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Generating the Silver Ratio γ Using Different Approaches

Naresh Patel*, C.L. Parihar**

*Department of Applied Sciences (Mathematics)
Institute of Engineering & Technology
Devi Ahilya University, Indore-452017(M.P.) India
n_patel_1978@yahoo.co.in

**Indian Academy of Mathematics
15-Kaushaliyapuri, Chitawad Road
Indore-452001 (M.P.) India.
Profparihar@hotmail.com

Abstract:

Pell and Pell-Lucas numbers are respectively defined by the recurrence relation

$$\begin{cases} P_n = 2P_{n-1} + P_{n-2}, & P_0 = 0,\ P_1 = 1, \\ Q_n = 2Q_{n-1} + Q_{n-2}, & Q_0 = 2,\ Q_1 = 2. \end{cases}$$

Their Binet formulae are $P_n = \dfrac{\gamma^n - \delta^n}{\gamma - \delta}$ and $Q_n = \gamma^n + \delta^n$, where γ and δ are the roots of $x^2 - 2x - 1 = 0$ i.e. $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$ so that $\gamma\delta = -1$ or $\delta = \dfrac{-1}{\gamma}$ or $\gamma = \dfrac{-1}{\delta}$. Also $\gamma = \lim\limits_{n\to\infty} \dfrac{P_{n+1}}{P_n}$ is called Silver ratio.

In this paper, we have generated Silver ratio γ using geometry, differential equation, Newton-Raphson method, infinite continued fractions and bilinear transformations. In the last, we have derived the formula for calculating number of digits in Pell and Pell-Lucas numbers, that is, P_n and Q_n. Some of the interesting results of this paper are as follows:

(1) $x_{n+1} = \frac{P_{2^n+1}}{P_{2^n}}$ where $n \geq 1$. Hence $\lim_{m\to\infty} \frac{P_{m+1}}{P_m} = \gamma$, it follows that the sequence of approximations $\{x_n\}$ approaches γ as $n \to \infty$.

(2) Taking a triangle in which the square of one side is four times the product of other two sides, then the square root of the ratio of two sides of such triangle lies between $\frac{-1}{\gamma}$ and γ.

(3) The infinite continued fraction $[2;2,2,2,2,.....]$ converges to a limit γ.

(4) The Bilinear transformation $w = \frac{az+b}{cz+d}$ has two fixed points γ and δ if and only if $a - d = 2b = 2c \neq 0$, where *a, b, c, d* are integers; *a, d* > 0 and *ad-bc* = 1.

(5) $\sum_{n=1}^{\infty} (-1)^{n+1} \tan^{-1}\left(\frac{1}{P_{2i}}\right) = \tan^{-1}\left(\frac{1}{\gamma}\right)$.

(6) Number of digits in P_n $= \left\lceil n\log\gamma - \frac{1}{2}\log 8 \right\rceil$.

(7) Number of digits in = $\lceil n\log\gamma \rceil$

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Key words: Silver Ratio, Pell and Pell-Lucas Numbers, Binet Formulae, Bilinear Transformation, Number of Digits.

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1. Introduction:

1.1. Pell and Pell-Lucas Numbers [1]

Pell and Pell-Lucas numbers are respectively defined by the recurrence relation

$$\begin{cases} P_n = 2P_{n-1} + P_{n-2}, & P_0 = 0,\ P_1 = 1, \\ Q_n = 2Q_{n-1} + Q_{n-2}, & Q_0 = 2,\ Q_1 = 2. \end{cases}$$

Their Binet formulae are $P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$ and $Q_n = \gamma^n + \delta^n$, where γ and δ are the roots of $x^2 - 2x - 1 = 0$ i.e. $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$ so that $\gamma\delta = -1$ or $\delta = \frac{-1}{\gamma}$ or $\gamma = \frac{-1}{\delta}$. Also $\gamma = \lim_{n\to\infty} \frac{P_{n+1}}{P_n}$ is called Silver ratio.

