 > 3, \alpha_3 > 1$

$E_{13} : \bar{N}_1 = 0, \ \bar{N}_2 = \frac{2}{a_{22}}, \ \bar{N}_3 = \frac{\alpha_3 - 1}{a_{33}}$, when $\beta_2 = 3, \alpha_3 > 1$

The states E_{11} and E_{12} coincide when $\beta_2 = 3$ and do not exist when $\beta_2 < 3$.

$E_{14} : \bar{N}_1 = 0, \ \bar{N}_2 = \frac{\gamma_2 - 1}{a_{22}}, \ \bar{N}_3 = \frac{(\alpha_3+1)+\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$, when $\gamma_2 > 1, \alpha_3 > 3$

where $\gamma_2 = \alpha_2 + a_{23}\left[\frac{(\alpha_3+1)+\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}\right]$

$E_{15} : \bar{N}_1 = 0, \ \bar{N}_2 = \frac{(\gamma_2+1)+\sqrt{(\gamma_2+1)(\gamma_2-3)}}{2a_{22}}, \ \bar{N}_3 = \frac{(\alpha_3+1)+\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$,

when $\gamma_2, \alpha_3 > 3$

$E_{16} : \bar{N}_1 = 0, \ \bar{N}_2 = \frac{(\gamma_2+1)-\sqrt{(\gamma_2+1)(\gamma_2-3)}}{2a_{22}}, \ \bar{N}_3 = \frac{(\alpha_3+1)+\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$,

when $\gamma_2, \alpha_3 > 3$

$E_{17} : \bar{N}_1 = 0, \ \bar{N}_2 = \frac{2}{a_{22}}, \ \bar{N}_3 = \frac{(\alpha_3+1)+\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$, when $\gamma_2 = 3, \alpha_3 > 3$

The states E_{15} and E_{16} coincide when $\gamma_2 = 3$ and do not exist when $\gamma_2 < 3$.

$E_{18} : \bar{N}_1 = 0,\ \bar{N}_2 = \frac{\mu_2 - 1}{a_{22}},\ \bar{N}_3 = \frac{(\alpha_3 + 1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$, when $\mu_2 > 1, \alpha_3 > 3$

where $\gamma_2 = \alpha_2 + a_{23}\left[\frac{(\alpha_3 + 1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}\right]$

$E_{19} : \bar{N}_1 = 0,\ \bar{N}_2 = \frac{(\mu_2 + 1) + \sqrt{(\mu_2+1)(\mu_2-3)}}{2a_{22}},\ \bar{N}_3 = \frac{(\alpha_3 + 1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$,

when $\mu_2, \alpha_3 > 3$

$E_{20} : \bar{N}_1 = 0,\ \bar{N}_2 = \frac{(\mu_2 + 1) - \sqrt{(\mu_2+1)(\mu_2-3)}}{2a_{22}},\ \bar{N}_3 = \frac{(\alpha_3 + 1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$,

when $\mu_2, \alpha_3 > 3$

$E_{21} : \bar{N}_1 = 0,\ \bar{N}_2 = \frac{2}{a_{22}},\ \bar{N}_3 = \frac{(\alpha_3 + 1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$, when $\mu_2 = 3, \alpha_3 > 3$

The states E_{19} and E_{20} coincide when $\mu_2 = 3$ and do not exist when $\mu_2 < 3$.

$E_{22} : \bar{N}_1 = 0,\ \bar{N}_2 = \frac{\delta_2 - 1}{a_{22}},\ \bar{N}_3 = \frac{2}{a_{33}}$, when $\delta_2 > 1, \alpha_3 = 3$

where $\delta_2 = \alpha_2 + \frac{2a_{23}}{a_{33}}$

$E_{23} : \bar{N}_1 = 0,\ \bar{N}_2 = \frac{(\delta_2 + 1) + \sqrt{(\delta_2+1)(\delta_2-3)}}{2a_{22}},\ \bar{N}_3 = \frac{2}{a_{33}}$, when $\delta_2 > 3, \alpha_3 = 3$

$E_{24} : \bar{N}_1 = 0,\ \bar{N}_2 = \frac{(\delta_2 + 1) - \sqrt{(\delta_2+1)(\delta_2-3)}}{2a_{22}},\ \bar{N}_3 = \frac{2}{a_{33}}$, when $\delta_2 > 3, \alpha_3 = 3$

$E_{25} : \bar{N}_1 = 0,\ \bar{N}_2 = \frac{2}{a_{22}},\ \bar{N}_3 = \frac{2}{a_{33}}$, when $\delta_2 = 3, \alpha_3 = 3$

The states E_{23} and E_{24} coincide when $\delta_2 = 3$ and do not exist when $\delta_2 < 3$.

3.2.1 Stability of Equilibrium States

The equilibrium states E_1, E_2 and E_6 are **stable** as established in **3.1.1**. Now we will discuss the stability of other equilibrium states except these three states.

