

**AN ENEMY AND TIME DELAYED AMMENSAL SPECIES PAIR-
LIAPUNOV'S STABILITY ANALYSIS****K.V.L.N.ACHARYULU**Faculty of Mathematics
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Email:kvlna@yahoo.com**ABSTRACT**

In the paper ,The global stability is proved at co-existence state with the help of a properly Liapunov's function for the mathematical model of a delayed Ammensal and enemy species pair with limited resources.

AMS Classification: 92 D 25, 92 D 40

Key words: Equilibrium states, Stability, Liapunov's function for global stability

1) Introduction:

The present investigation is principally concentrated on the establishment of the global stability of a delayed Ammensal and enemy species pair with limited resources at the co-existent equilibrium state by utilizing a property constructed by Liapunov's function. Acharyulu.K.V.L.N. and Pattabhi Ramacharyulu. N.Ch. [1-7] analyzed the global stability of an Ammensal- enemy system on the quasi-linear basic balancing equations at various resources with Liapunov's criteria. Local stability analysis for an Ammensal- enemy species with various resources in different cases has been also carried out in the author's earlier work. Many authors like Lotka[9], Kapur[8] etc. applied this method in various situations for building the global stability.

Notations adopted

N_1 and N_2 are the populations of the Ammensal and enemy species with natural growth rates a_1 and a_2 respectively.

a_{11} = The rate of decrease of the Ammensal due to insufficient food.

a_{12} = The rate of increase of the Ammensal due to inhibition by the enemy.

a_{22} = The rate of decrease of the enemy due to insufficient food.

$K_i = a_i/a_{ii}$ are the carrying capacities of N_i , $i = 1, 2$

$\alpha = a_{12}/a_{11}$ is the coefficient of Ammensalism.

ψ = the delay in receiving benefit from the enemy, which is a constant.

The state variables N_1 and N_2 as well as the model parameters $a_1, a_2, a_{11}, a_{22}, K_1, K_2, \alpha$ and ψ are assumed to be non-negative constants.

2) Basic equations:

(i). Equation for the growth rate of Ammensal species (S_1)

$$\frac{dN_1}{dt} = a_{11} [K_1 N_1 - N_1^2 - \alpha(t - \psi) N_1 N_2] \quad (1)$$

(ii). Equation for the growth rate of enemy species (S_2)

$$\frac{dN_2}{dt} = a_{22} [K_2 N_2 - N_2^2] \quad (2)$$

3) Equilibrium states:

The system under investigation has four equilibrium states:

$$E_1: \quad \bar{N}_1 = 0; \bar{N}_2 = 0$$

In this state both Ammensal and enemy are washed out

$$E_2: \quad \bar{N}_1 = 0; \bar{N}_2 = K_2$$

In this state, only enemy survives and the Ammensals are washed out

$$E_3: \quad \bar{N}_1 = K_1; \bar{N}_2 = 0$$

In this state, only Ammensal survives and enemies are washed out

$$E_4: \quad \bar{N}_1 = K_1 - \alpha K_2; \bar{N}_2 = K_2 \quad (3)$$

The state, in which the Ammensal and enemy co-exist, is called "The normal steady state".

4) Global stability by Liapunov's criteria:

The linearized perturbed equations over the perturbations (U_1, U_2) are

$$\frac{dU_1}{dt} = -a_{11} \bar{N}_1 U_1 - a_{12} \bar{N}_1 (1 + a_2 \psi) U_2 \quad (4)$$

$$\frac{dU_2}{dt} = -a_{22} \bar{N}_2 U_2 \quad (5)$$

The corresponding characteristic equation is

$$(\lambda + a_{11} \bar{N}_1)(\lambda + a_{22} \bar{N}_2) = 0 \quad (6)$$

$$\Rightarrow \lambda^2 + (a_{11} \bar{N}_1 + a_{22} \bar{N}_2) \lambda + a_{11} a_{22} \bar{N}_1 \bar{N}_2 = 0 \quad (7)$$

Equation (7) is of the form of $\lambda^2 + p\lambda + q = 0$

where

$$p = a_{11} \bar{N}_1 + a_{22} \bar{N}_2 > 0 \quad (8)$$

$$q = a_{11} a_{22} \bar{N}_1 \bar{N}_2 > 0 \quad (9)$$

\therefore The conditions for the existence of Liapunov's function are satisfied.

