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A GENERAL MATHEMATICAL MODEL FOR TWO ECOLOGICALLY INTERACTING SPECIES

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ABSTRACT
Competition between two or more species or individuals would occur when they are to strive together in a habitat if the resources for their growth or existence are in a short supply. It arises essentially during the struggle for existence. Popular histories with competition have been dealt with the researchers such as Gause , Paul Colinvaux .In this paper the structuring of basic model equations for two species interactions has been presented via block diagrams. Various concepts of stability of equilibrium states are reviewed. The concept of global stability using Liapunov's method is also sketched.

Key words :Model equations , Stability ,Interactions,Equilibrium states, Liapunov's method

AMS Classification : 92 D 25, 92 D 40

1.1 Introduction

Ecology relates to the study of living beings (animals and plants) in relation to their habits and habitats. This discipline of knowledge is a branch of evolutionary biology purported to explain how or to what extent the living beings are regulated in nature. Allied to the problem of population regulation there are problems of species distribution-prey-predator, competition and so on.

As models in any branch of science and technology, mathematical models in theoretical ecology are of great importance and utility because they both answer and raise several questions related to natural phenomena. The formulations of problems in ecology are quiet complex as one should have a deeper insight into a situation before we attempt to formulate a mathematical model.

It is natural that two or more species living in a common habitat interact in different ways with each other. Some of the possible interactions between different species is shown in the table 1.1. Definitions of different interactions along with various examples are given in the table.

Table 1.1: Interaction between species with examples

Sno	Interaction	Definition	Examples
1	Neutralism	Absence of any interaction between members of a mixed population.	Certain plant species, bacteria, animal species which do not affect each others.
2	Competition	A conflict for nutrients, space or some other factor which results in all members of the mixed populations growing when compared with the growth characteristic alone.	Honey bees & butterflies competing for honey, rabbits & deer competing for carrots, paramecium Aurelia & P. caudatum
3	Mutualism	Both members of a mixed population benefit from the presence of each other.	Lichens which are associations of an algae with a fungus, insects & flowers, wild bees (Nomia) & Lucerne plants
4	Syntrophism	This is a special case of mutualism. This occurs when an organism is growing in an environment that can grow only if it has a close association with second organism which grows on metabolic products extracted by the first one.	Photosynthetic bacterial cultures, Chlorobium & cyano bacteria
5	Commensalism	One member of a mixed population benefits from another member which is unaffected itself.	Sucker Fish is found attached sharks, whales & turtles. Oystercrab is found in mantle cavity of the oyster.Epiphytes & liaxas are commensal plants.
6	Ammansalism	One population adversely effects the growth of another population which itself being unaffected by it.	Beet & Mustard, Potato & Pumpkin, Tomato & Cucumber
7	Parasitism	One organism consumes another, often in subtle, non-delitating relationship.	Thiobacillus & lactic acid bacteria
8	Predation	One organism ingests another organism and consumes it, often in a violent destructive relationship.	Measles& human body, ox- tapeworm & cattle, Peach potato aphid & Peach tree, ants& wasp species Foxes & Rabits, Lion & Deer, Cats&Rats, Herbivores&Plants,Didiniumnasutum & Paramecium caudatum, snakes & frogs

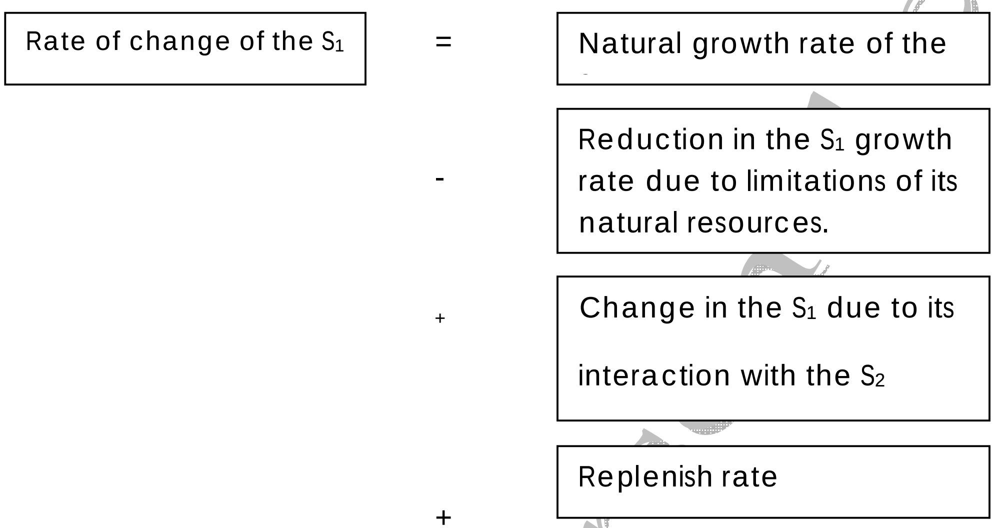
1.4 The Basic model equations for each of the interactions

The model equations for each of the interactions already listed in Chapter I have been presented in the treatises of the authors such as Kapur[23], Svirezhev and Logofet [46], Freedman [15], and Cushing [11].