1.2 Bilinear Transformation [6]

The transformation $w = \dfrac{az+b}{cz+d}$ where a, b, c, d are complex constants and $ad - bc \neq 0$ is called a bilinear transformation.

Such type of transformation was first studied by Mobius and hence it is sometimes called Mobius transformation. It can be written in the form $cwz + wd - az - b = 0$ which is linear in both w and z; and hence the name bilinear transformation.

1.3 Invariant or Fixed Points of Bilinear Transformation

The points which coincide with their transformations are called invariant Points of the transformation. If $z = f(z)$, that is, z maps into itself in the w-plane then

$$z = \frac{az+b}{cz+d} \quad \Rightarrow cz^2 - (a-d)z - b = 0$$

The roots of this equation are defined as fixed points of the bilinear transformation.

2. Generating the Silver Ratio γ by Newton-Raphson Method:

Let $f(x) = x^2 - 2x - 1$, then $f^{'}(x) = 2x - 2$. Since Newton's formula states that

$$x_{n+1} = x_n - \frac{f(x_n)}{f^{'}(x_n)}.$$

Taking $x_1 = 2$, we get

$$x_2 = x_1 - \frac{f(x_1)}{f^{'}(x_1)} = \frac{5}{2} = \frac{P_3}{P_2}$$

$$x_3 = x_2 - \frac{f(x_2)}{f^{'}(x_2)} = \frac{29}{12} = \frac{P_5}{P_4}$$

$$x_4 = x_3 - \frac{f(x_3)}{f^{'}(x_3)} = \frac{985}{408} = \frac{P_9}{P_8}$$

$$x_5 = x_4 - \frac{f(x_4)}{f^{'}(x_4)} = \frac{2378}{985} = \frac{P_{10}}{P_9}$$

These results can be generalized as given in the following theorem:

Theorem 2.1: $x_{n+1} = \dfrac{P_{2^n+1}}{P_{2^n}}$ where $n \geq 1$.

Proof: We prove it by Principle of Mathematical Induction (PMI). Since $x_2 = x_1 - \dfrac{f(x_1)}{f^{'}(x_1)} = \dfrac{5}{2} = \dfrac{P_3}{P_2}$, the formula is clearly true when $m = 1$. Assume it is true for an arbitrary positive integer k,

i.e. $x_{k+1} = \dfrac{P_{2^k+1}}{P_{2^k}} = \dfrac{P_{m+1}}{P_m}$ where $m = 2^k$. (1)

Now, $$x_{k+2} = x_{k+1} - \frac{f(x_{k+1})}{f^{'}(x_{k+1})} = x_{k+1} - \frac{x_{k+1}^2 - 2x_{k+1} - 1}{2x_{k+1} - 2}$$

$$= \frac{2x_{k+1}^2 - 2x_{k+1} - x_{k+1}^2 + 2x_{k+1} + 1}{2x_{k+1} - 2} = \frac{x_{k+1}^2 + 1}{2x_{k+1} - 2}$$

$$= \frac{\dfrac{P_{m+1}^2}{P_m^2} + 1}{2\left(\dfrac{P_{m+1}}{P_m}\right) - 2}$$ Using (1)

$$= \frac{P_{m+1}^2 + P_m^2}{2P_m\left(P_{m+1} - P_m\right)} = \frac{P_{2m+1}}{P_m\left(2P_{m+1} - 2P_m\right)}$$ (Since $P_{m+1}^2 + P_m^2 = P_{2m+1}$)

$$= \frac{P_{2m+1}}{P_m\left[2P_{m+1} - \left(P_{m+1} - P_{m-1}\right)\right]}$$ (Since $P_{n+1} = 2P_n + P_{n-1}$)

$$= \frac{P_{2m+1}}{P_m\left(P_{m+1} + P_{m-1}\right)} = \frac{P_{2m+1}}{P_m Q_m}$$ (Since $P_{m+1} + P_{m-1} = Q_m$)

$$= \frac{P_{2m+1}}{P_{2m}} = \frac{P_{2.2^k+1}}{P_{2.2^k}} = \frac{P_{2^{k+1}+1}}{P_{2^{k+1}}}$$ (Since $P_{2m} = P_m Q_m$)

Consequently, by PMI, theorem is true for every $n \geq 1$.