Stability of E_3:

$N_1(t) = N_1(t+1) = N_1(t+2) = ... = 0$; $N_2(t) = N_2(t+1) = N_2(t+2) = ... = 0$

i.e, $N_i(t+r)=0$, where r is an integer and $i=1,2$

$$N_3(t)=N_3(t+2)=N_3(t+4)=...=\frac{(\alpha_3+1)+\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

$$N_3(t+1)=N_3(t+3)=N_3(t+5)=...=\frac{(\alpha_3+1)-\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

i.e, $N_3(t+2r)=\frac{(\alpha_3+1)+\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$; $N_3(t+\overline{2r+1})=\frac{(\alpha_3+1)-\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$

where r is an integer.

Hence, E_3 **oscillates** between $\frac{(\alpha_3+1)+\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$ and $\frac{(\alpha_3+1)-\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$, when $\alpha_3>3$ and is stable when $\alpha_3=3$.

Stability of E_4:

$N_1(t)=N_1(t+1)=N_1(t+2)=...=0$; $N_2(t)=N_2(t+1)=N_2(t+2)=...=0$

i.e, $N_i(t+r)=0$, where r is an integer and $i=1,2$

$$N_3(t)=N_3(t+2)=N_3(t+4)=...=\frac{(\alpha_3+1)-\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

$$N_3(t+1)=N_3(t+3)=N_3(t+5)=...=\frac{(\alpha_3+1)+\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

i.e, $N_3(t+2r)=\frac{(\alpha_3+1)-\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$; $N_3(t+\overline{2r+1})=\frac{(\alpha_3+1)+\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$

where r is an integer.

Hence, E_4 **oscillates** between $\frac{(\alpha_3+1)-\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$ and $\frac{(\alpha_3+1)+\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$, when $\alpha_3>3$ and is stable when $\alpha_3=3$.

Stability of E_5:

$N_1(t)=N_1(t+1)=N_1(t+2)=...=0$; $N_2(t)=N_2(t+1)=N_2(t+2)=...=0$

$$N_3(t)=N_3(t+1)=N_3(t+2)=...=\frac{2}{a_{33}}$$

i.e, $N_i(t+r)=0,\ N_3(t+r)=\frac{2}{a_{33}}$, where r is an integer and $i=1,2$

Hence, E_5 is **stable**.

Stability of E_7:

$$N_1(t)=N_1(t+1)=N_1(t+2)=...=0;\ N_3(t)=N_3(t+1)=N_3(t+2)=...=0$$

i.e, $N_i(t+r)=0$, where r is an integer and $i=1,3$

$$N_2(t)=N_2(t+2)=N_2(t+4)=...=\frac{(\alpha_2+1)+\sqrt{(\alpha_2+1)(\alpha_2-3)}}{2a_{22}}$$

$$N_2(t+1)=N_2(t+3)=N_2(t+5)=...=\frac{(\alpha_2+1)-\sqrt{(\alpha_2+1)(\alpha_2-3)}}{2a_{22}}$$

i.e, $N_2(t+2r)=\frac{(\alpha_2+1)+\sqrt{(\alpha_2+1)(\alpha_2-3)}}{2a_{22}}$; $N_2(t+\overline{2r+1})=\frac{(\alpha_2+1)-\sqrt{(\alpha_2+1)(\alpha_2-3)}}{2a_{22}}$

where r is an integer.

Hence, E_7 **oscillates** between $\frac{(\alpha_2+1)+\sqrt{(\alpha_2+1)(\alpha_2-3)}}{2a_{22}}$ and $\frac{(\alpha_2+1)-\sqrt{(\alpha_2+1)(\alpha_2-3)}}{2a_{22}}$, when $\alpha_2>3$ and is stable when $\alpha_2=3$.

Stability of E_8:

$$N_1(t)=N_1(t+1)=N_1(t+2)=...=0;\ N_3(t)=N_3(t+1)=N_3(t+2)=...=0$$

i.e, $N_i(t+r)=0$, where r is an integer and $i=1,3$

$$N_2(t)=N_2(t+2)=N_2(t+4)=...=\frac{(\alpha_2+1)-\sqrt{(\alpha_2+1)(\alpha_2-3)}}{2a_{22}}$$

$$N_2(t+1)=N_2(t+3)=N_2(t+5)=...=\frac{(\alpha_2+1)+\sqrt{(\alpha_2+1)(\alpha_2-3)}}{2a_{22}}$$

i.e, $N_2(t+2r)=\frac{(\alpha_2+1)-\sqrt{(\alpha_2+1)(\alpha_2-3)}}{2a_{22}}$; $N_2(t+\overline{2r+1})=\frac{(\alpha_2+1)+\sqrt{(\alpha_2+1)(\alpha_2-3)}}{2a_{22}}$

where r is an integer.