Now we define

$$E(U_1, U_2) = \frac{1}{2} (aU_1^2 + 2bU_1U_2 + cU_2^2) \quad (10)$$

where

$$a = \frac{(a_{22} \bar{N}_2)^2 + (a_{11} a_{22} \bar{N}_1 \bar{N}_2)}{D} \quad (11)$$

$$b = \frac{-a_{12} a_{22} \bar{N}_1 \bar{N}_2 (1 + a_2 \psi)}{D} \quad (12)$$

$$c = \frac{(a_{11} \bar{N}_1)^2 + (a_{12} \bar{N}_1 (1 + a_2 \psi))^2 + (a_{11} a_{22} \bar{N}_1 \bar{N}_2)}{D} \quad (13)$$

$$\text{where } D = pq = (a_{11} \bar{N}_1 + a_{22} \bar{N}_2) (a_{11} a_{22} \bar{N}_1 \bar{N}_2) \quad (14)$$

From equations (8) and (9) it is clear that $D > 0$ and $a > 0$

Also $D^2 (ac - b^2)$

$$= D^2 \left[\left(\frac{(a_{22} \bar{N}_2)^2 + a_{11} a_{22} \bar{N}_1 \bar{N}_2}{D} \right) \left(\frac{(a_{11} \bar{N}_1)^2 + (a_{12} \bar{N}_1 (1 + a_2 \psi))^2 + a_{11} a_{22} \bar{N}_1 \bar{N}_2}{D} \right) - \frac{(a_{12} a_{22} \bar{N}_1 \bar{N}_2)^2}{D^2} \right]$$

$$= 2a_{11}^2 a_{22}^2 \bar{N}_1^2 \bar{N}_2^2 + a_{11} a_{22} \bar{N}_1 \bar{N}_2 (a_{22}^2 \bar{N}_2^2 + a_{11}^2 \bar{N}_1^2 + a_{12}^2 \bar{N}_1^2 (1 + a_2 \psi)^2) > 0 \quad (15)$$

$$\Rightarrow D^2 (ac - b^2) > 0$$

$$\Rightarrow ac - b^2 > 0 \quad \text{i.e., } b^2 - ac < 0 \quad (16)$$

\therefore The function $E(U_1, U_2)$ at (10) is positive definite.

Further

$$\frac{\partial E}{\partial U_1} \frac{dU_1}{dt} + \frac{\partial E}{\partial U_2} \frac{dU_2}{dt} = \left[(aU_1 + bU_2)(-a_{11} \bar{N}_1 U_1 - a_{12} \bar{N}_1 (1 + a_2 \psi) U_2) + (bU_1 + cU_2)(-a_{22} \bar{N}_2 U_2) \right]$$

$$= -aa_{11} \bar{N}_1 U_1^2 - (aa_{12} \bar{N}_1 (1 + a_2 \psi) - ba_{11} \bar{N}_1 - ba_{22} \bar{N}_2) U_1 U_2 + (-ba_{12} \bar{N}_1 (1 + a_2 \psi) - ca_{22} \bar{N}_2) U_2^2 \quad (17)$$

Substituting the values of a , b and c from (11) (12) and (13) in (17) we get

$$\frac{\partial E}{\partial U_1} \frac{dU_1}{dt} + \frac{\partial E}{\partial U_2} \frac{dU_2}{dt} = - \left[\frac{a_{22}^2 \bar{N}_2^2 + a_{11} a_{22} \bar{N}_1 \bar{N}_2}{D} \right] a_{11} \bar{N}_1 U_1^2 +$$

$$\left[- \left(\frac{a_{22}^2 \bar{N}_2^2 + a_{11} a_{22} \bar{N}_1 \bar{N}_2}{D} \right) a_{12} \bar{N}_1 (1 + a_2 \psi) + \left(\frac{a_{12} a_{22} \bar{N}_1 \bar{N}_2 (1 + a_2 \psi)}{D} \right) a_{11} \bar{N}_1 + \left(\frac{a_{12} a_{22} \bar{N}_1 \bar{N}_2 (1 + a_2 \psi)}{D} \right) a_{22} \bar{N}_2 \right] U_1 U_2$$

