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AN EXPONENTIALLY FITTED FINITE DIFFERENCE SCHEME FOR HEAT EQUATION

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ABSTRACT

An exponentially fitted finite difference scheme is presented for the unsteady one dimensional flow of heat in a finite bar subject to isothermal boundary conditions. The scheme is a modified form of classical explicit scheme and it is uniformly convergent. The scheme presented in this paper is new. The applicability of the explicit scheme is illustrated using four sample problems. Numerical results are presented.

Keywords: heat equation, exponentially fitted, uniformly convergent, finite difference schemes.

AMS (MOS) subject classification: 65F05, 65N30, 65N35, 65Y05.

1. INTRODUCTION

The numerical solution of the boundary value problem for the unsteady one dimensional flow of heat in a finite bar subject to isothermal conditions is given in this paper. The problem is to find the temperature u (x, t) at a point x at time t, which satisfies

$$u_t(x,t) = u_{xx}(x,t) + F(x,t),\ 0 < x < 1,\ t > 0, \tag{1.1}$$

$$u(x,0) = f(x),\ 0 \le x \le 1, \tag{1.2}$$

$$u(0,t) = 0,\ u(1,t) = 0,\ t \ge 0 \tag{1.3}$$

where $u_t = \partial u/\partial t$, $u_{xx} = \partial^2 u/\partial x^2$ and f(x) Is the initial prescribed temperature distribution . Assume f (x) can be expanded in a Fourier sine or cosine series and f (x) = sin ($a\pi x$) or cos ($a\pi x$) , a > 0.

Finite difference schemes for (1.1)-(1.3) have been discussed in [1,2,9-12]. A complete survey of finite difference schemes for (1.1)-(1.3) is given in [1,5]. The aim of this paper is to present an explicit scheme for (1.1)-(1.3) which is uniformly convergent. On applying the classical explicit scheme for (1.1)-(1.3), the scheme is stable if $r \le 1/2$ where $r = k/h^2$. To remove this restriction on r an exponential fitted finite difference scheme is presented in this paper based on the works related to uniformly convergent exponential fitted finite difference schemes [5-8].

Purely implicit schemes for the heat equation (1.1)-(1.3) will not give satisfactory result (refer Table 1) for the sample problem 1 given in section 3. It is observed that for large values of h and k the relative error increases exponentially whereas the absolute error decreases exponentially . In the method presented in this paper both the absolute and relative errors decreases exponentially even for large values of h and k.

In section 2 an explicit uniformly convergent exponentially fitted finite difference scheme is presented for the boundary value problem (1.1)-(1.3). The scheme is illustrated using four simple problems and the numerical results are given in section 3.

Throughout this paper $r = k/h^2$, where h and k are the mesh size along the x-axis and t-axis respectively.

2. EXPONENTIALLY FITTED SCHEME

The finite difference scheme for (1.1)-(1.3) is given by

$$((u_i^{j+1} - u_i^{j})/k) = \sigma_i^{j} \left((u_{i-1}^{j} - 2u_i^{j} + u_{i+1}^{j})/h^2 \right) + F(x_i , t_{j+1}), \qquad (2.1)$$

for i = 1(1)N-1 , j = 0(1))N-1,

$$u_i^{0} = f(x_i), \quad i = 0(1)N, \qquad (2.2)$$

$$u_0^{j} = 0, \quad u_N^{j} = 0, \quad j = 0(1)N \qquad (2.3)$$

where the fitting factor σ_i^{j} is defined as

$$\sigma_i^{j} = 1/ [\sigma(-(a\pi)^2 k)\ \sigma^*((a\pi/2) h)],$$

$$\sigma(-(a\pi)^2 k) = (a\pi)^2 k / [1- \exp(-- (a\pi)^2 k)] ,$$

$$\sigma^*((a\pi/2) h) = [\sin((a\pi/2) h) /((a\pi/2) h)]^2$$

and $f(x) = \sin (a\pi x)$ or $\cos (a\pi x)$.

As k goes to zero , σ goes to one and as h goes to zero , σ * goes to one and hence the fitting factor σ_i^{j} will goes to one as h and k goes to zero. Hence the scheme (2.1)-(2.3) is consistent with the given boundary value problem (1.1)-(1.3) as h and k goes to zero[3-8].

If $f(x) = \sin(\pi x)$ and $F(x,t) = 0$, then the scheme (2.1)-(2.3) will be of the form

$$u_i^{j+1} = ((B-A)/B)\, u_i^{j} + (A/(2B))\, u_{i-1}^{j} + (A/(2B))\, u_{i+1}^{j} \qquad (3)$$

where

$A = 1- \exp(-- \pi^2 k)$ and $B = 1 - \cos(\pi h)$.

On solving (3) for u_i^{j} we have

$$u_i^{j} = \exp(-- \pi^2 k)^j \sin(\pi ih), \text{ for all i and j.} \qquad (4)$$

The exact solution for the boundary value problem (1.1)-(1.3) with $f(x) = \sin(\pi x)$ and $F(x,t) = 0$ is of the form

$$u(x,t) = \exp(-- \pi^2 t)\ \sin(\pi x). \qquad (5)$$

From (4) and (5) , it is clear that

$$u_i^{j} = u(x_i, t_j) \text{ for all i and j.} \qquad (6)$$

That is, the scheme(2.1)-(2.3) solves the problem (1.1)-(1.3) exactly with $f(x) = \sin(\pi x)$ and $F(x,t) = 0$.