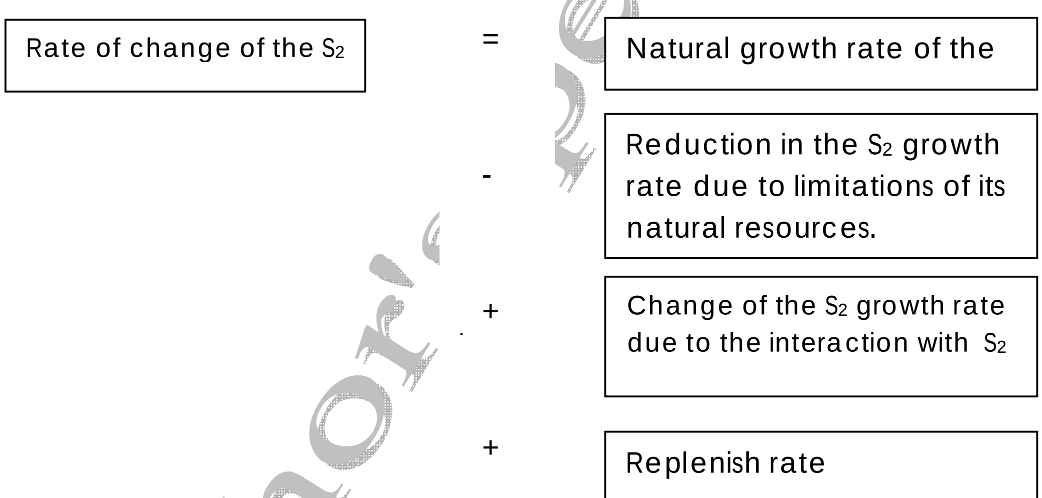
Based on the ideas suggested therein, the fundamental equations for a general interaction between two species say, S_1 and S_2 can be modeled as a pair of nonlinear coupled ordinary differential equations.

The following block diagrams explain the formulation of the basic model equations.

(i) Species S_1 :



(ii) Species S_2 :



The basic model equations are

$$\frac{dN_1(t)}{dt} = a_1 N_1(t) + a_{12} N_1(t) N \ldots {}_{11} N_1^{2}(t) + b_1 N_1(t) + H_1(t)$$
$$\frac{dN_2(t)}{dt} = a_2 N_2(t) + a_{21} N_1(t) N \ldots a_{22} N_2^{2}(t) + b_2 N_2(t) + H_2(t) \qquad (1.4.1)$$

$N_i(0) = N_{i0} > 0$ (1.4.2)

i = 1, 2 are the initial population siz… ecies S_1 & S_2 respectively.

In the above equation (1.4.1)

$N_1(t)$: Strength of the species S_1 at time

$N_2(t)$: Strength of the species S_2 at time t.

a_1, a_2 : Relative growth rates of S_1 & S_2

a_{11}, a_{22} : Self inhibition of S_1& S_2

a_{12}, a_{21} : Inhibition coefficients of S_1 due to S_2 and S_2 due to S_1

b_1, b_2 : Mutation coefficients of S_1 due to S_2 and S_2 due to S_1

$H_1(t)$, $H_2(t)$: The replenishments/renewals of S_1, S_2 per unit time H_1, H_2 are negative or positive or zero, according as, they are replenishments or renewals or none.

$\frac{a_1}{a_{11}}, \frac{a_2}{a_{22}}$: Carrying capacities of S_1 & S_2 (these parameters characterize the amount of resources available for the consumption of exclusively for the two species).