Hence, E_8 **oscillates** between $\frac{(\alpha_2+1)-\sqrt{(\alpha_2+1)(\alpha_2-3)}}{2a_{22}}$ and $\frac{(\alpha_2+1)+\sqrt{(\alpha_2+1)(\alpha_2-3)}}{2a_{22}}$, when $\alpha_2>3$ and is stable when $\alpha_2=3$.

Stability of E_9:

$N_1(t) = N_1(t+1) = N_1(t+2) = ... = 0$; $N_3(t) = N_3(t+1) = N_3(t+2) = ... = 0$

$N_2(t) = N_2(t+1) = N_2(t+2) = ... = \frac{2}{a_{22}}$

i.e, $N_i(t+r) = 0$, $N_2(t+r) = \frac{2}{a_{22}}$, where r is an integer and $i = 1, 3$

Hence, E_9 is **stable**.

Stability of E_{10}:

$N_1(t) = N_1(t+1) = N_1(t+2) = ... = 0$; $N_2(t) = N_2(t+1) = N_2(t+2) = ... = \frac{\beta_2 - 1}{a_{22}}$

$N_3(t) = N_3(t+1) = N_3(t+2) = ... = \frac{\alpha_3 - 1}{a_{33}}$

i.e, $N_1(t+r) = 0$, $N_2(t+r) = \frac{\beta_2 - 1}{a_{22}}$, $N_3(t+r) = \frac{\alpha_3 - 1}{a_{33}}$, where r is an integer

Hence, E_{10} is **stable**.

Stability of E_{11}:

$N_1(t) = N_1(t+1) = N_1(t+2) = ... = 0$; $N_3(t) = N_3(t+1) = N_3(t+2) = ... = \frac{\alpha_3 - 1}{a_{33}}$

i.e, $N_1(t+r) = 0$, $N_3(t+r) = \frac{\alpha_3 - 1}{a_{33}}$, where r is an integer

$N_2(t) = N_2(t+2) = N_2(t+4) = ... = \frac{(\beta_2 + 1) + \sqrt{(\beta_2 + 1)(\beta_2 - 3)}}{2a_{22}}$

$N_2(t+1) = N_2(t+3) = N_2(t+5) = ... = \frac{(\beta_2 + 1) - \sqrt{(\beta_2 + 1)(\beta_2 - 3)}}{2a_{22}}$

i.e, $N_2(t+2r) = \frac{(\beta_2 + 1) + \sqrt{(\beta_2 + 1)(\beta_2 - 3)}}{2a_{22}}$; $N_2(t + \overline{2r+1}) = \frac{(\beta_2 + 1) - \sqrt{(\beta_2 + 1)(\beta_2 - 3)}}{2a_{22}}$

where r is an integer.

Hence, E_{11} **oscillates** between $\frac{(\beta_2 + 1) + \sqrt{(\beta_2 + 1)(\beta_2 - 3)}}{2a_{22}}$ and $\frac{(\beta_2 + 1) - \sqrt{(\beta_2 + 1)(\beta_2 - 3)}}{2a_{22}}$,

when $\beta_2 > 3$ and is stable when $\beta_2 = 3$.

Stability of E_{12} :

$$N_1(t)=N_1(t+1)=N_1(t+2)=...=0\,;\;\; N_3(t)=N_3(t+1)=N_3(t+2)=...=\frac{\alpha_3-1}{a_{33}}$$

i.e, $N_1(t+r)=0,\; N_3(t+r)=\dfrac{\alpha_3-1}{a_{33}}$, where r is an integer

$$N_2(t)=N_2(t+2)=N_2(t+4)=...=\frac{(\beta_2+1)-\sqrt{(\beta_2+1)(\beta_2-3)}}{2a_{22}}$$

$$N_2(t+1)=N_2(t+3)=N_2(t+5)=...=\frac{(\beta_2+1)+\sqrt{(\beta_2+1)(\beta_2-3)}}{2a_{22}}$$

i.e, $N_2(t+2r)=\dfrac{(\beta_2+1)-\sqrt{(\beta_2+1)(\beta_2-3)}}{2a_{22}}$; $N_2(t+\overline{2r+1})=\dfrac{(\beta_2+1)+\sqrt{(\beta_2+1)(\beta_2-3)}}{2a_{22}}$

where r is an integer.

Hence, E_{12} **oscillates** between $\dfrac{(\beta_2+1)-\sqrt{(\beta_2+1)(\beta_2-3)}}{2a_{22}}$ and $\dfrac{(\beta_2+1)+\sqrt{(\beta_2+1)(\beta_2-3)}}{2a_{22}}$, when $\beta_2>3$ and is stable when $\beta_2=3$.