From (4) , (5) and (6) , it is clear that there is no restriction on r for numerical stability. That is , the scheme is unconditionally stable foe all values of h and k. Since the scheme (2.1)-(2.3) solves the problem(1.1)-(1.3) exactly with f (x) = sin (π x) and F (x,t) = 0, the scheme(2.1)-(2.3) is an exponentially fitted scheme[5-8] .

2.1. STABILITY ANALYSIS

The stability result presented in this paper is based on the analysis of Von Neumann[1,2, 9-12]. Set $U_p^{\,q} = A_{mn}$ exp (i m ph) exp (α n qk) (7)

where x_p = ph , t_q = qk , i = sqrt(-1) , m = aπ , n = m^2 , α = -1 and $U_p^{\,q}$ is the m^{th} Fourier component whose amplitude is A_{mn} and α is a complex constant whose real part must be negative . In order for $U_p^{\,q}$ to decay as q grows. For our simplicity set the expression (7) as

$$U_p^{\,q} = A_{mn} \exp (i\, m\, \theta)\ g^q$$

where θ = ph , g = exp (α n k) and g is called the amplification factor . For stability the absolute value of the amplification factor must not be greater than one. Now on inserting the above expression for $U_p^{\,q}$ into the relation (3) , one will get

$$g = 1 - 2r\ \sigma_i^{\,j} = \exp(\alpha\, n\, k).$$

The amplification factor satisfies the condition $|g| \le 1$ and so the scheme is unconditionally stable for all values of h and k.

2.3 UNIFORM CONVERGENT RESULT

The scheme (2.1)-(2.3) can be written in the form

$$u_i^{\,j+1} = ((B-A)/B)\, u_i^{\,j} + (A/(2B))\, u_{i-1}^{\,j} + (A/(2B))\, u_{i+1}^{\,j} + h\, F(x_i, t_{j+1}) + O(k^2 + \sigma_i^{\,j}\, k\, h^2) \qquad (8)$$

where

$$A = 1 - \exp(-(a\pi)^2 k) \quad \text{and} \quad B = 1 - \cos(a\pi h).$$

Now set $z_i^{\,j} = u_i^{\,j} - U_i^{\,j}$. (9)

where $u_i^{\,j}$ and $U_i^{\,j}$ are the solutions of schemes(2.1)-(2.3) and the boundary value problems (1.1)-(1.3) respectively.
Using (8) and (9) for i = 1(1)N-1, j = 0(1)N-1

$$z_i^{\,j+1} = ((B-A)/B)\, z_i^{\,j} + (A/(2B))\, z_{i-1}^{\,j} + (A/(2B))\, z_{i+1}^{\,j} + O(k^2 + \sigma_i^{\,j}\, k\, h^2) \qquad (10.1)$$

$$z_i^{\,0} = 0\ , i=0(1)N\ , \qquad (10.2)$$

$$z_0^{\,j} = 0\ , z_N^{\,j} = 0\ , j=0(1)N. \qquad (10.3)$$

The three coefficients on the right hand side of equation (10.1) sum to one for all values of r = k/h^2. Then ,

$$| z_i^{\,j+1} | \le ((B-A)/B)\, |z_i^{\,j}| + (A/(2B))\, |z_{i-1}^{\,j}| + (A/(2B))\, | z_{i+1}^{\,j} | + O(k^2 + \sigma_i^{\,j} k h^2)$$

$$\le \| z^j \| + C(k^2 + \sigma_i^{\,j} k h^2)$$

since $|\sigma_i^{\,j}| \le C$ and $\| z^j \| = \max |z_i^{\,j}|$, $i = 0(1)N$.

Consequently,

$$\| z^{j+1} \| \le \| z^j \| + C(k^2 + k h^2) \qquad (11)$$

and since $\| z^0 \| = 0$, we have

$$\| z^j \| \le C\, j\, (k^2 + k h^2)$$
$$\le C\, (jk)(k + k^2)$$
$$\le C\, (k + h^2) \qquad (12)$$

as $(jk) \le 1$, $0 \le t \le 1$.

Equation (12) means that the error $z_i^{\,j}$ tends to zero as h and k tends to zero . Thus the solution of the finite difference scheme converges uniformly to the solution of the given boundary value problem as h and k tend to zero. This can be stated as follows:

THEOREM 1

Let u and U be the solution of the difference scheme (2.1)-(2.3) and the boundary value problem (1.1)-(1.3) respectively. Then , for all $0 \le x \le 1$ and $0 \le t \le 1$,

$$\max |u_i^{\,j} - U(x_i, t_j)| \le C\,(k + h^2) \qquad (13)$$

where C is independent of i , j, k and h .

Hence the scheme (2.1)-(2.3) is $O(k + h^2)$ uniformly convergent.

Remark. The scheme can be simplified into the form

$$u_i^{\,j+1} = \exp(-(a\pi)^2 k)\, u_i^{\,j} + [1 - \exp(-(a\pi)^2 k)]\, F(x_i, t_{j+1}) / (a\pi)^2 \qquad (14)$$

since $u_{xx} = -(a\pi)^2 u$ for the heat equation (1.1)-(1.3) . As $u_{xx} = -(a\pi)^2 u$, the heat equation (1.1) reduces to the form

$$u_t + (a\pi)^2 u = F(x,t),\ 0 < x < 1,\ t > 0, \qquad (15)$$

and the difference scheme (2.1) reduces to the form (14) .

3 NUMERICAL RESULTS

In this section the scheme (2.1)-(2.3) is applied to four sample problems.

