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NUMERICAL STUDY OF ONE –DIMENSIONAL CONTACT PROBLEM

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Abstract:

When two bodies which have different velocities come into contact an impact occurs. Within an impact analysis one is interested in the displacement of the bodies after impact and in the impact force as a function of time‘t ‘in one-dimension. The wave propagation due to material nonlinearity and hysteresis is studied after impact. The objective of this paper is to present a numerical study of propagating pulsed and harmonic waves in nonlinear media using a Finite difference scheme. This study focuses on longitudinal, one-dimensional wave propagation.

A bar 1 of length L_1 impacts another bar 2 of length ‘ L_2‘ . Both bars have the same material properties. The left bar has an initial velocity of V_0, whereas the right bar is at rest.

The solution of this problem can be derived from the one-dimensional wave equation

$$EA\, \partial^2 u/\partial x^2 = \rho A\, \partial^2 u/\partial t^2 \tag{1}$$

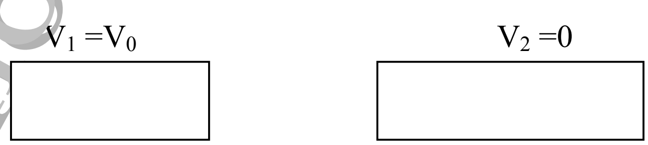

Fig-1 (LONGITUDINAL IMPACT OF TWO BARS)

$$\text{Here } c(u) \geq 0,\ R_N \leq 0 \ \text{ And } \ R_N\, c(u) = 0 \tag{2}$$

Here R_N, $c(u)$ are perpendicular to one another.

Furthermore one has to fulfill the initial and boundary conditions of the problem stated in the above figure and the standard contact conditions (2) which describe that no Penetration can occur at the contact point and also that the contact force has to be a compression force.

The solution of (1) is

$$u\,(t) = f(x-ct) + g(x+ct) \text{ with } c = \sqrt{\frac{E}{\rho}} \qquad (3)$$

Where c denotes the speed of wave travelling in the bars. Function f corresponds to a wave travelling in the X-direction of the bar, while g is associated with a wave travelling in the opposite direction.

$$\varepsilon = \frac{\partial u}{\partial x} = f'(x-ct) + g'(x-ct) \qquad (4)$$

$$V = \frac{\partial u}{\partial t} = c[-f'(x-ct)+g'(x-c)] \qquad (5)$$

The Normal stress σ in the bar is given by

$$\sigma = E\varepsilon = E/c\ \partial u/\partial t = c\ \rho\ \partial u/\partial t \qquad (6)$$

(6) Shows that there is a linear relationship between the stress at any point in the bar and the particle velocity.

When a wave travels with speed 'c 'along the bar, there is also a stress pulse which travels with the same velocity.

When such pulse reaches the free end of the bar, one can compute the behavior of the pulse from the condition that the end of the bar has to be stress free. This leads with (5) to the condition.

$$\sigma = E\ \partial u/\partial t = E\,[f'(x-ct) + g'(x-ct)] = 0 \quad \forall t \qquad (7)$$

From which a relation between f' and g' follows for the free ends

$x=0$ and $x= l_1 + l_2$

$$f'(x-ct) = -g'(x+ct) \qquad (8)$$

Thus a reflection occurs at the free ends with equal Amplitude in the stress pulse but with opposite velocity. Furthermore the initial conditions can be stated for the impact of two bars can be defined as below.

$v= [\partial u/\partial t]\ t=0 \quad = v_0$ for $0 \le x \le l_1$

$v= [\partial u/\partial t\ ]\ t=0 \quad = 0$ for $l_1 \le x \le l_2$

$$\sigma = E\left[\frac{\partial u}{\partial x}\right]_{at\ t=0} = f'(x) + g'(x) = 0 \text{ for } 0 \le x \le l_1 + l_2 \qquad (9)$$

From these conditions follow the initial values of f' and g' as

$$f'(x) = -\frac{v_0}{2c},\ g'(x) = \frac{v_0}{2c} \quad \text{for } 0 \le x \le l_1$$

since $f'(x) = 0$; $g'(x) = 0$ for $l_1 \le x \le l_1 + l_2$

The problem stated in Fig. (1) Can be solved with the relations above. Since the bar 1 has an initial velocity v_0, one has a distribution of f' and g' for t=0.

