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STABILITY OF A SYN- ECOSYSTEM CONSISTING OF A PREY-PREDATOR AND HOST COMMENSAL TO THE PREY (WITH PREDATOR MORTALITY RATE)

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ABSTRACT

The present paper deals with a three species ecosystem consisting of a prey (S_1), a predator (S_2) and a host (S_3) commensal to the prey with mortality rate for predator. The mathematical model equations constitute a set of three first order non-linear simultaneous equations in the strengths N_1, N_2 and N_3 of S_1, S_2 and S_3 .The equation for host is non-linear but de-coupled with the prey-predator pair. In all, there will be six equilibrium points of the model and the criteria for their stability are discussed. The trajectories on perturbations over the equilibrium states on species have been drawn.

Keywords: *Equilibrium point, Equilibrium state, Stability, Carrying capacity, Reversal time of dominance.*

1. INTRODUCTION

Research in the area of Theoretical Ecology was initiated by Lotka [11] in 1925 and by Volterra [18] in 1931. Since then many Mathematicians and Ecologists contributed to the growth of this area of knowledge reported in the treatises of Meyer [12], Cushing [5], Paul Colinvaux [13], Freedman [6], Kapur [7, 8] and several others. Ecology relates to the study of living beings in relation to their living styles. This is a branch of evolutionary biology purported to explain how or to what extent living beings are regulated in nature in their struggle for existence in the same environments for several generation, sharing the same habitat and interacting with each other in diverse ways such as prey-predator, competition, mutualism, commensalisms, Ammensalism, Neutralism and so on. N.C.Srinivas [17] studied the competitive ecosystems of two species and three species with limited and unlimited resources. Lakshminarayana and Pattabhi Ramacharyulu [9, 10] investigated prey-predator ecological models with a partial cover for the prey and alternative food for the predator and prey-predator model with cover for prey and alternate food for the predator and to me delay. Stability analysis of competitive species was carried out by Archana Reddy, Pattabhi Ramacharyulu and Gandhi [1, 2], by Bhaskara Rama Sarma and Pattabhi Ramacharyulu [3, 4]. While the mutualism between two species was examined by Ravindra Reddy [15]. Recently Phanikumar et.al [14] obtained the criteria for the stability of a host- A flourishing commensal species pair with limited resources. SeshagiriRao et.al [16] investigated the stability of a host- A decaying commensal species pair with limited resources.

The present investigation is related to an analytical study of three species system**:** Commensal-Prey-Predator and Host system. In all, six equilibrium points are identified based on the model equations and these are spread over three distinct classes (i) Fully washed out (ii).Semi/partially washed out and (iii).Co-existent states. Criteria for the asymptotic stability of the states have been derived. It is noticed that only the following two states are stable and the remaining states are unstable.

(i).Predator washed out equilibrium state.

(ii).Co-existent state.

2. Notations adopted:

N_1**:** The population of the prey-commensal S_1.

N_2**:** The population of the predation striving of the prey S_1.

N_3**:** The population of the host to the prey S_1.

d_1**:** The natural death/decay rate of prey S_1.

a_2**:** The natural growth rate of S_2.

a_3**:** The natural growth rate of S_3.

a_{ii}**:** The rate of decrease of S_i due to insufficient resources of S_i, i =1,2,3.

a_{12}**:** The decrease of prey (S_1) due to inhibition by the predator (S_2).

a_{13}**:** The rate of increase of the commensal (S_1) due to its successful promotion by the host (S_3).

a_{21}**:** The rate of increase of the predator (S_2) due to its successful attacks on the prey (S_1).

$k_1(= a_1 / a_{11})$ the carrying capacity of prey (S_1).

$e_2(= d_2 / a_{22})$ is the extinction coefficient of predator (S_2).
$k_3(= a_3 / a_{33})$ is the carrying capacity of host (S_3).

$p(= a_{12} / a_{11})$ is the coefficient of prey-commensal inhibition of the predator.

$q(= a_{21} / a_{22})$ is the coefficient predation consumption of the prey.

$c(= a_{13} / a_{11})$ is the coefficient of commensalism.