SAMPLE PROBLEM 1. [1,2,5,9,10,11]

Consider the heat equation

$$u_t(x,t) = u_{xx}(x,t),\ 0 < x < 1,\ t > 0, \qquad (16)$$

$$u(x,0) = \sin(\pi x),\ 0 \le x \le 1,$$

$$u(0,t) = 0,\ u(1,t) = 0,\ t \ge 0.$$

Numerical results are given using (14) in Table 2.

SAMPLE PROBLEM 2.

Consider the heat equation

$u_t(x,t) = u_{xx}(x,t),\ 0 < x < 1,\ t > 0,$ (17)

$u(x,0) = \sin(\pi x)[1 + 6\cos(\pi x)],\ 0 \le x \le 1,$

$u(0,t) = 0,\ u(1,t) = 0,\ t \ge 0.$

The initial condition can be written as

$u(x,0) = \sin(\pi x),\ + 3\sin(2\pi x).$

And hence the problem (17) can be split into two boundary value problems

$v_t(x,t) = v_{xx}(x,t),\ 0 < x < 1,\ t > 0,$ (18)

$v(x,0) = \sin(\pi x),\ ,\ 0 \le x \le 1,$

$v(0,t) = 0,\ v(1,t) = 0,\ t \ge 0$

and

$w_t(x,t) = w_{xx}(x,t),\ 0 < x < 1,\ t > 0,$ (19)

$w(x,0) = \sin(2\pi x),\ ,\ 0 \le x \le 1,$

$w(0,t) = 0,\ w(1,t) = 0,\ t \ge 0$

such that

$u(x,t) = v(x,t) + 3w(x,t)$. The boundary value problems(18) and (19) can be solved numerically using the scheme (2.1)-(2.3) and the numerical solution of (17) is given by

$u_i^j = v_i^j + 3w_i^j$.

The scheme solves the problem(17) exactly. Numerical results are given using(14) in Table 3.

SAMPLE PROBLEM 3.

Consider the heat equation

$u_t(x,t) = u_{xx}(x,t),\ -1 < x < 1,\ t > 0,$ (20)

$u(x,0) = \cos(\pi x/2),\ -1 \le x \le 1,$

$u(-1,t) = 0,\ u(1,t) = 0,\ t \ge 0.$

The scheme solves exactly the problem (20). Numerical results are given using (14) in Table 4.

SAMPLE PROBLEM 4 [11]

Consider the heat equation

$u_t(x,t) = u_{xx}(x,t) + \sin(\pi x),\ 0 < x < 1,\ t > 0,$ (21)

$u(x,0) = 0,\ 0 \le x \le 1,$

$u(0,t) = 0,\ u(1,t) = 0,\ t \ge 0.$

The scheme solves exactly the problem (21). Numerical results are given using (14) in Table 5.

The mesh sizes h = k = 1/8 is considered for the sample problem 1,2 and 4 and the numerical results are given in Tables 2,3 and 5. For the sample problem 3 the mesh sizes taken are h=1/4 and k=1/8 and the numerical results are given in Table 4. It must be noted that the scheme (2.1)-(2.3) solves the sample problems well even for large value of the mesh sizes.

In the Tables entries mean $(i,j)^{th}$ entries , APPROXIMATE means the approximate solution at the $(i,j)^{th}$ entry , ABSOLUTE and RELATIVE means absolute and relative errors respectively . The absolute and relative errors are defined as

ABSOLUTE ERROR = $|u_i^j - u(x_i, t_j)|$ for all i and j

and RELATIVE ERROR = $|1 - [u_i^j / u(x_i, t_j)]|$ for all i and j respectively .

On replacing the classical implicit scheme to the Test problem1 from Table1 it is observed that the absolute error decreases and the relative error increases as time proceeds.

And hence the applied mathematicians, scientists and engineers can use the simplified form (14) to solve heat equations with periodic boundary conditions.

From Tables 2-5 it is observed that scheme solves the sample problems exactly even for large values of h and k. The instability of the classical explicit scheme on the factor $r=k/h^2$ is completely removed on using explicit exponential fitted finite difference scheme presented in this paper. It must be noted that absolute and relative errors decrease exponentially even for large values of h and k

CONCLUSIONS

This paper presents an exponentially fitted finite difference scheme for the one dimensional heat equation with periodic boundary conditions. The scheme is the modified form of the classical explicit finite difference scheme. The scheme solves the heat equation exactly even for large values of the mesh sizes. There is no restriction on the factor $r=k/h^2$ for the numerical stability. The advantage of the scheme is one need not go for complicated implicit schemes to solve the given heat equation .And the present scheme is computationally simpler than the other schemes available in the literature.

To solve the heat equation very exactly a computationally simpler form (14) is given in this paper. Using this one can very easily solve the given heat equation using Micr Vax II Computer . We are able to advance in the t-direction with large t-steps are a definite computational advantage over the classical explicit methods.

The scheme presented in this paper is new, explicit, in nature, unconditionally stable and computationally simpler than all other methods.

All computations were performed in Pascal single precision on a Micro Vax II computer at Bharathidasan University, Tiruchirapalli-620 024, Tamil Nadu, India.