These are associated with two waves, one travelling in the X- direction and the other in the opposite direction. The two bodies remain in contact until

$$T_{imp.} = \frac{4\ l_1}{C}$$

Which corresponds to the time at which the reflected wave in bar 2 arrives at the contact point since the first bar is stress free this wave encounters a free end and hence does not enter bar 1, but reflects due to the stress free boundary condition. After that time the bars are no longer in contact. The final velocity of bar 1 after impact is $v_1=0$ and for bar 2 the velocity is then $v_2 = v_0/2$. It is clear that there is still an oscillation due to the travelling stress wave in bar-2where as bar 1 is at rest.

Which are different when compared to the wave solution above. This is due to the oscillations remaining in bar 2 after impact, which is, as also the impact time, neglected in the case of rigid body impact. So one has to study the Non-linear material behavior in order to get the wave propagation nature in impact problem. At the same time the impact time is very short and the stresses generated are high. Hence the Numerical methods to solve impact problems have to consider Non-linear material behavior and have to be designed for short time responses. Due to the possibility of high oscillatory responses near wave fronts we have to be careful when developing algorithms of impact problems. Moreover we have to consider the wave front characteristics within the Numerical scheme.

Here One –dimensional wave propagation in non-linear medium problem is considered. It is interesting to find that the well known non-linear elastic stress-strain relationship is a special case of integral relationship. By using this relationship from McCall and Guyer model of hysteretic materials can also be derived .Here we established a quadratic relation between stress and strain. Kurganov and Tadmor solved this nonlinear hyperbolic problem by high –resolution schemes. Here we made an attempt to solve the problem by Finite difference schemes. In the Finite difference scheme Iteration across the time level method is applied at selected time level. This process reduces the nonlinear partial differential equation to a linear partial differential equation. The reduced partial differential equation is solved by Finite difference method.

The study of wave propagation in materials and their characterization is a challenging area. Traditional ultrasonic techniques rely on wave propagation in linear elastic media. However, techniques based on the propagation of nonlinear waves are often more effective for evaluating and characterizing physical properties pertaining to reliability, durability and remaining life of materials and structure components. In addition some materials are inherently nonlinear such as rocks.

There has been extensive work in the published literature on nonlinear wave propagation. Kolsky (1963), Bland (1969), Debnath (1997), Drumheller (1998), Whitham (1999), Hamilton and Blackstock (1998) and Naugolnykh and Ostrovsky (1998) for an introduction and review of suitable techniques. The revise article by Norris (1998) provides a comprehensive review of nonlinear wave propagation in solids.

1. Nonlinear Governing laws:

For materials under plastic deformation, Materials with distributed damage, linear elastic Hooke's law is usually inadequate to describe their nonlinear, inelastic behavior. Various constitutive laws have been proposed. Here we study the class of materials whose behavior can be described by the following stress-strain relationship.

$$\frac{\partial\sigma(\varepsilon,\varepsilon^{'})}{\partial\varepsilon} = g(\varepsilon) - \alpha s[\sigma(\varepsilon_0) - f(\varepsilon_0)]\,e^{\alpha s[\varepsilon_0-\varepsilon]} - \alpha s\int_{\varepsilon_0}^{\varepsilon}[g\{\tau) - \frac{df(\tau)}{d\tau}]\,e^{\alpha s[\tau-\varepsilon]}\,d\tau$$

(10)

Integration gives

$$\sigma(\varepsilon,\varepsilon^{'}) = f(\varepsilon) + [\sigma(\varepsilon_0 - f(\varepsilon_0)]\,e^{\alpha s[\varepsilon_0-\varepsilon]} + \int_{\varepsilon_0}^{\varepsilon}[g(\tau) - \frac{df(\tau)}{d\tau}]\,d\tau \qquad (11)$$

Where ε_0 is the initial strain, s = sign ($\varepsilon^{'}$), α is a constant , and f(ε) and g(ε) are functions to be determined for a given material. In the above governing equations (10) , (11) a dot overhead denotes derivative with respect to time. Macki et al. (1993) and Mayergoyz (1991) show that with proper selection of α, f(ε) and g(ε) , the constitutive law described by (10) or (11) can be used to

describe a vast range of material behavior.

The traditional nonlinear elastic stress-strain law which is related to (10) or (11) is obtained by taking the assumption that no initial stress and strain. So that

$$\frac{\partial\sigma(\varepsilon,\varepsilon^{'})}{\partial\varepsilon} = g(\varepsilon) - \alpha s[f(0)]\,e^{\alpha s\varepsilon} - \alpha s\int_{0}^{\varepsilon}[g\{\tau) - \frac{df(\tau)}{d\tau}]\,e^{\alpha s[\tau-\varepsilon]}\,d\tau \qquad (13)$$

By selecting $\alpha = 0$ and

$$g(\varepsilon) = E(1 - \gamma\varepsilon - \delta\,\varepsilon^2) \qquad (14)$$

One can reduce the stress-strain relationship of (13) to the well-known nonlinear elastic constitutive law,

$$\frac{\partial\sigma(\varepsilon,\varepsilon')}{\partial\varepsilon} = E(1-\gamma\varepsilon-\delta\varepsilon^{2}-.....) \qquad (15)$$

Where E is the second order Elastic (Young's) modulus. $E\gamma$ is called the third order elastic constant, Equation (15) was derived by Landau and Lifshitz (1959) by expanding the strain energy density function for hyper-elastic materials.