3. BASIC BALANCE EQUATIONS OF THE MODEL:

The model equations for a three species multi-reactive ecosystem are given by the following system of non-linear ordinary differential equations**:**

1. Equation for the growth rate of the prey commensal species (S_1):

$$\frac{dN_1}{dt} = a_{11} N_1 [k_1 - N_1 - pN_2 + cN_3] \qquad \text{.................. (3.1)}$$

2. Equation for the growth rate of predator species (S_2):

$$\frac{dN_2}{dt} = a_{22} N_2 [-e_2 - N_2 + qN_1] \qquad \text{.................. (3.2)}$$

3. Equation for the growth rate of host species (S_3):

$$\frac{dN_3}{dt} = a_{33} N_3 [k_3 - N_3] \qquad \text{..................... (3.3)}$$

Further the variables N_1, N_2, N_3 are non-negative and the model parameters a_1, d_2, a_3, a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{33} are non-negative and assumed to be constants.

4. EQUILIBRIUM STATES:

These are given by $\frac{dN_i}{dt} = o, i = 1,2,3$ .The system under investigation has six equilibrium states that can put in four categories A,B,C,D as follows:

A. Fully washed out state.

(i). $\overline{N_1}=0$; $\overline{N_2}=0$; $\overline{N_3}=0$ ………………………(4.1)

B. States in which two of three species are washed out and third is not.

(ii). $\overline{N_1}=0$; $\overline{N_2}=0$; $\overline{N_3}=k_3$. …………………….(4.2)

(iii). $\overline{N_1}=k_1$; $\overline{N_2}=0$; $\overline{N_3}=0$ ……………………… (4.3)

C. Only one of the three species is washed out while the other two are not.

(iv). $\overline{N_1}=k_1+ck_3$; $\overline{N_2}=0$; $\overline{N_3}=k_3$ …………………. (4.4)

(v). $\overline{N_1}=\dfrac{k_1+pe_2}{1+pq}$; $\overline{N_2}=\dfrac{qk_1-e_2}{1+pq}$; $\overline{N_3}=0$ ………………….. (4.5)

This would exists only when $e_2 < qk_1$. When $e_2 = qk_1$ this equilibrium state merges with the equilibrium state **No. (4.3).**

D. The co-existence state or normal steady state

(vi). $\overline{N_1}=\dfrac{k_1+ck_3+pe_2}{1+pq}$; $\overline{N_2}=\dfrac{qk_1+qck_3-e_2}{1+pq}$; $\overline{N_3}=k_3$ …………… (4.6)

This would exist when $e_2 < qk_1 + cqk_3$.

5. THE STABILITY OF THE EQUILIBRIUM STATES:

To this end, we consider slight deviations $U_1(t), U_2(t), U_3(t)$ over the steady state $\left(\overline{N_1},\overline{N_2},\overline{N_3}\right)$**:**

$$N_1=\overline{N_1}+U_1(t) \;;\; N_2=\overline{N_2}+U_2(t) \;;\; N_3=\overline{N_3}+U_3(t) \quad \ldots\ldots\ldots\ldots (5.1)$$

where $U_1(t), U_2(t)$ and $U_3(t)$ are small so that their second and higher powers and products can be neglect.

5.1. FULLY WASHED OUT EQUILIBRIUM STATE:

In this case, we have

$$\frac{dU_1}{dt}=k_1a_{11}U_1 \;;\; \frac{dU_2}{dt}=-e_2a_{22}U_2 \;;\; \frac{dU_3}{dt}=k_3a_{33}U_3 \quad \ldots\ldots\ldots\ldots (5.1.1)$$

the characteristic roots of the system are $k_1a_{11}, -e_2a_{22}, k_3a_{33}$. Since two of the three roots are positive, hence the state is **unstable**. The equation (5.1.1) yield the solution curves

$$U_1 = U_{10}e^{k_1a_{11}t} \;;\; U_2 = U_{20}e^{-e_2a_{22}t} \;;\; U_3 = U_{30}e^{k_3a_{33}t} \quad \ldots\ldots\ldots\ldots (5.1.2)$$

where U_{10}, U_{20}, U_{30} are initial values of U_1, U_2, U_3 respectively.