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TABLE 1

ENTRICES	APPROXIMATE	EXACT	ABSOLUTE	RELATIVE
U [1, 1]	1.72541E-01	1.11442E-01	6.10986E-02	5.48253E-01
U [2, 1]	3.18814E-01	2.05919E-01	1.12896E-01	5.48253E-01
U [3, 1]	4.16551E-01	2.69046E-01	1.47505E-01	5.48253E-01
U [4, 1]	4.50871E-01	2.91213E-01	1.59658E-01	5.48253E-01
U [5, 1]	4.16551E-01	2.69046E-01	1.47505E-01	5.48253E-01
U [6, 1]	3.18814E-01	2.05919E-01	1.12895E-01	5.48253E-01
U [7, 1]	1.72541E-01	1.11442E-01	6.10986E-02	5.48253E-01
U [1, 2]	7.77938E-02	3.24535E-02	4.53403E-02	1.39709E+00
U [2, 2]	1.43744E-01	5.99662E-02	8.37780E-02	1.39709E+00
U [3, 2]	1.87811E-01	7.83496E-02	1.09463E-01	1.39709E+00
U [4, 2]	2.03285E-01	8.48050E-02	1.18480E-01	1.39709E+00
U [5, 2]	1.87811E-01	7.83496E-02	1.09461E-01	1.39709E+00
U [6, 2]	1.43744E-01	5.99662E-02	8.37780E-02	1.39709E+00
U [7, 2]	7.77918E-02	3.24535E-02	4.53403E-02	1.39709E+00
U [1, 3]	3.50750E-02	9.45087E-03	2.56241E-02	2.71130E+00
U [2, 3]	6.48101E-02	1.74629E-02	4.73472E-02	2.71130E+00
U [3, 3]	8.46785E-02	2.28164E-02	6.18621E-02	2.71130E+00
U [4, 3]	9.16554E-02	2.46963E-02	6.69591E-02	2.71130E+00
U [5, 3]	8.46785E-02	2.28164E-02	6.18621E-02	2.71130E+00
U [6, 3]	6.48101E-02	1.74629E-02	4.73472E-02	2.71130E+00
U [7, 3]	3.50750E-02	9.45087E-03	2.56241E-02	2.71130E+00
U [1, 4]	1.58143E-02	2.75222E-03	1.30621E-02	4.74603E+00
U [2, 4]	2.92210E-02	5.08543E-03	2.41356E-02	4.74603E+00
U [3, 4]	3.81791E-02	6.64444E-03	3.15347E-02	4.74603E+00
U [4, 4]	4.13248E-02	7.19189E-03	3.41329E-02	4.74603E+00
U [5, 4]	3.81791E-02	6.64444E-03	3.15347E-02	4.74603E+00
U [6, 4]	2.92210E-02	5.08543E-03	2.41356E-02	4.74603E+00
U [7, 4]	1.58143E-02	2.75222E-03	1.30621E-02	4.74603E+00
U [1, 5]	7.13022E-03	8.01481E-04	6.32874E-03	7.89630E+00
U [2, 5]	1.31749E-02	1.08543E-03	1.16940E-02	7.89630E+00
U [3, 5]	1.72139E-02	1.93495E-03	1.52789E-02	7.89630E+00
U [4, 5]	1.86322E-02	2.09437E-03	1.65378E-02	7.89630E+00
U [5, 5]	1.72139E-02	1.93495E-03	1.52789E-02	7.89630E+00
U [6, 5]	1.31749E-02	1.48094E-03	1.16940E-02	7.89630E+00
U [7, 5]	7.13022E-03	8.01481E-04	6.32874E-03	7.89630E+00
U [1, 6]	3.21481E-03	2.33402E-04	2.98141E-03	1.27737E+01
U [2, 6]	5.94020E-03	4.31270E-04	5.50893E-03	1.27737E+01
U [3, 6]	7.76124E-03	5.63481E-04	7.19776E-03	1.27737E+01
U [4, 6]	8.40070E-03	6.09908E-04	7.79080E-03	1.27737E+01
U [5, 6]	7.76124E-03	5.63481E-04	7.19776E-03	1.27737E+01
U [6, 6]	5.94019E-03	4.32170E-04	5.50892E-03	1.27737E+01
U [7, 6]	3.21481E-03	2.33402E-04	2.98141E-03	1.27737E+01
U [1, 7]	1.44947E-03	6.79696E-05	1.38150E-03	2.03252E+01
U [2, 7]	2.67826E-03	1.25591E-04	2.55267E-03	2.03252E+01
U [3, 7]	3.49932E-03	1.64093E-04	3.33523E-03	2.03252E+01
U [4, 7]	3.78764E-03	1.77613E-04	3.61002E-03	2.03252E+01
U [5, 7]	3.49932E-03	1.64093E-04	3.33523E-03	2.03252E+01

U [6, 7]	2.67826E-03	1.25591E-04	2.55267E-03	2.03252E+01
U [7, 7]	1.44940E-03	6.79696E-05	1.38150E-03	2.03252E+01
U [1, 8]	6.53523E-04	1.97936E-05	6.33729E-04	3.20168E+01
U [2, 8]	1.20755E-03	3.65739E-05	1.17098E-03	3.20168E+01
U [3, 8]	1.57774E-03	4.77861E-05	1.52996E-03	3.20168E+01
U [4, 8]	1.70774E-03	5.17233E-05	1.65601E-03	3.20168E+01
U [5, 8]	1.57774E-03	4.77861E-05	1.52996E-03	3.20168E+01
U [6, 8]	1.20755E-03	3.65739E-05	1.70980E-03	3.20168E+01
U [7, 8]	6.53522E-04	1.97936E-05	6.33729E-04	3.20168E+01