$$+ \left[\left(\frac{a_{12} a_{22} \bar{N}_1 \bar{N}_2 (1 + a_2 \psi)}{D} \right) a_{12} \bar{N}_1 (1 + a_2 \psi) - \left(\frac{a_{11}^2 \bar{N}_1^2 + a_{12}^2 \bar{N}_1^2 (1 + a_2 \psi)^2 + a_{11} a_{22} \bar{N}_1 \bar{N}_2}{D} \right) a_{22} \bar{N}_2 \right] U_2^2$$

$$\frac{\partial E}{\partial U_1} \frac{dU_1}{dt} + \frac{\partial E}{\partial U_2} \frac{dU_2}{dt} = - \left[\left(\frac{a_{11} a_{22} \bar{N}_1 \bar{N}_2 (a_{11} \bar{N}_1 + a_{22} \bar{N}_2)}{D} \right) U_1^2 + \left(\frac{a_{11} a_{22} \bar{N}_1 \bar{N}_2 + (a_{11} \bar{N}_1 + a_{22} \bar{N}_2)}{D} \right) U_2^2 \right]$$

$$= - \frac{1}{D} [DU_1^2 + DU_2^2] \quad (18)$$

$$= - (U_1^2 + U_2^2) \quad (19)$$

$$\therefore \frac{\partial E}{\partial U_1} \frac{dU_1}{dt} + \frac{\partial E}{\partial U_2} \frac{dU_2}{dt} = -(U_1^2 + U_2^2) \quad (20)$$

which is clearly negative definite

So, $E(U_1, U_2)$ at (10) is a Liapunov function for the Linear system.

Next we have prove that E (U₁, U₂) is also a Liapunov function for the non-linear system.

If f₁ and f₂ are two functions in N₁ and N₂ are defined by

$$f_1(N_1, N_2) = N_1 [a_1 - a_{11} N_1 - a_{12} N_2(t - \psi)] \quad (21)$$

$$f_2(N_1, N_2) = N_2 [a_2 - a_{22} N_2] \quad (22)$$

Now we have to show that $\frac{\partial E}{\partial U_1} f_1 + \frac{\partial E}{\partial U_2} f_2$ is negative definite

Putting N₁ = $\bar{N}_1 + U_1$; N₂ = $\bar{N}_2 + U_2$ in (1) and (2) we get

$$\begin{aligned} \frac{dU_1}{dt} &= (\bar{N}_1 + U_1) \left[a_1 - a_{11} (\bar{N}_1 + U_1) - a_{12} \left[(1 - a_2 \psi) (\bar{N}_2 + U_2) + a_{22} (\bar{N}_2 + U_2)^2 \psi \right] \right] \\ &= a_1 \bar{N}_1 - a_{11} \bar{N}_1^2 - a_{11} \bar{N}_1 U_1 - a_{12} (1 - a_2 \psi) \bar{N}_1 \bar{N}_2 - a_{12} (1 - a_2 \psi) \bar{N}_1 U_2 \\ &\quad - a_{12} a_{22} \bar{N}_2^2 \bar{N}_1 \psi - 2a_{12} a_{22} \bar{N}_1 \bar{N}_2 U_2 \psi - a_{12} a_{22} \bar{N}_1 U_2^2 \psi \\ &\quad + a_1 U_1 - a_{11} \bar{N}_1 U_1 - a_{12} (1 - a_2 \psi) \bar{N}_2 U_1 - a_{12} (1 - a_2 \psi) U_1 U_2 - a_{11} U_1^2 - a_{12} a_{22} \bar{N}_2^2 U_1 \psi \\ &\quad - 2a_{12} a_{22} \bar{N}_2 U_1 U_2 \psi - a_{12} a_{22} U_2^2 U_1 \psi \\ \Rightarrow f_1(u_1, u_2) &= \frac{dU_1}{dt} - a_{11} \bar{N}_1 U_1 - a_{12} (1 - a_2 \psi) \bar{N}_1 U_2 + F(U_1, U_2) \end{aligned} \quad (23)$$