Trajectories of Perturbations:

The trajectories (solution curves of (5.1.1)) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane are given by

$$\left(\frac{U_1}{U_{10}}\right)^{e_2 a_{22}} = \left(\frac{U_2}{U_{20}}\right)^{-k_1 a_{11}} \; ; \left(\frac{U_2}{U_{20}}\right)^{k_3 a_{33}} = \left(\frac{U_3}{U_{30}}\right)^{-e_2 a_{22}} ; \left(\frac{U_3}{U_{30}}\right)^{k_1 a_{11}} = \left(\frac{U_1}{U_{10}}\right)^{k_3 a_{33}} \qquad \ldots\ldots\ldots\ldots (5.1.3)$$

We have observed different types of solution curves of which only a few of them are discussed in the following figures.

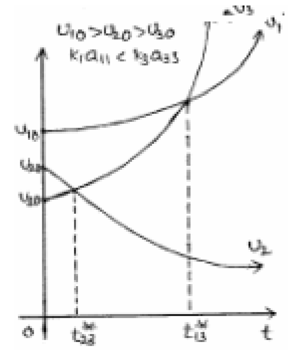

Figure: 1

Case: (5.1.i) $\quad U_{10} > U_{20} > U_{30} \quad ; \; k_1 a_{11} < k_3 a_{33}$

The prey-commensal out-numbers the host till the time instant

$$t_{13}^{\ *} = \frac{1}{(k_3 a_{33} - k_1 a_{11})} \log\left(\frac{U_{10}}{U_{30}}\right)$$ after which the host out-numbers the prey.

Here the predator even though declining dominates over the host up to the time instant

$$t_{23}^{\ *} = \frac{1}{(k_3 a_{33} + e_2 a_{22})} \log\left(\frac{U_{20}}{U_{30}}\right)$$ and there after the host dominates the predator.

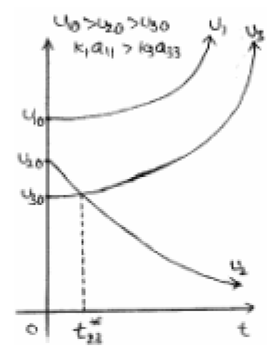

Figure: 2

Case: (5.1.ii) $\quad U_{10} > U_{20} > U_{30} \quad ; \quad k_1 a_{11} > k_3 a_{33}$

The prey always out-numbers both the predator and the host in natural growth rates as well as in their initial population strengths. In this case the predator dominates the host till the time instant $t_{23}^{\ *} = \dfrac{1}{(k_3 a_{33} + e_2 a_{22})} \log\left(\dfrac{U_{20}}{U_{30}}\right)$ after which the dominance is reversed.

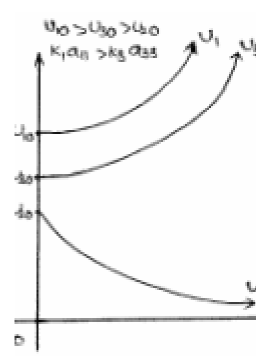

Figure: 3

Case: (5.1.iii) $\quad U_{10} > U_{30} > U_{20} \quad ; \; k_1 a_{11} > k_3 a_{33}$

In this case both the prey and its host always out-numbers the predator in natural growth rates as well as in its initial population strengths. Here the predator is asymptotic to the equilibrium point while the other two species go away from the equilibrium point.

5.2 PREY-COMMENSAL AND PREDATOR WASHED OUT STATE:
In this case, we get

$$\frac{dU_1}{dt} = a_{11}(k_1 + ck_3)U_1 \quad ; \quad \frac{dU_2}{dt} = -e_2 a_{22} U_2 \quad ; \quad \frac{dU_3}{dt} = -k_3 a_{33} U_3 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5.2.1)$$

Since the characteristic roots of which are $a_{11}(k_1 + ck_3), -e_2 a_{22}, -k_3 a_{33}$, the state is **unstable.**

The equation (5.2.1) yield

$$U_1 = U_{10} e^{a_{11}(k_1+ck_3)t} \quad ; \quad U_2 = U_{20} e^{-e_2 a_{22} t} \quad ; \quad U_3 = U_{30} e^{-k_3 a_{33} t} \qquad \ldots\ldots\ldots\ldots (5.2.2)$$

where U_{10}, U_{20}, U_{30} are initial values of U_1, U_2, U_3 respectively.