TABLE 2

ENTRICES	APPROXIMATE	EXACT	ABSOLUTE	RELATIVE
U [1, 1]	1.11442E-01	1.11442E-01	0.00000E+00	0.00000E+00
U [2, 1]	2.05919E-01	2.05919E-01	0.00000E+00	0.00000E+00
U [3, 1]	2.69046E-01	2.69046E-01	0.00000E+00	0.00000E+00
U [4, 1]	2.91213E-01	2.91213E-01	0.00000E+00	0.00000E+00
U [5, 1]	2.69046E-01	2.69046E-01	0.00000E+00	0.00000E+00
U [6, 1]	2.05919E-01	2.05919E-01	0.00000E+00	0.00000E+00
U [7, 1]	1.11442E-01	1.11442E-01	0.00000E+00	0.00000E+00
U [1, 2]	3.24535E-02	3.24535E-02	0.00000E+00	0.00000E+00
U [2, 2]	5.99662E-02	5.99662E-02	0.00000E+00	0.00000E+00
U [3, 2]	7.83496E-02	7.83496E-02	7.45058E-09	1.19209E-07
U [4, 2]	8.48050E-02	8.48050E-02	0.00000E+00	0.00000E+00
U [5, 2]	7.83496E-02	7.83496E-02	0.00000E+00	0.00000E+00
U [6, 2]	5.99662E-02	5.99662E-02	0.00000E+00	0.00000E+00
U [7, 2]	3.24535E-02	3.24535E-02	3.72529E-09	1.19209E-07
U [1, 3]	9.45087E-03	9.45087E-03	9.31323E-10	1.19209E-07
U [2, 3]	1.74629E-02	1.74629E-02	1.86265E-09	1.19209E-07
U [3, 3]	2.28164E-02	2.28164E-02	3.72529E-09	1.19209E-07
U [4, 3]	2.46963E-02	2.46963E-02	1.86265E-09	1.19209E-07
U [5, 3]	2.28164E+01	2.28164E+01	1.86265E-09	1.19209E-07
U [6, 3]	1.74629E-02	1.74629E-02	1.86265E-09	1.19209E-07
U [7, 3]	9.45087E-03	9.45087E-03	0.00000E+00	0.00000E+00
U [1, 4]	2.75222E-03	2.75222E-03	2.32831E-10	5.96046E-08
U [2, 4]	5.08543E-03	5.08543E-03	4.65661E-10	1.19209E-07
U [3, 4]	6.64444E-03	6.64444E-03	0.00000E+00	0.00000E+00
U [4, 4]	7.19189E-03	7.19189E-03	9.31323E-10	1.19209E-07
U [5, 4]	6.64444E-03	6.64444E-03	4.65661E-10	5.96046E-08
U [6, 4]	5.08543E-03	5.08543E-03	4.65661E-10	1.19209E-07
U [7, 4]	2.75222E-03	2.75222E-03	4.65661E-10	1.78814E-07
U [1, 5]	8.01481E-04	8.01481E-04	1.74623E-10	2.38419E-07
U [2, 5]	1.48094E-03	1.48094E-03	2.32831E-10	1.78814E-07
U [3, 5]	1.93495E-03	1.93495E-03	1.16415E-10	5.96046E-08
U [4, 5]	2.09437E-03	2.09437E-03	4.65661E-10	2.38419E-07
U [5, 5]	1.93495E-03	1.93495E-03	2.32831E-10	1.19209E-07
U [6, 5]	1.48094E-03	1.48094E-03	2.32831E-10	1.78814E-07
U [7, 5]	8.01381E-04	8.01381E-04	2.32831E-10	2.98023E-07
U [1, 6]	2.33402E-04	2.33402E-04	2.91038E-06	1.19209E-07
U [2, 6]	4.31270E-04	4.31270E-04	5.82077E-11	1.19209E-07

U [3, 6]	5.63482E-04	5.63482E-04	1.16415E-05	2.38419E-07
U [4, 6]	6.09908E-04	6.09908E-04	5.82077E-11	1.19209E-07
U [5, 6]	5.63482E-04	5.63482E-04	5.82077E-11	1.19209E-07
U [6, 6]	4.31270E-04	4.31270E-04	5.82077E-11	1.19209E-07
U [7, 6]	2.33402E-04	2.33402E-04	1.45519E-11	1.19209E-07
U [1, 7]	6.79696E-05	6.79696E-05	2.91038E-11	4.17233E-07
U [2, 7]	1.25591E-04	1.25591E-04	5.82077E-11	4.76837E-07
U [3, 7]	1.64093E-04	1.64093E-04	5.82077E-11	3.57628E-07
U [4, 7]	1.77613E-04	1.77613E-04	8.73115E-11	4.76837E-07
U [5, 7]	1.64093E-04	1.64093E-04	7.27596E-11	4.17233E-07
U [6, 7]	1.25591E-04	1.25591E-04	5.82077E-11	4.76837E-07
U [7, 7]	6.79696E-05	6.79696E-05	3.63798E-11	5.36442E-07
U [1, 8]	1.97936E-05	1.97936E-05	3.63798E-12	1.78814E-07
U [2, 8]	3.65739E-05	3.65739E-05	3.63798E-12	1.19209E-07
U [3, 8]	4.77860E-05	4.77860E-05	3.63798E-12	5.96046E-08
U [4, 8]	5.17232E-05	5.17232E-05	7.27596E-12	1.19201E-07
U [5, 8]	4.77860E-05	4.77860E-05	7.27596E-12	1.78814E-07
U [6, 8]	3.65739E-05	3.65739E-05	3.63798E-12	1.19201E-07
U [7, 8]	1.97936E-05	1.97936E-05	3.63798E-12	1.78814E-07