By appropriating proper signs and values to the parameters, the specific interactions can be realized such as the following mentioned below:

a. Prey-predation :

$a_1 > 0, a_{12} < 0, b_1 = 0$;
$a_2 < 0, a_{21} > 0, b_2 = 0$
(With S_1 and S_2 as prey and predator respectively)

b. Competition :

$a_1 > 0, a_{12} < 0, b_1 = 0$;
$a_2 > 0, a_{21} < 0, b_2 = 0$
(With S_1 and S_2 both competing with each other)

c. Mutualism :

$a_1 > 0, a_{12} > 0, b_1 = 0$;
$a_1 > 0, a_{21} > 0, b_2 = 0$
(With S_1 and S_2 both helping each other)

d. Mutation :

$a_1 > 0, a_{12} > 0, b_1 > 0$ and small ;
$a_2 < 0, a_{21} > 0, b_2 > 0$ and small.
(With S_1 and S_2 as mutant species)

Further, the self-inhibition coefficients a_{11}, a_{22} are positive or zero depending upon the carrying capacities of each of the species.

The non-linearity of the set (2.1) makes the equations intractable to obtain an exact analytical solution enabling a thorough investigation by the mathematical scientists/bio-mathematicians. However, one can examine the character of the solution in special cases. Adopting a method such as space-portrait analysis, the behaviour of the species S_1 and S_2, in some special cases like prey-predator, competition, etc., has been extensively dealt within the treatise of Braun [7], Freedman [16], Kapur [24], Varma [48], etc.

The model equations distinctive of the various types of basic interactions mentioned above between two species form a nucleus in setting- up the model equations for the composite interactions among several species. A general discussion on the multi-species population models can be referred in the monographs like the one by Kapur [24].

In the foregoing chapters, some theoretical investigations on the species interactions related to two competing species are presented. Stability analysis of the equilibrium states of the models proposed has been carried out and approximate solutions of some systems perturbed from their possible equilibrium states have been carried out.

1.4.3 Equilibrium States:

The basic model equations for a two interactive species system is given by the following set of non-linear first order simultaneous differential equations

$$\frac{dN_1}{dt} = f_1(N_1, N_2) \;\&\; \frac{dN_2}{dt} = f_2(N_1, N_2) \qquad (1.4.3)$$

where N_1 is population of the S N_2 is population of the S_2 and f_1, f_2 are functions of N_1, N_2

A solution $\left(\overline{N}_1, \overline{N}_2\right)$ of the system (1.4.3) is obtained by solving simultaneous equations $\frac{dN_1}{dt} = 0,\ \frac{dN_2}{dt} = 0$ is called an equilibrium state $\left(\overline{N}_1, \overline{N}_2\right)$ of the system

An equilibrium state $\left(\overline{N}_1, \overline{N}_2\right)$ may be called

i) a fully washed out state if $\overline{N}_1 = 0; \overline{N}_2 = 0$

ii) S_1 washed out state if $\overline{N}_1 = 0; \overline{N}_2 \neq 0$

iii) S_2 washed out state if $\overline{N}_1 \neq 0; \overline{N}_2 = 0$

iv) Co-existent state or the normal steady state if $\overline{N}_1 \neq 0; \overline{N}_2 \neq 0$

1.4.5 Stability of a Steady State :

A steady state of a physical system is said to be stable if it is back to the state after a small perturbation i.e., it has got a reasonable degree of permanence. If a small disturbance from the steady state leads to a larger and larger departure from that point then the steady state is said to be unstable.In other words an equilibrium state is said to be stable if all paths that get sufficiently close to the point stay close to that point. Further, the point is said to be asymptotically stable if every path around the point approaches the point as $t \rightarrow \propto$.

1.4.6. Stability of the Equilibrium States- General Analysis :

To examine the stability of the equilibrium state $\left(\overline{N}_1, \overline{N}_2\right)$ we consider perturbation (u_1, u_2) such that $N_1 = \overline{N}_1 + u_1,\ N_2 = \overline{N}_2 + u_2$ (1.4.4)

Here u_1 and u_2 are assumed to be small.

After linearization we get $\frac{dU}{dt} = AU$ (1.4.5)

where $A = \begin{bmatrix} \frac{\partial f_1}{\partial \overline{N}_1} & \frac{\partial f_1}{\partial \overline{N}_2} \\ \frac{\partial f_2}{\partial \overline{N}_1} & \frac{\partial f_2}{\partial \overline{N}_2} \end{bmatrix}$ and $U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ (1.4.6)

The solution is of the form : $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{\lambda t}$ (1.4.7)

Where λ satisfies the secular equation

$$\text{i.e., } \lambda^2 - \left[\frac{\partial f_1}{\partial \overline{N}_1} + \frac{\partial f_2}{\partial \overline{N}_2}\right]\lambda + \left[\frac{\partial f_1}{\partial \overline{N}_1}\frac{\partial f_2}{\partial \overline{N}_2} - \frac{\partial f_1}{\partial \overline{N}_2}\frac{\partial f_2}{\partial \overline{N}_1}\right] = 0 \qquad (1.4.8)$$

The equilibrium state (N_1, N_2) is stable if both the roots of the equation (1.4.8) have negative real parts.