Hence $\lim_{m\to\infty} \dfrac{P_{m+1}}{P_m} = \gamma$, it follows that the sequence of approximations $\{x_n\}$ approaches γ as $n \to \infty$.

3. Silver Ratio γ in Solution of Pell Differential Equation:

Consider the second order differential equation $y'' - 2y' - y = 0$. Its characteristic equation is $m^2 - 2m - 1 = 0$ so that roots are γ and δ. Thus the general solution is

$y = Ae^{\gamma x} + Be^{\delta x} = Ae^{\gamma x} + Be^{\left(\frac{-1}{\gamma}\right)x}$ where A and B are constants.

4. Silver Ratio γ in Geometry [4]:

Take a triangle in which the square of one side is four times the product of other two sides. Suppose the sides are x, y and $2\sqrt{xy}$ units long where $x > y$. Taking $x = a^2$, $y = b^2$, the three sides are a^2, b^2 and $2ab$.

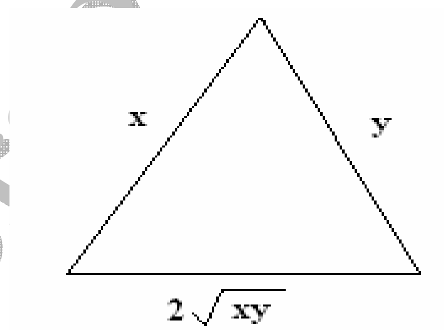
Then by the triangle inequality,

$$2ab + b^2 > a^2$$

$$\Rightarrow \left(\frac{a}{b}\right)^2 - 2\left(\frac{a}{b}\right) - 1 < 0$$

$$\Rightarrow \delta < \left(\frac{a}{b}\right) < \gamma$$

$$\Rightarrow \delta < \sqrt{\frac{x}{y}} < \gamma$$



That is, the square root of the ratio of two sides of such triangle lies between $\frac{-1}{\gamma}$ and γ.

5. Silver ratio γ as a Fixed Point of a Bilinear Transformation [6]:

Theorem 5.1: The Bilinear transformation $w = \frac{az+b}{cz+d}$ has two fixed points γ and δ if and only if $a - d = 2b = 2c \neq 0$, where a, b, c, d are integers; $a, d > 0$ and $ad - bc = 1$.

Proof: Suppose the Bilinear transformation has a fixed point, then

$$z = \frac{az+b}{cz+d}$$

$$\Rightarrow \quad cz^2 - (a-d)z - b = 0$$

Since there are two fixed points γ and δ, $c \neq 0$;

Also, $z^2-\left(\frac{a-d}{c}\right)z-\frac{b}{c}=(z-\gamma)(z-\delta)=z^2-2z-1$

Equating coefficients of like terms, we get

$$(a-d)=2c=2b$$

Thus, $a-d=2b=2c\neq 0$.

Conversely, let $a-d=2b=2c\neq 0$, then

$$w=\frac{az+b}{bz+(a-2b)}$$

Its fixed point are given by

$$z=\frac{az+b}{bz+(a-2b)}$$

$$\Rightarrow \quad bz^2-2bz-b=0$$

$$\Rightarrow \quad z^2-2z-1=0$$

$$\Rightarrow \quad z=\gamma,\delta$$

So, the fixed points are γ and δ.