TABLE 3

ENTRICES	APPROXIMATE	EXACT	ABSOLUTE	RELATIVE
U [1, 1]	1.26699E-01	1.26699E-01	0.00000E+00	0.00000E+00
U [2, 1]	2.27494E-01	2.27494E-01	0.00000E+00	0.00000E+00
U [3, 1]	2.84302E-01	2.84302E-01	0.00000E+00	0.00000E+00
U [4, 1]	2.91213E-01	2.91213E-01	0.00000E+00	0.00000E+00
U [5, 1]	2.53789E-01	2.53789E-01	0.00000E+00	0.00000E+00
U [6, 1]	1.84343E-01	1.84343E-01	0.00000E+00	0.00000E+00
U [7, 1]	9.61861E-01	9.61861E-01	0.00000E+00	0.00000E+00
U [1, 2]	3.25632E-02	3.25632E-02	0.00000E+00	0.00000E+00
U [2, 2]	6.01214E-02	6.01214E-02	0.00000E+00	0.00000E+00
U [3, 2]	7.84593E-02	7.84593E-02	7.45058E-09	1.19209E-07
U [4, 2]	8.48050E-02	8.48050E-02	0.00000E+00	0.00000E+00
U [5, 2]	7.82399E-02	7.82399E-02	0.00000E+00	0.00000E+00
U [6, 2]	5.98110E-02	5.98110E-02	0.00000E+00	0.00000E+00
U [7, 2]	3.23437E-02	3.23437E-02	3.72529E-09	1.19209E-07
U [1, 3]	9.45166E-03	9.45166E-03	9.31323E-10	1.19209E-07
U [2, 3]	1.74640E-02	1.74640E-02	1.86265E-09	1.19209E-07
U [3, 3]	2.28172E-02	2.28172E-02	3.72529E-09	1.19209E-07
U [4, 3]	2.46963E-02	2.46963E-02	1.86265E-09	1.19209E-07
U [5, 3]	2.28156E-02	2.28156E-02	1.86265E-09	1.19209E-07
U [6, 3]	1.74618E-02	1.74618E-02	1.86265E-09	1.19209E-07
U [7, 3]	9.45008E-03	9.45008E-03	0.00000E+00	0.00000E+00
U [1, 4]	2.75222E-03	2.75222E-03	2.32831E-10	5.96046E-08
U [2, 4]	5.08544E-03	5.08544E-03	4.65661E-10	1.19209E-07
U [3, 4]	6.44440E-03	6.44440E-03	0.00000E+00	0.00000E+00
U [4, 4]	7.19189E-03	7.19189E-03	9.32323E-10	1.19209E-07
U [5, 4]	6.44440E-03	6.44440E-03	4.56610E-10	5.96046E-08
U [6, 4]	5.08544E-03	5.08544E-03	4.56610E-10	1.19209E-07

U [7, 4]	2.75222E-03	2.75222E-03	4.56610E-10	1.78814E-07
U [1, 5]	8.01481E-04	8.01481E-04	1.74623E-10	2.38419E-07
U [2, 5]	1.48094E-03	1.48094E-03	2.32831E-10	1.78814E-07
U [3, 5]	1.93495E-03	1.93495E-03	1.16415E-10	5.96046E-08
U [4, 5]	2.09437E-03	2.09437E-03	4.65661E-10	2.38419E-07
U [5, 5]	1.93495E-03	1.93495E-03	2.32831E-10	1.19209E-07
U [6, 5]	1.48094E-03	1.48094E-03	2.32831E-10	1.78814E-07
U [7, 5]	8.01481E-04	8.01481E-04	2.32831E-10	2.98023E-07
U [1, 6]	2.33402E-04	2.33402E-04	2.91038E-11	1.19209E-07
U [2, 6]	4.31270E-04	4.31270E-04	5.82077E-11	1.19209E-07
U [3, 6]	5.63482E-04	5.63482E-04	1.16415E-10	2.38419E-07
U [4, 6]	6.09908E-04	6.09908E-04	5.82077E-11	1.19209E-07
U [5, 6]	5.63482E-04	5.63482E-04	5.82077E-11	1.19209E-07
U [6, 6]	4.31270E-04	4.31270E-04	5.82077E-11	1.19209E-07
U [7, 6]	2.33402E-04	2.33402E-04	1.45519E-11	1.19209E-07
U [1, 7]	6.79696E-05	6.79696E-05	2.91038E-11	4.17233E-07
U [2, 7]	1.25591E-04	1.25591E-04	5.82077E-11	4.76837E-07
U [3, 7]	1.64093E-04	1.64093E-04	5.82077E-11	3.57628E-07
U [4, 7]	1.77661E-04	1.77661E-04	8.73115E-11	4.76837E-07
U [5, 7]	1.64093E-04	1.64093E-04	7.27596E-11	4.17233E-07
U [6, 7]	1.25591E-04	1.25591E-04	5.82077E-11	4.76837E-07
U [7, 7]	6.79696E-05	6.79696E-05	3.63798E-11	5.36442E-07
U [1, 8]	1.97936E-05	1.97936E-05	3.63798E-12	1.78814E-07
U [2, 8]	3.65739E-05	3.65739E-05	3.63798E-12	1.19209E-07
U [3, 8]	4.77860E-05	4.77860E-05	3.63798E-12	5.96046E-07
U [4, 8]	5.17232E-05	5.17232E-05	7.27596E-12	1.19209E-07
U [5, 8]	4.77860E-05	4.77860E-05	7.27596E-12	1.78814E-07
U [6, 8]	3.65739E-05	3.65739E-05	3.63798E-12	1.19209E-07
U [7, 8]	1.97936E-05	1.97936E-05	3.63798E-12	1.78814E-07

