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Homotopy Analysis Method for Solving Eighth Order Boundary Value Problems

Hymavathi T*, P.Vijay Kumar and W.Sridhar

Department of Mathematics, Adikavi Nannaya University, Rajahmundry

*Author for Correspondence: talla.hymavathianur@gmail.com

ABSTRACT

In this article, homotopy analysis method (HAM) is demonstrated to solve eighth order boundary value problems. HAM solution contains an auxiliary parameter 'h' which provides a convenient way to control the convergence region of the series solutions. Numerical examples are considered to check the efficiency of the method. Comparisons are made to confirm the reliability and accuracy of the technique.

Keywords: Boundary value problem, Series solution, Error estimate, Homotopy Analysis Method.

1. Introduction

A wide class of problems, which arise in many physical processes such as hydrodynamic, hydromagnetic stability and fluid dynamics, can be modeled by eighth-order boundary-value problems.

Several numerical and analytic techniques including differential quadrature method, spline method, variational iteration method, homotopy perturbation method (HPM) [1], Adomian's decomposition method [2] have been developed for solving these problems. It is well known that the above methods have their inbuilt deficiencies like use of small parameters, identification of Lagrange multiplier, divergent results and huge computational work. These facts motivated us to utilize alternate technique known as HAM proposed by Liao [3] for solving higher order boundary and initial-value problems.

In this paper, we consider eighth-order boundary-value problems of the form

$$y^{(8)}(x) + f(x)y(x) = g(x), \qquad x \in [a, b] \tag{1}$$

with boundary conditions

$$y(a) = \alpha_0, y(b) = \alpha_1,\ y^{(1)}(a) = \gamma_0, y^{(1)}(b) = \gamma_1,$$
$$y^{(2)}(a) = \delta_0,\ y^{(2)}(b) = \delta_1, y^{(3)}(a) = v_0, y^{(3)}(b) = v_1 \tag{2}$$

where $\alpha_i,\ \gamma_i,\ \delta_i,\ and\ v_i$, $i = 0, 1$ are finite real constants and the functions $f(x)\ and\ g(x)$ are continuous on [a, b].

2. Basic Idea of HAM

To have a basic idea of HAM, let us consider the following differential equation

$$N[u(x)] = 0, \tag{3}$$

where N is a nonlinear operator, x denotes independent variable, $u(x)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [4] constructed the following zero deformation equation

$$(1 - p)L\big(\phi(x; p) - u_0(x)\big) = phH(x)N\big(\phi(x; p)\big), \tag{4}$$

where $p \in [0, 1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, $H(x) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x)$ is an initial guess of $u(x)$, $\phi(x;\ p)$ is an unknown function of the independent variables x, p.

It is important to note that one has great freedom to choose the above mentioned auxiliary functions and parameters in HAM. When $p = 0$ and $p = 1$, $\phi(x;\ p)$ becomes

$$\phi(x;\ 0) = u_0(x), \qquad \phi(x;\ 1) = u(x), \tag{5}$$

respectively. Thus as p increases from 0 to 1, the solution $\phi(x; p)$ varies from the initial guess $u_0(x)$ to the solution $u(x)$. Expanding $\phi(x; p)$ in Taylor series with respect to p, we have

$$\phi(x;\ p) = u_0(x) + \sum_{m=1}^{\infty} u_m(x)\, p^m, \tag{6}$$

where

$$u_m(x) = \frac{1}{m!}\frac{\partial^m \emptyset(x;p)}{\partial p^m}\bigg|_{p=0} \qquad (7)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter and the auxiliary function are properly chosen, series (6) converges at $p = 1$ and we have

$$u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) \qquad (8)$$

which must be one of the solutions of original nonlinear equation as proved by Liao [3]. Define the vector $\bar{u}_n$ as $\bar{u}_n = \{u_0, u_1, \dots, u_n\}$. Differentiating equation (4) m times with respect to the embedding parameter p, setting $p = 0$ and finally dividing with $m!$, we obtain the m^{th} order deformation equation as below

$$L(u_m - \chi_m u_{m-1}) = hH(x)R_m(\bar{u}_{m-1}), \qquad (9)$$

where

$$R_m(\bar{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\phi(x;p)}{\partial p^{m-1}}\bigg|_{p=0} \qquad (10)$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \qquad (11)$$

Thus, we can obtain $u_0, u_1, \dots, u_n$ by solving the linear higher order differential equation (9) one after the other in order. The m^{th} order approximation of $u(x)$ is given by $u(x) = \sum_{m=0}^{\infty} u_m(x)$.