Stability of E_{13} :

$$N_1(t)=N_1(t+1)=N_1(t+2)=...=0\,;\;\; N_2(t)=N_2(t+1)=N_2(t+2)=...=\frac{2}{a_{22}}$$

$$N_3(t)=N_3(t+1)=N_3(t+2)=...=\frac{\alpha_3-1}{a_{33}}$$

i.e, $N_1(t+r)=0,\; N_2(t+r)=\dfrac{2}{a_{22}}, N_3(t+r)=\dfrac{\alpha_3-1}{a_{33}}$, where r is an integer

Hence, E_{13} is **stable**.

Stability of E_{14} :

$N_1(t)=N_1(t+1)=N_1(t+2)=...=0$, i.e, $N_1(t+r)=0$, where r is an integer

$N_2(t)=N_2(t+1)=\dfrac{\gamma_2-1}{a_{22}}$, but $N_2(t+2)\neq N_2(t+3)\neq N_2(t+4)\neq ...\neq \dfrac{\gamma_2-1}{a_{22}}$

i.e, $N_2(t+r)\neq \dfrac{\gamma_2-1}{a_{22}}$, where r is an integer except 0 and 1

$$N_3(t)=N_3(t+2)=N_3(t+4)=...=\frac{(\alpha_3+1)+\sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

$$N_3(t+1) = N_3(t+3) = N_3(t+5) = \ldots = \frac{(\alpha_3+1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

i.e, $N_3(t+2r) = \dfrac{(\alpha_3+1) + \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$; $N_3(t+\overline{2r+1}) = \dfrac{(\alpha_3+1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$

where r is an integer.

Hence, E_{14} is **unstable**, when $\alpha_3 > 3$ and is stable when $\alpha_3 = 3$.

Stability of E_{15}:

$N_1(t) = N_1(t+1) = N_1(t+2) = \ldots = 0$, i.e, $N_1(t+r) = 0$, where r is an integer

$N_2(t+1) \neq N_2(t+2) \neq N_2(t+3) \neq \ldots \neq N_2(t)$

i.e, $N_2(t+r) \neq \dfrac{(\gamma_2+1) + \sqrt{(\gamma_2+1)(\gamma_2-3)}}{2a_{22}}$, where r is an integer except 0

$$N_3(t) = N_3(t+2) = N_3(t+4) = \ldots = \frac{(\alpha_3+1) + \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

$$N_3(t+1) = N_3(t+3) = N_3(t+5) = \ldots = \frac{(\alpha_3+1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

i.e, $N_3(t+2r) = \dfrac{(\alpha_3+1) + \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$; $N_3(t+\overline{2r+1}) = \dfrac{(\alpha_3+1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$

where r is an integer.

Hence, E_{15} is **unstable**, when $\alpha_3 > 3$ and is oscillatory when $\alpha_3 = 3$.

Stability of E_{16}:

$N_1(t) = N_1(t+1) = N_1(t+2) = \ldots = 0$, i.e, $N_1(t+r) = 0$, where r is an integer

$N_2(t+1) \neq N_2(t+2) \neq N_2(t+3) \neq \ldots \neq N_2(t)$

i.e, $N_2(t+r) \neq \dfrac{(\gamma_2+1) - \sqrt{(\gamma_2+1)(\gamma_2-3)}}{2a_{22}}$, where r is an integer except 0

$$N_3(t) = N_3(t+2) = N_3(t+4) = \ldots = \frac{(\alpha_3+1) + \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

$$N_3(t+1) = N_3(t+3) = N_3(t+5) = \ldots = \frac{(\alpha_3+1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

i.e, $N_3(t+2r) = \dfrac{(\alpha_3+1) + \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$; $N_3(t+\overline{2r+1}) = \dfrac{(\alpha_3+1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$

where r is an integer.

Hence, E_{16} is **unstable**, when $\alpha_3 > 3$ and is oscillatory when $\alpha_3 = 3$.

Stability of E_{17}:

$N_1(t) = N_1(t+1) = N_1(t+2) = ... = 0$, i.e, $N_1(t+r) = 0$, where r is an integer

$N_2(t) = N_2(t+1) = \dfrac{2}{a_{22}}$, but $N_2(t+2) \neq N_2(t+3) \neq N_2(t+4) \neq ... \neq \dfrac{2}{a_{22}}$

i.e, $N_2(t+r) \neq \dfrac{2}{a_{22}}$, where r is an integer except 0 and 1

$$N_3(t) = N_3(t+2) = N_3(t+4) = ... = \frac{(\alpha_3 + 1) + \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

$$N_3(t+1) = N_3(t+3) = N_3(t+5) = ... = \frac{(\alpha_3 + 1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

i.e, $N_3(t+2r) = \dfrac{(\alpha_3 + 1) + \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$; $N_3(t+\overline{2r+1}) = \dfrac{(\alpha_3 + 1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$

where r is an integer.

Hence, E_{17} is **unstable**, when $\alpha_3 > 3$ and is stable when $\alpha_3 = 3$.