TABLE 4

ENTRICES	APPROXIMATE	EXACT	ABSOLUTE	RELATIVE
U [1, 1]	2.81120E-01	2.81120E-01	0.00000E+00	0.00000E+00
U [2, 1]	5.19443E-01	5.19443E-01	0.00000E+00	0.00000E+00
U [3, 1]	6.78685E-01	6.78685E-01	0.00000E+00	0.00000E+00
U [4, 1]	7.34603E-01	7.34603E-01	0.00000E+00	0.00000E+00
U [5, 1]	6.78685E-01	6.78685E-01	0.00000E+00	0.00000E+00
U [6, 1]	5.19443E-01	5.19443E-01	0.00000E+00	0.00000E+00
U [7, 1]	2.81120E-01	2.81120E-01	0.00000E+00	0.00000E+00
U [1, 2]	2.06512E-01	2.06512E-01	2.98023E-08	1.19209E-07
U [2, 2]	3.81584E-01	3.81584E-01	2.98023E-08	1.19209E-07
U [3, 2]	4.98564E-01	4.98564E-01	5.96046E-08	1.19209E-07
U [4, 2]	5.39642E-01	5.39642E-01	5.96046E-08	1.19209E-07
U [5, 2]	4.98564E-01	4.98564E-01	5.96046E-08	1.19209E-07
U [6, 2]	3.81584E-01	3.81584E-01	2.98023E-08	1.19209E-07
U [7, 2]	2.06512E-01	2.06512E-01	2.98023E-08	1.19209E-07
U [1, 3]	1.51704E-01	1.51704E-01	1.49012E-08	1.19209E-07
U [2, 3]	2.80313E-01	2.80313E-01	2.98023E-08	1.19209E-07
U [3, 3]	3.66246E-01	3.66246E-01	2.98023E-08	1.19209E-07

U [4, 3]	3.96422E-01	3.96422E-01	2.98023E-08	1.19209E-07
U [5, 3]	3.66246E-01	3.66246E-01	2.98023E-08	1.19209E-07
U [6, 3]	2.80313E-01	2.80313E-01	2.98023E-08	1.19209E-07
U [7, 3]	1.51704E-01	1.51704E-01	1.49012E-08	1.19209E-07
U [1, 4]	1.11442E-01	1.11442E-01	1.49012E-08	1.19209E-07
U [2, 4]	2.05919E-01	2.05919E-01	2.98023E-08	1.19209E-07
U [3, 4]	2.69046E-01	2.69046E-01	2.98023E-08	1.19209E-07
U [4, 4]	2.91213E-01	2.91213E-01	2.98023E-08	1.19209E-07
U [5, 4]	2.69046E-01	2.69046E-01	2.98023E-08	1.19209E-07
U [6, 4]	2.05919E-01	2.05919E-01	2.98023E-08	1.19209E-07
U [7, 4]	1.11442E-01	1.11442E-01	1.49012E-08	1.19209E-07
U [1, 5]	8.18659E-02	8.18659E-02	7.45058E-09	1.19209E-07
U [2, 5]	1.51268E-01	1.51268E-01	1.49012E-08	1.19209E-07
U [3, 5]	1.97642E-01	1.97642E-01	1.49012E-08	1.19209E-07
U [4, 5]	2.13926E-01	2.13926E-01	1.49012E-08	1.19209E-07
U [5, 5]	1.97642E-01	1.97642E-01	1.49012E-08	1.19209E-07
U [6, 5]	1.51268E-01	1.51268E-01	1.49012E-08	1.19209E-07
U [7, 5]	8.18659E-02	8.18659E-02	7.45058E-09	1.19209E-07
U [1, 6]	6.01389E-02	6.01389E-02	1.11759E-08	2.38419E-07
U [2, 6]	1.11122E-01	1.11122E-01	1.49012E-08	1.19209E-07
U [3, 6]	1.45188E-01	1.45188E-01	1.49012E-08	1.19209E-07
U [4, 6]	1.57151E-01	1.57151E-01	2.98023E-08	2.38419E-07
U [5, 6]	1.45188E-01	1.45188E-01	1.49012E-08	1.19209E-07
U [6, 6]	1.11122E-01	1.11122E-01	1.49012E-08	1.19209E-07
U [7, 6]	6.01389E-02	6.01389E-02	1.11759E-08	2.38419E-07
U [1, 7]	4.41782E-02	4.41782E-02	3.72529E-09	1.19209E-07
U [2, 7]	8.16308E-02	8.16308E-02	7.45058E-09	1.19209E-07
U [3, 7]	1.06656E-01	1.06656E-01	0.00000E+00	0.00000E+00
U [4, 7]	1.15443E-01	1.15443E-01	1.49012E-08	1.19209E-07
U [5, 7]	1.06656E-01	1.06656E-01	0.00000E+00	0.00000E+00
U [6, 7]	8.16308E-02	8.16308E-02	7.45058E-09	1.19209E-07
U [7, 7]	4.41782E-02	4.41782E-02	3.72529E-09	1.19209E-07
U [1, 8]	3.24535E-02	3.24535E-02	7.45058E-09	2.38419E-07
U [2, 8]	5.99662E-02	5.99662E-02	1.49012E-08	2.38419E-07
U [3, 8]	7.83496E-02	7.83496E-02	7.45058E-09	1.19209E-07
U [4, 8]	8.48050E-02	8.48050E-02	2.23517E-08	2.38419E-07
U [5, 8]	7.83496E-02	7.83496E-02	7.45058E-09	1.19209E-07
U [6, 8]	5.99662E-02	5.99662E-02	1.49012E-08	2.38419E-07
U [7, 8]	3.24535E-02	3.24535E-02	7.45058E-09	2.38419E-07