Trajectories of Perturbations:

The trajectories (solution curves of (5.2.1)) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane respectively are given by

$$\left(\frac{U_1}{U_{10}}\right)^{-e_2 a_{22}} = \left(\frac{U_2}{U_{20}}\right)^{a_{11}(k_1+ck_3)} ; \left(\frac{U_2}{U_{20}}\right)^{k_3 a_{33}} = \left(\frac{U_3}{U_{30}}\right)^{e_2 a_{22}} ; \left(\frac{U_3}{U_{30}}\right)^{-a_{11}(k_1+ck_3)} = \left(\frac{U_1}{U_{10}}\right)^{k_3 a_{33}} \quad \ldots\ldots (5.2.3)$$

Some solution curves of (5.2.2) are illustrated hereunder with passing some remarks.

Case: **(5.2.i)** $U_{10} > U_{20} > U_{30}$; $e_2 a_{22} < k_3 a_{33}$

The prey-commensal always out-number both the predator and the host in natural growth rates as well as in their initial population strengths. In this case both the predator and the host converge asymptotically to the equilibrium point while the prey goes far away from the equilibrium point.

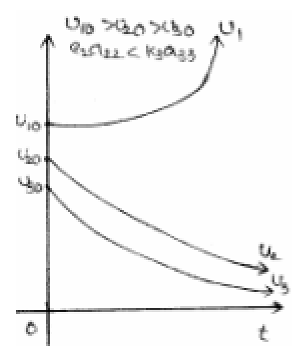

Figure: 4

Case: (5.2.ii) $U_{10} > U_{20} > U_{30}$; $e_2 a_{22} > k_3 a_{33}$

In this case the prey-commensal out-number both the predator and the host in natural growth rates as well as in their initial population strengths. The predator dominates the host up to the time instant $t_{23}^* = \frac{1}{(e_2 a_{22} - k_3 a_{33})} \log\left(\frac{U_{20}}{U_{30}}\right)$ and there after the host dominates the predator and then both declines further.

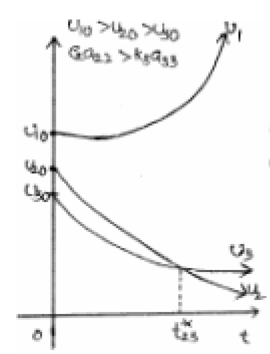

Figure: 5

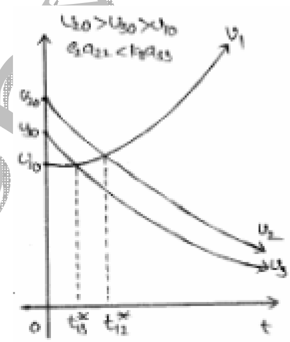

Figure: 6

Case: (5.2.iii) $U_{20} > U_{30} > U_{10}$; $e_2 a_{22} < k_3 a_{33}$

The predator and the host dominates the prey till the time instants $t_{12}^{\ *} = \dfrac{1}{a_{11}(k_1 + ck_3) + e_2 a_{22}} \log\left(\dfrac{U_{20}}{U_{10}}\right)$ and $t_{13}^{\ *} = \dfrac{1}{(k_3 a_{33} + a_{11}(k_1 + ck_3)} \log\left(\dfrac{U_{30}}{U_{10}}\right)$ after which the prey dominates the both the predator and the host. In this case both the predator and the host converge asymptotically to the equilibrium point.

5.3 PREDATOR AND HOST WASHED OUT EQUILIBRIUM STATE:

In this case, we have

$$\frac{dU_1}{dt} = -a_{11}k_1U_1 - pq_{11}k_1U_2 + cq_{11}k_1U_3 \quad ; \quad \frac{dU_2}{dt} = a_{22}(q - e_2)U_2 \quad ; \quad \frac{dU_3}{dt} = k_3 a_{33} U_3 \qquad \text{……….. (5.3.1)}$$

the characteristic roots of which are $-k_1 a_{11}, a_{22}(q - e_2), k_3 a_{33}$ .Since one of its three roots is positive and hence the state is **unstable**.