Stability of E_{18}:

$N_1(t) = N_1(t+1) = N_1(t+2) = ... = 0$, i.e, $N_1(t+r) = 0$, where r is an integer

$N_2(t) = N_2(t+1) = \dfrac{\mu_2 - 1}{a_{22}}$, but $N_2(t+2) \neq N_2(t+3) \neq N_2(t+4) \neq ... \neq \dfrac{\mu_2 - 1}{a_{22}}$

i.e, $N_2(t+r) \neq \dfrac{\mu_2 - 1}{a_{22}}$, where r is an integer except 0 and 1

$$N_3(t) = N_3(t+2) = N_3(t+4) = ... = \frac{(\alpha_3 + 1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

$$N_3(t+1) = N_3(t+3) = N_3(t+5) = ... = \frac{(\alpha_3 + 1) + \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

i.e, $N_3(t+2r) = \dfrac{(\alpha_3 + 1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$; $N_3(t+\overline{2r+1}) = \dfrac{(\alpha_3 + 1) + \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$

where r is an integer.

Hence, E_{18} is **unstable**, when $\alpha_3 > 3$ and is stable when $\alpha_3 = 3$.

Stability of E_{19}:

$N_1(t) = N_1(t+1) = N_1(t+2) = ... = 0$, i.e, $N_1(t+r) = 0$, where r is an integer

$N_2(t+1) \neq N_2(t+2) \neq N_2(t+3) \neq ... \neq N_2(t)$

i.e, $N_2(t+r) \neq \dfrac{(\mu_2+1)+\sqrt{(\mu_2+1)(\mu_2-3)}}{2a_{22}}$, where r is an integer except 0

$$N_3(t) = N_3(t+2) = N_3(t+4) = ... = \frac{(\alpha_3+1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

$$N_3(t+1) = N_3(t+3) = N_3(t+5) = ... = \frac{(\alpha_3+1) + \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

i.e, $N_3(t+2r) = \dfrac{(\alpha_3+1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$; $N_3(t+\overline{2r+1}) = \dfrac{(\alpha_3+1) + \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$

where r is an integer.

Hence, E_{19} is **unstable**, when $\alpha_3 > 3$ and is oscillatory when $\alpha_3 = 3$.

Stability of E_{20}:

$N_1(t) = N_1(t+1) = N_1(t+2) = ... = 0$, i.e, $N_1(t+r) = 0$, where r is an integer

$N_2(t+1) \neq N_2(t+2) \neq N_2(t+3) \neq ... \neq N_2(t)$

i.e, $N_2(t+r) \neq \dfrac{(\mu_2+1)-\sqrt{(\mu_2+1)(\mu_2-3)}}{2a_{22}}$, where r is an integer except 0

$$N_3(t) = N_3(t+2) = N_3(t+4) = ... = \frac{(\alpha_3+1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

$$N_3(t+1) = N_3(t+3) = N_3(t+5) = ... = \frac{(\alpha_3+1) + \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

i.e, $N_3(t+2r) = \dfrac{(\alpha_3+1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$; $N_3(t+\overline{2r+1}) = \dfrac{(\alpha_3+1) + \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$

where r is an integer.

Hence, E_{20} is **unstable**, when $\alpha_3 > 3$ and is oscillatory when $\alpha_3 = 3$.

Stability of E_{21}:

$N_1(t) = N_1(t+1) = N_1(t+2) = ... = 0$, i.e, $N_1(t+r) = 0$, where r is an integer

$N_2(t) = N_2(t+1) = \dfrac{2}{a_{22}}$, but $N_2(t+2) \neq N_2(t+3) \neq N_2(t+4) \neq ... \neq \dfrac{2}{a_{22}}$

i.e, $N_2(t+r) \neq \dfrac{2}{a_{22}}$, where r is an integer except 0 and 1

$$N_3(t) = N_3(t+2) = N_3(t+4) = \ldots = \frac{(\alpha_3 + 1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

$$N_3(t+1) = N_3(t+3) = N_3(t+5) = \ldots = \frac{(\alpha_3 + 1) + \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$$

i.e, $N_3(t+2r) = \frac{(\alpha_3 + 1) - \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$; $N_3(t+\overline{2r+1}) = \frac{(\alpha_3 + 1) + \sqrt{(\alpha_3+1)(\alpha_3-3)}}{2a_{33}}$

where r is an integer.

Hence, E_{21} is **unstable**, when $\alpha_3 > 3$ and is stable when $\alpha_3 = 3$.

Stability of E_{22}:

$N_1(t) = N_1(t+1) = N_1(t+2) = \ldots = 0$; $N_2(t) = N_2(t+1) = N_2(t+2) = \ldots = \frac{\delta_2 - 1}{a_{22}}$

$$N_3(t) = N_3(t+1) = N_3(t+2) = \ldots = \frac{2}{a_{33}}$$

i.e, $N_1(t+r) = 0$, $N_2(t+r) = \frac{\delta_2 - 1}{a_{22}}$, $N_3(t+r) = \frac{2}{a_{33}}$, where r is an integer

Hence, E_{22} is **stable**.