6. Silver ratio γ as Infinite Continued Fractions [3]:

Let $x=[2;2,2,2,2,....]$. Since the infinite continued fraction converges to a limit,

So $[2;2,2,2,2,....]=[2;[2;2,2,2,2,.....]]$

That is

$$x=[2;x]$$

Or $\quad x=2+\frac{1}{x}$

Or $\quad x^2-2x-1=0$

Therefore $x=\gamma$, since $x>0$. Thus,

$$\gamma=[2;2,2,2,2,.....].$$

7. Silver ratio γ as convergent series of inverse tangent functions [7]:

Theorem: $\sum_{n=1}^{\infty}(-1)^{n+1}\tan^{-1}\left(\frac{1}{P_{2i}}\right)=\tan^{-1}\left(\frac{1}{\gamma}\right)$

Proof: Since inverse tangent function is a continuous increasing function, so $\tan^{-1}\left(\frac{1}{P_{2n}}\right) > \tan^{-1}\left(\frac{1}{P_{2n+2}}\right)$.

Also $\lim_{n\to\infty}\tan^{-1}\left(\frac{1}{P_{2n}}\right) = \tan^{-1}0 = 0.$ Therefore, the series converges and

$$\sum_{n=1}^{\infty}(-1)^{n+1}\tan^{-1}\left(\frac{1}{P_{2n}}\right) = \lim_{n\to\infty}\sum_{n=1}^{m}(-1)^{n+1}\tan^{-1}\left(\frac{1}{P_{2n}}\right)$$

$$= \lim_{m\to\infty}\tan^{-1}\left(\frac{P_m}{P_{m+1}}\right) = \tan^{-1}\left(\lim_{m\to\infty}\frac{P_m}{P_{m+1}}\right) = \tan^{-1}\left(\frac{1}{\gamma}\right).$$

8. Number of Digits in P_n [6]:

Using Binet formula, we can write

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{Or} \quad P_n = \frac{\gamma^n}{\sqrt{8}}\left[1 - \left(\frac{\delta}{\gamma}\right)^n\right]$$

Since $|\delta| < |\gamma|$, $\left(\frac{\delta}{\gamma}\right)^n \to 0$ as $n \to \infty$.

Therefore, when n is sufficiently large $P_n \approx \frac{\gamma^n}{\sqrt{8}}$

Or $\log P_n \approx n\log\gamma - \frac{1}{2}\log 8$

$\therefore$ Number of digits in P_n = 1 + Characteristic of $\log P_n$

$$= \lceil \log P_n \rceil$$

$$= \left\lceil n\log\gamma - \frac{1}{2}\log 8 \right\rceil$$

$$= \left\lceil n\log(1+\sqrt{2}) - \frac{1}{2}\log 8 \right\rceil$$

For example,

The number of digits in P_{15} is given by

$$\left\lceil 15\log(1+\sqrt{2}) - \frac{1}{2}\log 8 \right\rceil = \lceil 5.290090287 \rceil = 6$$

Notice that $P_{15} = 195025$ does indeed contains six digits. Likewise, P_{40} consists of 15 digits.

9. Number of Digits in Q_n [6]:

Using Binet formula, we can write

$$Q_n = \gamma^n + \delta^n$$

Or $$Q_n = \gamma^n \left[1 + \left(\frac{\delta}{\gamma} \right)^n \right]$$

Since $|\delta| < |\gamma|$, $\left(\frac{\delta}{\gamma} \right)^n \to 0$ as $n \to \infty$.

Therefore, when *n* is sufficiently large

$$Q_n \approx \gamma^n$$

Or $$\ln Q_n \approx n \log \gamma$$

$\therefore$ Number of digits in Q_n = 1 + Characteristic of $\log Q_n$

$$= \lceil \log Q_n \rceil$$

$$= \lceil n \log \gamma \rceil$$

$$= \left| n \log(1+\sqrt{2}) \right|$$

For example, The number of digits in Q_{16} is given by

$$\left| 16 \log(1+\sqrt{2}) \right| = \lceil 6.124410965 \rceil = 7$$

Notice that $Q_{16} = 1331714$ does indeed contains seven digits. Likewise, Q_{50} consists of 20 digits.

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