3. Numerical Examples

In order to illustrate the ability of HAM in solving eighth order boundary value problems, we consider the following examples. The obtained HAM results are compared with the available results in literature and are seen in good agreement.

Example 1

For $x \in [0, \frac{\pi}{2}]$, consider the following nonlinear eighth order boundary value problem [5]

$$y^{(8)}(x) = y(x) \qquad (12)$$

with boundary conditions

$$y^{(0)}(0) = 1, y^{(0)}\left(\frac{\pi}{2}\right) = 1, y^{(1)}(0) = 1, y^{(1)}\left(\frac{\pi}{2}\right) = -1,$$

$$y^{(2)}(0) = -1, y^{(2)}\left(\frac{\pi}{2}\right) = -1, y^{(3)}(0) = -1, y^{(3)}\left(\frac{\pi}{2}\right) = 1 \quad (13)$$

Exact solution of the above differential equation is

$$y(x) = \sin x + \cos x \quad (14)$$

For the zeroth order deformation equation (4) the auxiliary linear operator is given by

$$L(\phi(x;\, p)) = y^{(8)}(x) \quad (15)$$

We also choose $H(x) = 1$, for simplicity and the nonlinear operator N as

$$N(\phi(x;\, p)) = \emptyset^{(8)}(x) - \emptyset(x) \quad (16)$$

In view of the boundary conditions (13), the initial guess is obtained as

$$y_0 = 0.000544627\,(1836.12 + 1836.12\,x - 918.059\,x^2 - 306.02\,x^3 + 76.1261x^4 + 16.259\,x^5 - 3.45027\,x^6) \quad (17)$$

Following the explanation given in the previous section, the m^{th} order deformation equation becomes

$$L(y_m - \chi_m y_{m-1}) = hR_m(\bar{y}_{m-1}), \quad (18)$$

with

$$y^{(0)}(0) = 0, y^{(0)}\left(\frac{\pi}{2}\right) = 0, y^{(1)}(0) = 0, y^{(1)}\left(\frac{\pi}{2}\right) = 0,$$

$$y^{(2)}(0) = 0, y^{(2)}\left(\frac{\pi}{2}\right) = 0, y^{(3)}(0) = 0, y^{(3)}\left(\frac{\pi}{2}\right) = 0 \quad (19)$$

where

$$R_m(\bar{y}_{m-1}) = {y^{(8)}}_{m-1}(x) - y_{m-1}(x) \quad (20)$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \quad (21)$$

Now the solution of m^{th} order deformation equation for $m \geq 1$ becomes

$$y_m = \chi_m y_{m-1} + L^{-1}\big(h * R_m(\bar{y}_{m-1})\big). \quad (22)$$

Using symbolic computation software's such as Matlab or Maple or Mathematica, we can recursively obtain $y_0, y_1, y_2, \ldots$

The approximate solution of (12) can be obtained by setting h = - 1, as below

$y(x) = 1 + x - 0.5x^2 - 0.166667x^3 - 0.041666621x^4 + 0.008333331x^5 - 0.001388888x^6 - 0.000198413x^8 - 0.000000248016x^8 + 2.75573 * 10^{-6}x^9 - 2.75573 * 10^{-7}x^{10} - 2.50521 * 10^{-8}x^{11} + 2.08768 * 10^{-9}x^{12} + 1.6059 * 10^{-10}x^{13} - 114707 * 10^{-11}x^{14} - 7.64716 * 10^{-12}x^{15} + 4.77948 * 10^{-14}x^{16} + O(17)$ (23)

The comparison of the exact solution with the series solution, of the problem (12), obtained using HAM and Variational Iteration Technique (VIT) solutions are shown in Table 1.