The equations (5.3.1) yield the solution curves

$$U_1 = \alpha_1 e^{a_{22}(q-e_2)t} + \alpha_2 e^{k_3 a_{33} t} + [U_{10} - (\alpha_1 + \alpha_2)] e^{-k_1 a_{11} t} ; U_2 = U_{20} e^{a_{22}(q-e_2)t} ;$$
$$U_3 = U_{30} e^{k_3 a_{33} t} \qquad \text{…………………… (5.3.2)}$$

Case: A When $q < e_2$ and $U_{10} = \alpha_1 + \alpha_2$ then (5.3.2) becomes

$$U_1 = U_{10}\left[e^{-a_{22}(e_2-q)t} + e^{k_3 a_{33} t}\right] - \alpha_1 e^{k_3 a_{33} t} - \alpha_2 e^{-a_{22}(e_2-q)t} ; U_2 = U_{20} e^{-a_{22}(e_2-q)t} ;$$
$$U_3 = U_{30} e^{k_3 a_{33} t} \qquad \text{…………………… (5.3.3)}$$

<u>Trajectories of Perturbations:</u>

In this case, the trajectories of (5.3.3) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane are respectively given by

$$\left(\frac{U_2}{U_{20}}\right)^{-k_3 a_{33}} = \left(\frac{U_3}{U_{30}}\right)^{a_{22}(e_2-q)} \quad ; \quad x = (1\text{-}M)\, y + (1\text{-}N)\, z \qquad \text{………………… (5.3.A)}$$

where $x = \dfrac{U_1}{U_{10}}$; $y = \dfrac{U_2}{U_{20}}$; $z = \dfrac{U_3}{U_{30}}$; $M = \dfrac{\alpha_2}{U_{10}}$ and $N = \dfrac{\alpha_1}{U_{10}}$

Some solution curves of (5.3.3) are illustrated in following figures and the conclusions are presented.

Case: **(5.3.A (i))** $U_{10} > U_{20} > U_{30}$

Initially the prey and the predator out-number the host up to the time instants $t_{13}{}^{*} = \dfrac{1}{(k_3 a_{33} + a_{22}(e_2 - q)} \log\left(\dfrac{U_{10} - \alpha_2}{U_{30} - U_{10} + \alpha_1}\right)$ and $t_{23}{}^{*} = \dfrac{1}{(k_3 a_{33} + a_{22}(e_2 - q))} \log\left(\dfrac{U_{20}}{U_{30}}\right)$ respectively after which the host out-number both the prey and the predator. In this case the prey and the predator converge asymptotically to the equilibrium point while the host goes away from the equilibrium point.

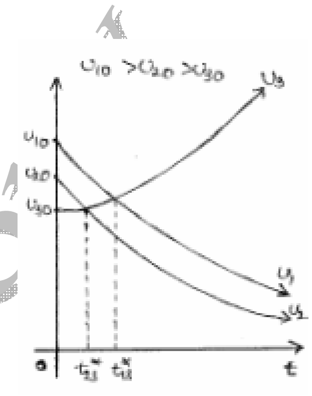

Figure: 7

Case: **(5.3.A (ii))** $U_{10} > U_{30} > U_{20}$

The host dominates the prey in natural growth rate but its initial population strength is less than that of the prey. In this case the prey out-numbers the host up to the time instant $t_{13}{}^{*} = \dfrac{1}{(k_3 a_{33} + a_{22}(e_2 - q)} \log\left(\dfrac{U_{10} - \alpha_2}{U_{30} - U_{10} + \alpha_1}\right)$ after which the dominance is reversed. Here both the prey and the predator declines further.

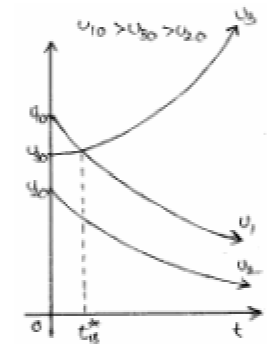

Figure: 8

Case: **(5.3.A (iii))** $U_{20} > U_{10} > U_{30}$

The predator out-numbers the prey and the host till the time instants

$t_{12}{}^{*} = \dfrac{1}{(k_3 a_{33} + a_{22}(e_2 - q)} \log\left(\dfrac{U_{10} - U_{20} - \alpha_2}{\alpha_1 - U_{10}}\right)$ and

$t_{23}{}^{*} = \dfrac{1}{(k_3 a_{33} + a_{22}(e_2 - q))} \log\left(\dfrac{U_{20}}{U_{30}}\right)$ respectively after which the host out-number both the prey and the predator. The predator even thought decline dominates over the prey till the time instant $t_{13}{}^{*} = \dfrac{1}{(k_3 a_{33} + a_{22}(e_2 - q)} \log\left(\dfrac{U_{10} - \alpha_2}{U_{30} - U_{10} + \alpha_1}\right)$ and there after the prey dominates the predator. Here both the predator and prey declines together as shown in the figure 9.