Equations (15) do not show any hysteresis in the stress-strain relationship. The hysteretic behavior is accounted for by using a nonzero α . Means, call α the hysteresis parameter. B y substituting

$$\frac{df(\varepsilon)}{d\varepsilon} = g(\varepsilon) - E(1+\alpha\Delta\varepsilon), \quad f(0) = -\frac{E\Delta\varepsilon}{s} \qquad (16)$$

In the equation (13) we get

$$\frac{\partial\sigma(\varepsilon,\varepsilon')}{\partial\varepsilon} = g(\varepsilon) - E(1+\alpha\Delta\varepsilon) + E\, e^{-\alpha s\varepsilon} \qquad (17)$$

For small values of α

$$e^{\pm\alpha s\varepsilon} \approx 1 \pm \alpha s\varepsilon \qquad (18)$$

Equation (17) together with (14) reduces to

$$\frac{\partial\sigma(\varepsilon,\varepsilon')}{\partial\varepsilon} = E\left[1 - \gamma\varepsilon - \delta\varepsilon^{2} - \alpha(\Delta\varepsilon + s\varepsilon)\right]$$ (Special case of (11) or (12))

(19)

This is identical to the stress-strain relationship derived in McCall and Guyer (1994); Guyer and McCall (1995) and Van Den Abeele et al. (2000a, b).

Now, we consider a one-dimensional problem of wave propagation through a nonlinear medium. For small strain deformation, the equation of motion can be written as (Achenbach, 1999)

$$\frac{1}{\rho}\frac{\partial\sigma}{\partial x} = \frac{\partial^{2}u}{\partial t^{2}} \qquad (20)$$

Where u(x, t) is the displacement in the x-direction, ρ is the mass density, and $\sigma(x,t)$ is the normal stress in the x-direction. For the small strain deformation considered here, the normal strain in the x-direction is

$$\varepsilon = \frac{\partial u}{\partial x} \qquad (21)$$

Next assume that the nonlinear constitutive relationship of the medium be described by

$$\sigma = \sigma(\varepsilon, \varepsilon') \qquad (22)$$

Apply (22) and $c = \sqrt{\frac{E}{\rho}}$ in (20) yields

$$\frac{1}{C^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \left[\frac{1}{E}\frac{\partial \sigma}{\partial \varepsilon} - 1\right]\frac{\partial^2 u}{\partial x^2}$$
(23)

Where E is the elastic Young's modulus and $c = \sqrt{\frac{E}{\rho}}$ can be considered as the phase velocity.

This nonlinear equation is solved by applying finite difference method. In the middle Iteration across the time –step concept is introduced.

From equation (15) the case of a nonlinear material defined by the first two terms so that

$$\frac{\partial \sigma(\varepsilon, \varepsilon')}{\partial \varepsilon} = E(1 - \gamma\varepsilon) \qquad (24)$$

So that $\sigma = E\left(\varepsilon - \frac{1}{2}\gamma\,\varepsilon^2\right)$ (25)

Clearly, when $\gamma = 0$, the material is linear elastic. The parameter γ indicates the amount of material nonlinearity. The parameter γ defined here is identical to the acoustic nonlinear parameter (Cantrell and Yost, 1990). The acoustic nonlinear parameter arises in metals due to lattice anharmonicity which is usually very small in comparison to the elastic deformation of the metals. So we can study stress-strain curves for various values of Υ. For choice γ =10000, γ=5000 and γ =2500 respectively. From (25) dictates that the material behaves differently in tension and Compression, although the difference is only to the second order. In the literature, such material behavior is sometimes referred to as pseudo elastic. To model materials with identical nonlinear tensile and compressive behavior, only the quadratic terms in (15) should be used

Apply (25) in (23)

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(1 - \gamma\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2} \qquad (26)$$

This is the same equation derived by Gol'dberg (1961) based on the first principles of classical elasticity. Numerical methods can be applied to study the solution behavior (26) . All evaluations are based on the conservation law.