Stability of E_{23}:

$N_1(t) = N_1(t+1) = N_1(t+2) = \ldots = 0$; $N_3(t) = N_3(t+1) = N_3(t+2) = \ldots = \frac{2}{a_{33}}$

i.e, $N_1(t+r) = 0$, $N_3(t+r) = \frac{2}{a_{33}}$, where r is an integer

$$N_2(t) = N_2(t+2) = N_2(t+4) = \ldots = \frac{(\delta_2 + 1) + \sqrt{(\delta_2+1)(\delta_2-3)}}{2a_{22}}$$

$$N_2(t+1) = N_2(t+3) = N_2(t+5) = \ldots = \frac{(\delta_2 + 1) - \sqrt{(\delta_2+1)(\delta_2-3)}}{2a_{22}}$$

i.e, $N_2(t+2r) = \frac{(\delta_2 + 1) + \sqrt{(\delta_2+1)(\delta_2-3)}}{2a_{22}}$; $N_2(t+\overline{2r+1}) = \frac{(\delta_2 + 1) - \sqrt{(\delta_2+1)(\delta_2-3)}}{2a_{22}}$

where r is an integer.

Hence, E_{23} **oscillates** between $\frac{(\delta_2+1)+\sqrt{(\delta_2+1)(\delta_2-3)}}{2a_{22}}$ and $\frac{(\delta_2+1)-\sqrt{(\delta_2+1)(\delta_2-3)}}{2a_{22}}$, when $\delta_2 > 3$ and is stable when $\delta_2 = 3$.

Stability of E_{24}:

$$N_1(t) = N_1(t+1) = N_1(t+2) = ... = 0; \; N_3(t) = N_3(t+1) = N_3(t+2) = ... = \frac{2}{a_{33}}$$

i.e, $N_1(t+r) = 0, \; N_3(t+r) = \frac{2}{a_{33}}$, where r is an integer

$$N_2(t) = N_2(t+2) = N_2(t+4) = ... = \frac{(\delta_2+1)-\sqrt{(\delta_2+1)(\delta_2-3)}}{2a_{22}}$$

$$N_2(t+1) = N_2(t+3) = N_2(t+5) = ... = \frac{(\delta_2+1)+\sqrt{(\delta_2+1)(\delta_2-3)}}{2a_{22}}$$

i.e, $N_2(t+2r) = \frac{(\delta_2+1)-\sqrt{(\delta_2+1)(\delta_2-3)}}{2a_{22}}$; $N_2(t+\overline{2r+1}) = \frac{(\delta_2+1)+\sqrt{(\delta_2+1)(\delta_2-3)}}{2a_{22}}$

where r is an integer.

Hence, E_{24} **oscillates** between $\frac{(\delta_2+1)-\sqrt{(\delta_2+1)(\delta_2-3)}}{2a_{22}}$ and $\frac{(\delta_2+1)+\sqrt{(\delta_2+1)(\delta_2-3)}}{2a_{22}}$, when $\delta_2 > 3$ and is stable when $\delta_2 = 3$.

Stability of E_{25}:

$$N_1(t) = N_1(t+1) = N_1(t+2) = ... = 0; \; N_2(t) = N_2(t+1) = N_2(t+2) = ... = \frac{2}{a_{22}}$$

$$N_3(t) = N_3(t+1) = N_3(t+2) = ... = \frac{2}{a_{33}}$$

i.e, $N_1(t+r) = 0, N_2(t+r) = \frac{2}{a_{22}}, \; N_3(t+r) = \frac{2}{a_{33}}$, where r is an integer

Hence, E_{25} is **stable**, when $\delta_2 = 3$, $\alpha_3 = 3$.

At this stage, in all twenty five equilibrium states, only the nine states E_1, E_2, E_5, E_6, E_9, E_{10}, E_{13}, E_{22}, E_{25} are **stable** and E_3, E_4, E_7, $E_8, E_{11}, E_{12}, E_{23}, E_{24}$ are **oscillatory** and remaining all eight are **unstable**.

4. NUMERICAL EXAMPLES

The numerical solutions of the discrete model equations computed for specific values of the various parameters and the initial conditions. The results are illustrated in Figures 4.1 to 4.4.