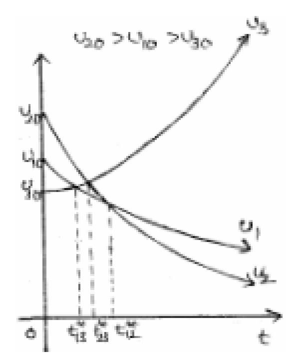

Figure: 9

Case: B When $q > e_2$ and $U_{10} = \alpha_1 + \alpha_2$ then (5.3.2) becomes

$$U_1 = U_{10}\left[e^{a_{22}(q-e_2)t} + e^{k_3 a_{33} t}\right] - \alpha_1 e^{k_3 a_{33} t} - \alpha_2 e^{a_{22}(q-e_2)t}; U_2 = U_{20} e^{a_{22}(q-e_2)t};$$

$$U_3 = U_{30} e^{k_3 a_{33} t} \qquad \text{……………………. (5.3.4)}$$

Trajectories of Perturbations:

The trajectories of (5.3.4) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane are respectively given by

$$\left(\frac{U_2}{U_{20}}\right)^{k_3 a_{33}} = \left(\frac{U_3}{U_{30}}\right)^{a_{22}(q-e_2)} \quad ; \quad x = (1-M)\, y + (1-N)\, z \qquad \text{………………… (5.3.B)}$$

where $x = \dfrac{U_1}{U_{10}}$; $y = \dfrac{U_2}{U_{20}}$; $z = \dfrac{U_3}{U_{30}}$; $M = \dfrac{\alpha_2}{U_{10}}$ and $N = \dfrac{\alpha_1}{U_{10}}$

Some solution curves of (5.3.4) are illustrated in following figures and the conclusions are presented.

Case: **(5.3.B (i)**) $U_{10} > U_{20} > U_{30}$ $a_{22}(q-e_2) < k_3 a_{33}$

In this case the prey always out- numbers both the predator and the host in natural growth rates as well as in their initial population strengths. Here the predator dominates the host till the time instant

$$t_{23}^{\ *} = \frac{1}{(k_3 a_{33} - a_{22}(q-e_2))} \log\left(\frac{U_{20}}{U_{30}}\right)$$

after which the host dominates the predator. In this case all the three species go far away from the equilibrium point.

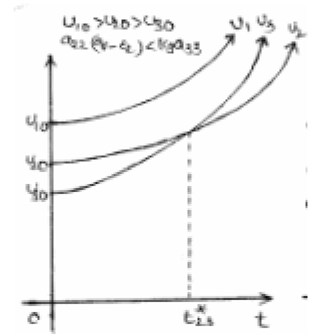

Figure: 10

Case: **(5.3.B (ii)**) $U_{10} > U_{20} > U_{30}$ $a_{22}(q-e_2) > k_3 a_{33}$

In this case the prey always out-numbers the predator, the host and the predator always out-numbers the host in natural growth rate. In this case all the three species go away from the equilibrium point

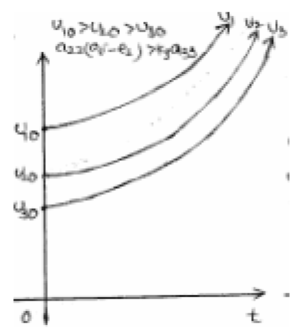

Figure: 11

Case: C When $q = e_2$ and $U_{10} = \alpha_1 + \alpha_2$ then (5.3.2) becomes

$$U_1 = U_{10};\ U_2 = U_{20};\ U_3 = U_{30} e^{k_3 a_{33} t} \qquad \text{……………………. (5.3.5)}$$

Trajectories of Perturbations:

The trajectories of (5.3.5) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane are respectively given by

$$\frac{U_1}{U_{10}}=1 \quad ; \qquad \frac{U_2}{U_{20}}=1 \qquad \text{..................... (5.3.C)}$$

Some solution curves of (5.3.5) are illustrated in following figures with some remarks.