Table 1:

x	Exact solution	Error of HAM	Error of VIT
0	1.000000000000000	0.00	0.00000
π/10	1.260073510670101	0.00	9.19931×10^{-12}
2π/10	1.396802246667421	2.22×10^{-16}	5.58589×10^{-11}
3π/10	1.396802246667421	4.44×10^{-16}	6.9188×10^{-11}
4π/10	1.260073510670101	4.44×10^{-16}	1.7613×10^{-11}
5π/10	1.000000000000000	1.22×10^{-15}	2.22045×10^{-16}

Error = Exact solution − Approximate solution

Example 2: For $x \in [0, 1]$, consider the linear eighth order boundary value problem [5]

$$y^{(8)}(x) = y(x) - 48e^x - 16xe^x \tag{24}$$

with boundary conditions

$$y^{(0)}(0) = 0, y^{(0)}(1) = 0, y^{(1)}(0) = 1, y^{(1)}(1) = -e,$$

$$y^{(2)}(0) = 0, y^{(2)}(1) = -4\mathrm{e}, y^{(3)}(0) = -3, y^{(3)}(1) = -9e \tag{25}$$

Exact solution of the above differential equation is

$$y(x) = x(1-x)e^x \tag{26}$$

For the zeroth order deformation equation (4) the auxiliary linear operator is given by

$$L(\phi(x;\, p)) = y^{(8)}(x) \tag{27}$$

We also choose $H(x) = 1$, for simplicity and the nonlinear operator N as

$$N(\phi(x;\, p)) = \emptyset^{(8)}(x) - \emptyset(x) + 48e^x + 16xe^x \tag{28}$$

In view of the boundary conditions (13), the initial guess is obtained as

$$y_0 = 0.5\,(2\,x - x^3 - 0.662336x^4 - 0.266737\,x^5 - 0.0429543\,x^6 - 0.0279728x^7) \tag{29}$$

Following the explanation given in the previous section, the m^{th} order deformation equation becomes

$$L(y_m - \chi_m y_{m-1}) = hR_m(\bar{y}_{m-1}), \tag{30}$$

with

$$y^{(0)}(0) = 0, y^{(0)}(1) = 0, y^{(1)}(0) = 0, y^{(1)}(1) = 0,$$
$$y^{(2)}(0) = 0, y^{(2)}(1) = 0, y^{(3)}(0) = 0, y^{(3)}(1) = 0 \tag{31}$$

where

$$R_m(\bar{y}_{m-1}) = y^{(8)}{}_{m-1}(x) - y_{m-1}(x) + ((1-\chi_m) * (48e^x + 16xe^x)) \tag{32}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{33}$$

Now the solution of m^{th} order deformation equation for $m \geq 1$ becomes

$$y_m = \chi_m y_{m-1} + L^{-1}(h * R_m(\bar{y}_{m-1})). \tag{34}$$

Using symbolic computation software's such as Matlab or Maple or Mathematica, we can recursively obtain $y_0, y_1, y_2, \ldots$.

The approximate solution of (24) can be obtained by setting h = - 1, as below

$$\begin{aligned} y(x) = {} & -6560 + 6560e^x - 6399x - 160e^x x - 3120x^2 - 1012.33x^3 - 204.337832x^4 \\ & - 47.8582315x^5 - 7.7681529x^6 - 1.05833136x^7 - 0.132143x^8 - 0.014283x^9 \\ & - 0.00138889x^{10} - 000012273x^{11} - 9.93734 * 10^{-6}x^{12} - 7.4241 * 10^{-7}x^{13} \\ & - 5.148.7 * 10^{-8}x^{14} - 3.33034 * 10^{-9}x^{15} - 2.01885 * 10^{-10}x^{16} + O(17) \end{aligned} \tag{35}$$

Solution obtained through HAM is compared with exact and Variational Iteration Technique (VIT) solutions and are presented in the following table2.

Table 2:

x	Exact solution	Error of HAM	Error of VIT
0.0	0.000000000000000	0.00	0.00
0.1	0.099465382626808	9.89×10^{-13}	3.73×10^{-9}
0.2	0.195424441305627	2.09×10^{-13}	6.61×10^{-9}
0.3	0.283470349590961	2.07×10^{-13}	2.33×10^{-8}
0.4	0.358037927433905	1.14×10^{-12}	5.17×10^{-8}

0.5	0.412180317675032	4.83×10^{-12}	9.76×10^{-8}
0.6	0.437308512093722	8.53×10^{-12}	1.78×10^{-6}
0.7	0.422888068568800	4.86×10^{-11}	4.12×10^{-6}
0.8	0.356086548558795	7.27×10^{-11}	1.83×10^{-4}

Error = Exact solution − Approximate solution

4. Conclusions

In this paper, solution of eighth order boundary value problem has obtained through HAM. The numerical examples considered, revealed that HAM is both accurate and effective for solving eighth order boundary value problems. It can be concluded that HAM is a highly efficient method for solving high-order boundary value problems arising in various fields of engineering and science.

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