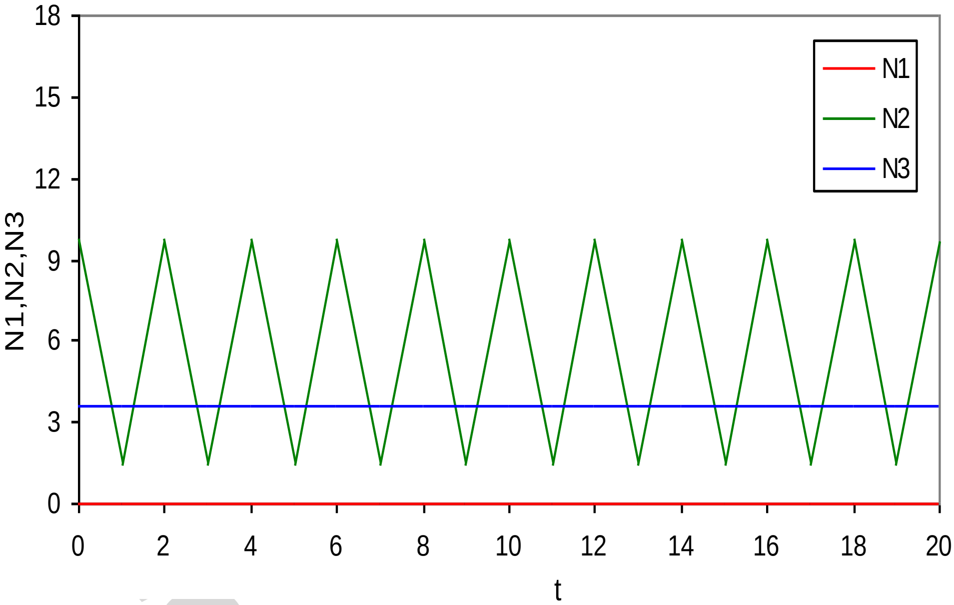

Figure 4.1: Variation of N_1, N_2 and N_3 against time(t) for

$$\alpha_1 = 1.9, \alpha_2 = 3.6, \alpha_3 = 2.8, a_{12} = 0.3, a_{13} = 4.7, a_{22} = 0.8, a_{33} = 0.5, a_{23} = 1.2,$$

$$N_1(0) = 0, N_2(0) = 9.71, N_3(0) = 3.6$$

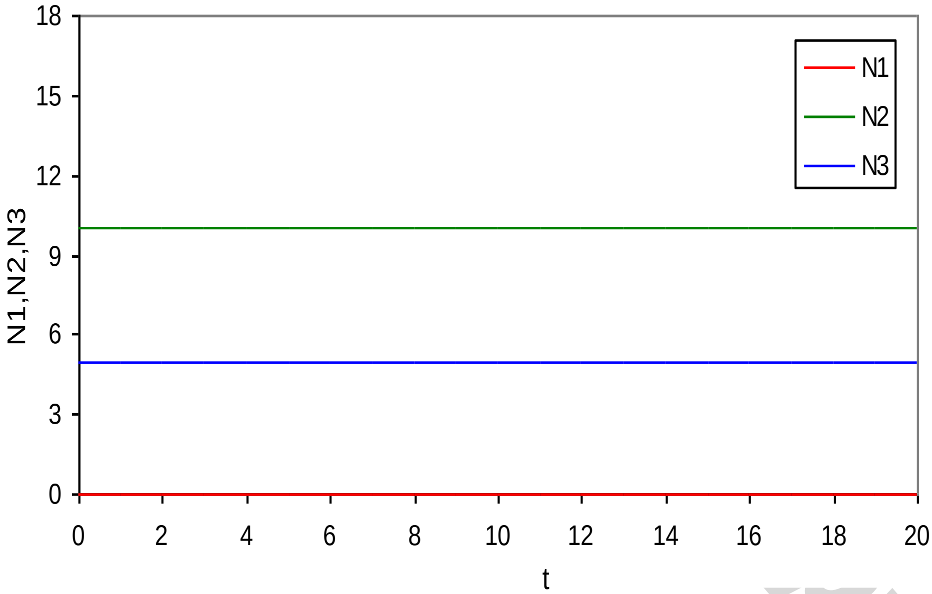

Figure 4.2: Variation of N_1, N_2 and N_3 against time(t) for

$$\alpha_1 = 2.5, \alpha_2 = 1.6, \alpha_3 = 3, a_{12} = 0.53, a_{13} = 3.7, a_{22} = 0.76, a_{33} = 0.4, a_{23} = 1.2,$$