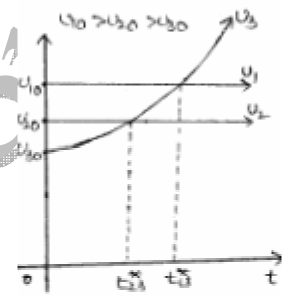

Figure: 12

Case: **(5.3.C (i)**) $U_{10}>U_{20}>U_{30}$

The prey and the predator out-number the host till the time instants $t_{23}^{\,*}=\frac{1}{k_3a_{33}}\log\left(\frac{U_{20}}{U_{30}}\right)$ and $t_{13}^{\,*}=\frac{1}{k_3a_{33}}\log\left(\frac{U_{10}}{U_{30}}\right)$ respectively after which the host out-number both the prey and the predator. In this case both the prey and the predator go away form the equilibrium point.

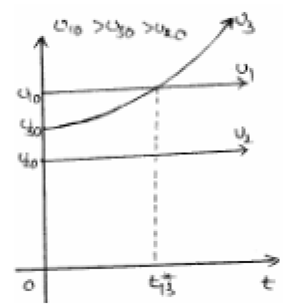

Figure: 13

Case: **(5.3.C (ii)**) $U_{10}>U_{30}>U_{20}$

The prey dominates the host till the time instant $t_{13}^{\,*}=\frac{1}{k_3a_{33}}\log\left(\frac{U_{10}}{U_{30}}\right)$ after which the dominance is reversed. In this case all the three species go away form the equilibrium.

5.4 PREDATOR WASHED OUT EQUILIBRIUM STATE:

In this case, we have

$$\frac{dU_1}{dt}=-a_{11}(k_1+ck_3)U_1-a_{11}p(k_1+ck_3)U_2+a_{11}c(k_1+ck_3)U_3 \;;\qquad \frac{dU_2}{dt}=a_{22}[q(k_1+ck_3)-e_2]U_2 \;;$$

$$\frac{dU_3}{dt}=-k_3a_{33}U_3 \qquad \text{..................... (5.4.1)}$$

the characteristic roots are of which are $-a_{11}(k_1+ck_3),\; a_{22}(q(k_1+ck_3)-e_2)\;,-k_3a_{33}$.

The equations (5.4.1) yield

$$U_1=\beta_1\,e^{a_{22}[q(k_1+ck_3)-e_2]t}+\beta_2\,e^{-k_3a_{33}t}+\left[U_{10}-\left(\beta_1+\beta_2\right)\right]e^{-a_{11}(k_1+ck_3)t}\;;$$

$$U_2=U_{20}e^{a_{22}[q(k_1+ck_3)-e_2]t};U_3=U_{30}e^{-k_3a_{33}t} \qquad \text{..................... (5.4.2)}$$

where

$$\beta_1=\frac{a_{11}p(k_1+ck_3)U_{20}}{a_{22}[e_2-q(k_1+ck_3)]-a_{11}(k_1+ck_3)}\;\; \beta_2=\frac{ca_{11}(k_1+ck_3)U_{30}}{a_{11}(k_1+ck_3)-k_3a_{33}}$$

Case: A When $q(k_1+ck_3)>e_2$, one of the three roots is positive so that the state is **unstable**.

When $q(k_1 + ck_3) > e_2$ and $U_{10} = \beta_1 + \beta_2$ then (5.4.2) becomes

$$U_1 = U_{10}\left[e^{a_{22}[q(k_1+ck_3)-e_2]t} + e^{-k_3 a_{33} t}\right] - \beta_2\, e^{a_{22}[q(k_1+ck_3)-e_2]t} - \beta_1\, e^{-k_3 a_{33} t};$$

$$U_2 = U_{20} e^{a_{22}[q(k_1+ck_3)-e_2]t}; U_3 = U_{30} e^{-k_3 a_{33} t} \qquad \text{…….. …..……….. (5.4.3)}$$

Trajectories of Perturbations:

In this case, the trajectories of (5.4.3) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane are given by

$$x = (1\text{-}M)\, y + (1\text{-}N)\, z \qquad \text{……………..…… (5.4.A)}$$

where $x = \dfrac{U_1}{U_{10}}, y = \dfrac{U_2}{U_{20}}, z = \dfrac{U_3}{U_{30}}, M = \dfrac{\beta_2}{U_{10}}, N = \dfrac{\beta_1}{U_{10}}$

We have observed several different types of solution curves of which only few typical of them are discussed in the following figures.