2. Numerical solution of wave equation:

Apply Finite difference Scheme on equation (26) so that it becomes a difference equation , $\frac{\partial u}{\partial x}$ value write as it is and it's value is calculated at each iteration. This process helps us to solve the Numerical scheme with out Non-linear equations. This method is called iteration across the time step.

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

(27)

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \qquad (28)$$

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = c^2\left(1-\gamma\frac{\partial u}{\partial x}\right)\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \frac{k^2}{h^2}c^2\left(1-\gamma\frac{\partial u}{\partial x}\right)(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \beta(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) \qquad (29)$$

Where $\beta = \frac{k^2}{h^2}c^2\left(1-\gamma\frac{\partial u}{\partial x}\right)$

$$u_{i,j+1} = -u_{i,j-1} + 2(1-\beta)u_{i,j} + \beta(u_{i+1,j} + u_{i-1,j}) \qquad (30)$$

Formula (30) shows that the function values at the j th and j-1th time levels are required in order to determine those at the (j+1)th time level. We have to select the mesh ratio

$\alpha = \frac{k}{h} \le 1$ In order to get the convergent solution.

The boundary conditions are u(0,t) =0 $\Rightarrow$ u(0,jk) =0 for j = 1,2,3……

U(15, t) = -0.03 $\Rightarrow$ u(15,jk)=-0.03 for j = 1,2,3….. For t>0

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = v_0;\ 0 \le x \le 7.5$$ so that

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0;\ 7.5 \le x \le 15$$

$$\Rightarrow u_{i,1} = u_{i,0} + v_0 k \qquad (31)$$

$$\Rightarrow u_{i,1} = u_{i,0} \qquad (32)$$

Initial displacement u (l_1,0) = 0.5 sinx

$$\Rightarrow u(7.5,0) = 0.5 \sin(ih) \qquad (33)$$

Apply the iteration across the time-step method with specific numerical code we can get the following results. Also the wave propagation is also plotted with various time levels with v_0 =0.5 m/s

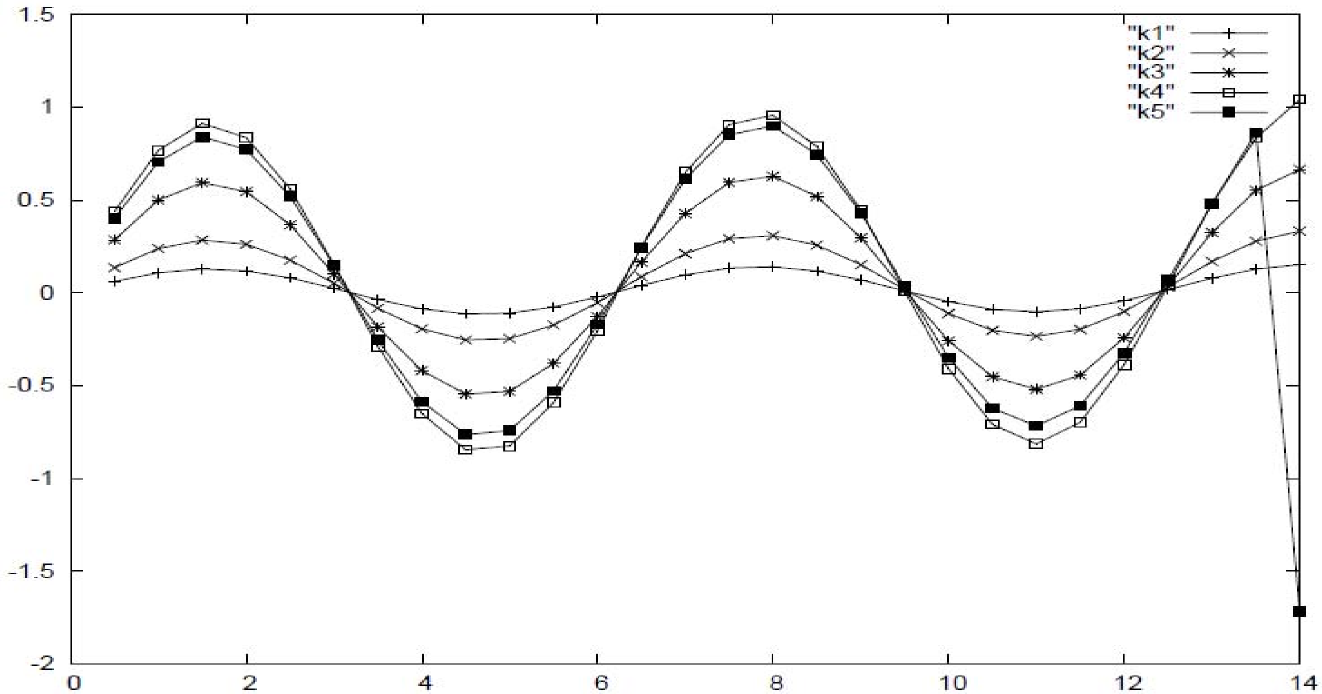

Figure 1Non -linear wave propagation

3. Numerical results:

Level-1:

0.5	0.06092	5.5	-0.0771925	10.5	-0.088962
1	0.10718	6	-0.0229269	11	-0.102999
1.5	0.127687	6.5	0.03989	11.5	-0.0864315
2	0.117662	7	0.0961233	12	-0.0430716
2.5	0.079809	7.5	0.13225	12.5	0.0167098
3	0.02364	8	0.13967	13	0.0785209
3.5	-0.0368479	8.5	0.116811	13.5	0.127473
4	-0.0866003	9	0.0695148	14	0.151826
4.5	-0.113191	9.5	0.00960611	14.5	0.145862
5	-0.109866	10	-0.0480026	15	0.111286

Level-2:

0.5 0.135289	**5.5-0.174153**	**10.5 .202572**
1 0.237945	**6 -0.0536828**	**11-0.234016**
1.5 0.283322	**6.5 .0858074**	**11.50.197392**
2 0.260802	**7 0.210655**	**12 -0.101177**
2.5 0.176386	**7.5 0.290782**	**12.50.0315613**
3 0.051234	**8 0.30706**	**13 0.168814**
3.5 .0835243	**8.5 0.255994**	**13.5 0.277467**

4 -0.194405 4.5-0.253772 5 -0.246599	9 0.150577 9.5 .0171066 10 -0.111248	14 0.331407 14.5 0.317918 15 0.524829

Level-3:

0.5 0.284054 1 0.499296 1.5 .593763 2 0.545058 2.5 0.365843 3 0.100729 3.5 -0.18464 4 -0.419661 4.5 0.546058 5 -0.532151	5.5 0.380609 6 -0.127802 6.5 0.16511 7 0.427146 7.5 0.594885 8 0.627992 8.5 0.519098 9 0.295597 9.5 0.012944 10 -0.258922	10.50.45270411 11 -0.520226 11.5 -0.444217 12 -0.242556 12.5 0.0361203 13 0.324315 13.5 0.552204 14 0.664725 14.5-0.0125258 15 2.6067

Level-4:

0.5 0.436569 1 0.76723 1.5 0.912005 2 0.836427 2.5 0.559981 3 0.151328 3.5 -0.2885 4 -0.650837 4.5 -0.845991 5 -0.825204	5.5 -0.592584 6 -0.204107 6.5 0.246096 7 0.648776 7.5 0.906326 8 0.956663 8.5 0.788448 9 0.44384 9.5 0.00819363 10 -0.410852	10.5 -0.709718 11 -0.814258 11.5 -0.697889 12 -0.388132 12.5 0.0401606 13 0.483102 13.5 0.833227 14 1.04108 14.5 -0.522624 15 4.97338

Level-5:

0.5 0.401638 1 0.706164 1.5 0.840245 2 0.77228 2.5 0.520126 3 0.146751 3.5 0.255209 4 -0.586116 4.5-0.763729 5 -0.743334	5.5 0.528708 6 -0.171163 6.5 0.24297 7 0.61354 7.5 .851021 8 0.898517 8.5 0.745602 9 0.430953 9.5 0.032826 10 -0.350077	10.5 .622793 11 -0.717303 11.50.609284 12 -0.323911 12.50.070126 13 0.477615 13.50.862448 14 -1.71805 14.5 11.4623 15 -14.2961

4. Result Analysis:

When ever impact occurs the velocities of the two objects are changes according to the starting compression force applied at the impact point. When ever an impact occurs a longitudinal sound wave is generated and it propagates in the region upto free end of the second object. When it reaches to the free end a reflection occurs. That is why the boundary condition at the free end is assumed as negative but small in magnitude.

The displacement in terms of the length of the impact system with respective to time is drawn in Figure-1. It gives the following inferences.

1) The Longitudinal sound waves harmonic in the given slot of time interval, since the amplitude of u(x, t) fluctuates about the mean position zero.

2) The progression of waves exhibits higher harmonic generation, since the u(x, t) repeats at the certain length of propagation.

3) The impact of collision of the material shows the vibrational state with in the system with Non-linear behavior.

4) The smooth curve interprets the uniform distribution of molecules of materials in other words the defects and imperfections are might be absent (almost) in the system.

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