$$N_1(0) = 0, N_2(0) = 10, N_3(0) = 5$$

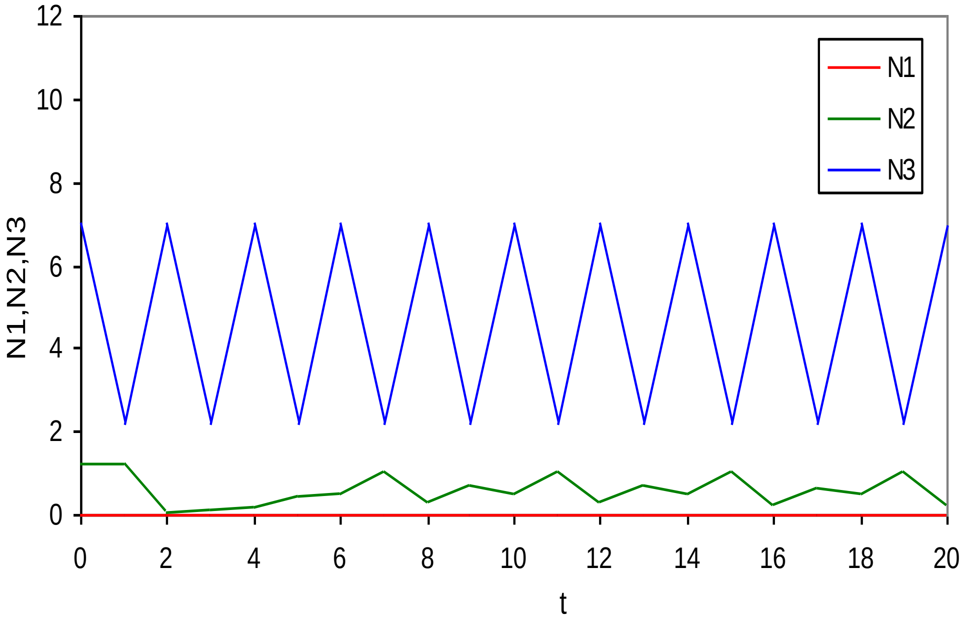

Figure 4.3: Variation of N_1, N_2 and N_3 against time(t) for
$\alpha_1 = 0.8, \alpha_2 = 1.3, \alpha_3 = 4.5, a_{12} = 2.8, a_{13} = 3.3, a_{22} = 1.4, a_{33} = 0.6, a_{23} = 0.2,$
$N_1(0) = 0, N_2(0) = 1.21, N_3(0) = 6.98$

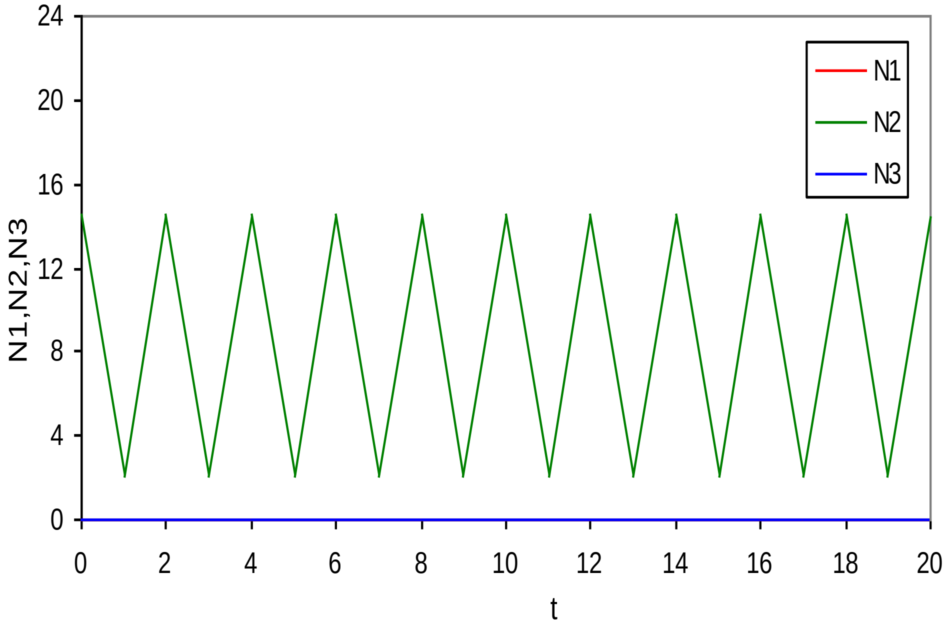

Figure 4.4: Variation of N_1, N_2 and N_3 against time(t) for
$\alpha_1 = 3.6, \alpha_2 = 4.8, \alpha_3 = 1.1, a_{12} = 0.3, a_{13} = 2, a_{22} = 0.35, a_{33} = 6.3, a_{23} = 3.2,$
$N_1(0) = 0, N_2(0) = 14.47, N_3(0) = 0$

5. CONCLUSION

The present paper deals with an investigation on a discrete model of three species syn eco-system with unlimited resources for the first species. The system comprises of a commensal (S_1) common to two hosts S_2 and S_3 ie., S_2 and S_3 both benefit S_1, without getting themselves effected either positively or adversely. Further S_2 is a commensal of S_3, S_3 is a host of both S_1, S_2. All possible equilibrium points of the model are identified based on the model equations at two stages.

$$\text{Stage-I} \;:\; N_i(t+1) = N_i(t);\; i = 1,2,3$$
$$\text{Stage-II} :\; N_i(t+2) = N_i(t);\; i = 1,2,3$$

In Stage-I there are only five equilibrium points, while the Stage-II there would be twenty five equilibrium points. All the five equilibrium points in Stage-I are found to be **stable** while in stage-II only nine are **stable.** Further the numerical solutions for the discrete model equations are computed.

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