1.4.7. Liapunov's method of Global Stability :

If the total energy of physical system has a local minimum at a certain equilibrium point then the point is said to be stable. Liapunov generalized this principle by constructing a function $E(N_1, N_2)$ whose rate of change is given by

$$\frac{dE}{dt} = \frac{\partial E}{\partial N_1}\frac{dN_1}{dt} + \frac{\partial E}{\partial N_2}\frac{dN_2}{dt} = \frac{\partial E}{\partial N_1} f_1 + \frac{\partial E}{\partial N_2} f_2 \qquad (1.4.9)$$

corresponding to the autonomous system (1.4.3)

E is said to be positive definite

If $E(N_1,N_2)>0$ for $(N_1,N_2)\neq(0,0)$
and $E(N_1,N_2)\neq 0$ for $(N_1,N_2)=(0,0)$ (1.4.10)

E is said to be positive semi-definite

if $E(N_1, N_2) \geq 0$ for $(N_1, N_2) \neq (0, 0)$
and $E(N_1, N_2) = 0$ if $(N_1, N_2) = (0, 0)$ (1.4.11)

E is said to be negative definite

if $E (N_1, N_2) < 0$ for $(N_1, N_2) \neq (0, 0)$
and $E(N_1, N_2) \neq 0$; if $(N_1, N_2) = (0, 0)$ (1.4.12)

E is said to be negative semi-definite

if $E(N_1, N_2) \leq 0$ for $(N_1, N_2) \neq (0, 0)$
and $E(N_1, N_2) = 0$ if $(N_1, N_2) = (0, 0)$ (1.4.13)

Liapunov's function: A positive definite function $E(N_1, N_2)$ with the property that $\frac{dE}{dt} = \frac{\partial E}{\partial N_1} f_1 + \frac{\partial E}{\partial N_2} f_2$ is negative semi-definite is called a Liapunov's function for the system (1.4.3).

Following theorem characterizes the global stability.

Theorem (A): If there exists a Liapunov's function E (N_1, N_2) for the system (1.4.3) the equilibrium point (0, 0) is stable. Further more if this function has additional property that dE/dt is negative definite then (0, 0) is asymptotically stable.

The following theorem helps to ascertain definiteness of a Liapunov's function:

Theorem (B): The function $E(x, y) = ax^2+bxy+cy^2$ is positive definite if $a > 0$ and $b^2-4ac<0$ and negative definite if $a < 0$, $b^2-4ac < 0$.

1.4.8 Simple Equilibrium points of Non-Linear Systems (Linearization) :

Consider an autonomous system: $\frac{dN_1}{dt} = f_1(N_1, N_2)$ & $\frac{dN_2}{dt} = f_2(N_1, N_2)$ with an isolated equilibrium point (0, 0)

If f_1 and f_2 can be expanded in power series in N_1, N_2 then the above system takes the form

$$\left.\begin{aligned} \frac{dN_1}{dt} &= a_1N_1 + b_1N_2 + c_1N_1^2 + d_1N_1N_2 + e_1N_2^2 + \ldots.. \\ \frac{dN_2}{dt} &= a_2N_1 + b_2N_2 + c_2N_1^2 + d_2N_1N_2 + e_2N_2^2 + \ldots. \end{aligned}\right\} \qquad (1.4.14)$$

when $|N_1|$ and $|N_2|$ are small, second and higher degree terms are discarded from (1.4.14) and a linear system is obtained as

$$\frac{dN_1}{dt} = a_1N_1 + b_1N_2 \quad \& \quad \frac{dN_2}{dt} = a_2N_1 + b_2N_2 \qquad (1.4.15)$$

The process of replacing system (1.4.14) by system (1.4.15) is called linearization. An equivalence of these systems is obtained in the following theorem without proof.

Theorem (C): Let (0, 0) be equilibrium point of non- linear system (1.4.14) and consider the related linear system (1.4.15). If the point (0, 0) of (1.4.15) is asymptotically stable then so also (0, 0) of (1.4.14).

CONCLUSIONS

This paper presents a brief introduction to some concepts of Mathematical modeling, Ecology and Competition. A few interesting examples for ecological symbiosis are given. Various ecological interactions are also explained . The local stability analysis for a general mathematical model of two ecologically interacting species and construction of Liapunov's function for Global Stability are also outlined here.

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