TABLE 5

ENTRICES	APPROXIMATE	EXACT	ABSOLUTE	RELATIVE
U [1, 1]	2.74825E-02	2.74825E-02	0.00000E+00	0.00000E+00
U [2, 1]	5.07810E-02	5.07810E-02	0.00000E+00	0.00000E+00
U [3, 1]	6.63485E-02	6.63485E-02	0.00000E+00	0.00000E+00
U [4, 1]	7.18151E-02	7.18151E-02	0.00000E+00	0.00000E+00
U [5, 1]	6.63485E-02	6.63485E-02	0.00000E+00	0.00000E+00
U [6, 1]	5.07810E-02	5.07810E-02	0.00000E+00	0.00000E+00
U [7, 1]	2.74825E-02	2.74825E-02	0.00000E+00	0.00000E+00
U [1, 2]	3.54857E-02	3.54857E-02	0.00000E+00	0.00000E+00

U [2, 2]	6.55691E-02	6.55691E-02	0.00000E+00	0.00000E+00
U [3, 2]	8.56701E-02	8.56701E-02	0.00000E+00	0.00000E+00
U [4, 2]	9.27287E-02	9.27287E-02	0.00000E+00	0.00000E+00
U [5, 2]	8.56701E-02	8.56701E-02	7.45058E-09	1.19209E-07
U [6, 2]	6.55691E-02	6.55691E-02	0.00000E+00	0.00000E+00
U [7, 2]	3.54857E-02	3.54857E-02	3.72529E-09	1.19209E-07
U [1, 3]	3.78164E-02	3.78164E-02	3.72529E-09	1.19209E-07
U [2, 3]	6.98755E-02	6.98755E-02	0.00000E+00	0.00000E+00
U [3, 3]	9.12968E-02	9.12968E-02	0.00000E+00	0.00000E+00
U [4, 3]	9.88189E-02	9.88189E-02	0.00000E+00	0.00000E+00
U [5, 3]	9.12968E-02	9.12968E-02	0.00000E+00	0.00000E+00
U [6, 3]	6.98755E-02	6.98755E-02	0.00000E+00	0.00000E+00
U [7, 3]	3.78164E-02	3.78164E-02	0.00000E+00	0.00000E+00
U [1, 4]	3.84951E-02	3.84951E-02	0.00000E+00	0.00000E+00
U [2, 4]	7.11296E-02	7.11296E-02	7.45058E-09	1.19209E-07
U [3, 4]	9.29354E-02	9.29354E-02	0.00000E+00	0.00000E+00
U [4, 4]	1.00593E-01	1.00593E-01	7.45058E-09	1.19209E-07
U [5, 4]	9.29354E-02	9.29354E-02	7.45058E-09	1.19209E-07
U [6, 4]	7.11296E-02	7.11296E-02	7.45058E-09	1.19209E-07
U [7, 4]	3.84951E-02	3.84951E-02	3.72529E-09	1.19209E-07
U [1, 5]	3.86927E-02	3.86927E-02	0.00000E+00	0.00000E+00
U [2, 5]	7.14949E-02	7.14949E-02	0.00000E+00	0.00000E+00
U [3, 5]	9.34125E-02	9.34125E-02	0.00000E+00	0.00000E+00
U [4, 5]	1.01109E-01	1.01109E-01	0.00000E+00	0.00000E+00
U [5, 5]	9.34125E-02	9.34125E-02	7.45058E-09	1.19209E-07
U [6, 5]	7.14949E-02	7.14949E-02	0.00000E+00	0.00000E+00
U [7, 5]	3.86927E-02	3.86927E-02	0.00000E+00	0.00000E+00
U [1, 6]	3.87503E-02	3.87503E-02	0.00000E+00	0.00000E+00
U [2, 6]	7.16012E-02	7.16012E-02	7.45058E-09	1.19209E-07
U [3, 6]	9.35515E-02	9.35515E-02	7.45058E-09	1.19209E-07
U [4, 6]	1.01259E-01	1.01259E-01	0.00000E+00	0.00000E+00
U [5, 6]	9.35515E-02	9.35515E-02	7.45058E-09	1.19209E-07
U [6, 6]	7.16012E-02	7.16012E-02	7.45058E-09	1.19209E-07
U [7, 6]	3.87503E-02	3.87503E-02	3.72529E-09	1.19209E-07
U [1, 7]	3.87671E-02	3.87671E-02	0.00000E+00	0.00000E+00
U [2, 7]	7.16322E-02	7.16322E-02	7.45058E-09	1.19209E-07
U [3, 7]	9.35920E-02	9.35920E-02	0.00000E+00	0.00000E+00
U [4, 7]	1.01303E-01	1.01303E-01	0.00000E+00	0.00000E+00
U [5, 7]	9.35920E-02	9.35920E-02	7.45058E-09	1.19209E-07
U [6, 7]	7.16322E-02	7.16322E-02	7.45058E-09	1.19209E-07
U [7, 7]	3.87671E-02	3.87671E-02	0.00000E+00	0.00000E+00
U [1, 8]	3.87719E-02	3.87719E-02	3.72529E-09	1.19209E-07
U [2, 8]	7.16412E-02	7.16412E-02	0.00000E+00	0.00000E+00
U [3, 8]	9.36037E-02	9.36037E-02	7.45058E-09	1.19209E-07
U [4, 8]	1.01316E-01	1.01316E-01	0.00000E+00	0.00000E+00
U [5, 8]	9.36037E-02	9.36037E-02	7.45058E-09	1.19209E-07
U [6, 8]	7.16412E-02	7.16412E-02	0.00000E+00	0.00000E+00
U [7, 8]	3.87719E-02	3.87719E-02	3.72529E-09	1.19209E-07