where F(U₁, U₂)

$$= -2a_{12} a_{22} \bar{N}_1 \bar{N}_2 U_2 \psi - a_{12} a_{22} \bar{N}_1 U_2^2 \psi - (a_{12} (1 - a_2 \psi) + 2a_{12} a_{22} \bar{N}_2 \psi) U_1 U_2 - a_{11} U_1^2 - a_{12} a_{22} U_2^2 U_1 \psi$$

$$(24) \text{ Similarly } \frac{dU_2}{dt} = (\bar{N}_2 + U_2) (a_2 - a_{22} (\bar{N}_2 + U_2))$$

$$= a_2 \bar{N}_2 - a_{22} \bar{N}_2^2 - a_{22} \bar{N}_2 U_2 + a_2 U_2 - a_{22} \bar{N}_2 U_2 - a_{22} U_2^2$$

$$= -a_{22} \bar{N}_2 U_2 + U_2 (a_2 - a_{22} \bar{N}_2) - a_{22} U_2^2$$

$$\Rightarrow f_2(U_1, U_2) = \frac{dU_2}{dt} = -a_{22} \bar{N}_2 U_2 + G(U_1, U_2) \quad (25)$$

$$\text{where, } G(U_1, U_2) = -a_{22} U_2^2 \quad (26)$$

From (10)

$$\frac{\partial E}{\partial u_1} = a U_1 + b U_2 \quad (27)$$

$$\frac{\partial E}{\partial u_2} = b U_1 + c U_2 \quad (28)$$

Now

$$\begin{aligned} \frac{\partial E}{\partial U_1} f_1 + \frac{\partial E}{\partial U_2} f_2 &= (a U_1 + b U_2) [-a_{11} \bar{N}_1 U_1 - a_{12} (1 - a_2 \psi) \bar{N}_1 U_2 + F(U_1, U_2)] \\ &\quad + (b U_1 + c U_2) [-a_{22} \bar{N}_2 U_2 + G(U_1, U_2)] \\ &= [(a U_1 + b U_2) (-a_{11} \bar{N}_1 U_1 - a_{12} (1 - a_2 \psi) \bar{N}_1 U_2) + (b U_1 + c U_2) (-a_{22} \bar{N}_2 U_2)] \\ &\quad + (a U_1 + b U_2) F(U_1, U_2) + (b U_1 + c U_2) G(U_1, U_2) \end{aligned} \quad (29)$$

$$\frac{\partial E}{\partial U_1} f_1 + \frac{\partial E}{\partial U_2} f_2 = - (U_1^2 + U_2^2) + (a U_1 + b U_2) F(U_1, U_2) + (b U_1 + c U_2) G(U_1, U_2) \quad (30)$$

By introducing the polar Co-ordinates $U_1 = r \cos \theta$ $U_2 = r \sin \theta$, equation (30) can be written as

$$\frac{\partial E}{\partial U_1} f_1 + \frac{\partial E}{\partial U_2} f_2 = -r^2 + r [(a \cos \theta + b \sin \theta) + F(U_1, U_2) + (b \cos \theta + c \sin \theta) G(U_1, U_2)] \quad (31)$$

Let us denote the largest of the numbers $|a|, |b|, |c|$ by M

Our assumptions imply that $|F(U_1, U_2)| < \frac{r}{6M}$ and $|G(U_1, U_2)| < \frac{r}{6M}$, for all sufficiently small $r > 0$.

$$\text{So, } \frac{\partial E}{\partial U_1} f_1 + \frac{\partial E}{\partial U_2} f_2 < -r^2 + \frac{4Mr^2}{6M} = -\frac{r^2}{3} < 0. \quad (32)$$

Thus $E(U_1, U_2)$ is a positive definite function with the condition

$$\frac{\partial E}{\partial U_1} f_1 + \frac{\partial E}{\partial U_2} f_2 \text{ is negative definite} \quad (33)$$

Hence, it is **asymptotically stable**.

Conclusion: The global stability is proved with the help of a properly Liapunov's function for the mathematical model of a delayed Ammensal and enemy species pair with limited resources.

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