Case: **(5.4.A (i))**

In this case the prey out-numbers the predator till the time instant $t_{12}^* = \dfrac{1}{(k_3 a_{33} + a_{22}[q(k_1 + ck_3) - e_2])} \log\left(\dfrac{\beta_1 - U_{10}}{U_{10} - U_{20} - \beta_2}\right)$ after which the predator out-numbers the prey. In this case the host is asymptotic to the equilibrium point while other two species go away from the equilibrium point.

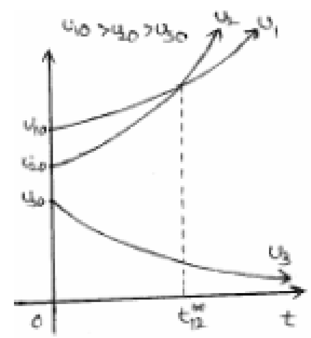

Figure: 14

Case: **(5.4.A (ii))** $U_{10} > U_{30} > U_{20}$

The prey out-numbers the predator till the time instant $t_{12}^* = \dfrac{1}{(k_3 a_{33} + a_{22}[q(k_1 + ck_3) - e_2])} \log\left(\dfrac{\beta_1 - U_{10}}{U_{10} - U_{20} - \beta_2}\right)$ after which the predator out-numbers the prey. Here the host out-numbers the predator up to the time instant $t_{23}^* = \dfrac{1}{(k_3 a_{33} + a_{22}[q(k_1 + ck_3) - e_2])} \log\left(\dfrac{U_{30}}{U_{20}}\right)$ after which the predator out-numbers the host and then the host declines further.

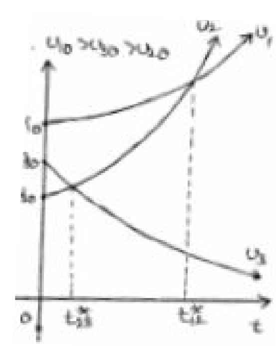

Figure: 15

Case: B When $q(k_1 + ck_3) < e_2$, all the three roots are negative and hence the state is **stable**.

When $q(k_1 + ck_3) < e_2$ and $U_{10} = \beta_1 + \beta_2$ then (5.4.2) becomes

$$U_1 = U_{10}\left[e^{-a_{22}[e_2 - q(k_1+ck_3)]t} + e^{-k_3 a_{33} t}\right] - \beta_2\, e^{-a_{22}[e_2 - q(k_1+ck_3)]t} - \beta_1\, e^{-k_3 a_{33} t}$$
$$U_2 = U_{20} e^{-a_{22}[e_2 - q(k_1+ck_3)]t} ; U_3 = U_{30} e^{-k_3 a_{33} t} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5.4.4)$$

<u>Trajectories of Perturbations:</u>

In this case, the trajectories of (5.4.4) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane are given by

$$x = (1\text{-}M)\, y + (1\text{-}N)\, z \; ; \; \left(\frac{U_2}{U_{20}}\right)^{k_3 a_{33}} = \left(\frac{U_3}{U_{30}}\right)^{a_{22}(e_2 - qk_1 + ck_3)} \qquad \ldots\ldots\ldots\ldots (5.4.B)$$

where $x = \frac{U_1}{U_{10}}, y = \frac{U_2}{U_{20}}, z = \frac{U_3}{U_{30}}, M = \frac{\beta_2}{U_{10}}, N = \frac{\beta_1}{U_{10}}$

Some solution curves of (5.4.4) are illustrated in the following figures and the conclusions are presented

Case: **(5.4.B (i))** $U_{10} > U_{20} > U_{30}$; $a_{22}\left[e_2 - q(k_1 + ck_3)\right] < k_3 a_{33}$

The prey out-numbers both the predator and the host till the time instants

$$t_{12}^{\ *} = \frac{1}{(k_3 a_{33} - a_{22}[e_2 - q(k_1 + ck_3)])} \log\left(\frac{\beta_1 - U_{10}}{U_{10} - U_{20} - \beta_2}\right) \text{ and}$$

$$t_{13}^{\ *} = \frac{1}{(k_3 a_{33} - a_{22}[q(k_1 + ck_3) - e_2])} \log\left(\frac{U_{30} - U_{10} + \beta_1}{U_{10} - \beta_2}\right)$$ respectively after which the predator and the host out-numbers the prey. Here the predator always out-members the host. In this case the three species converge asymptotically to equilibrium point.

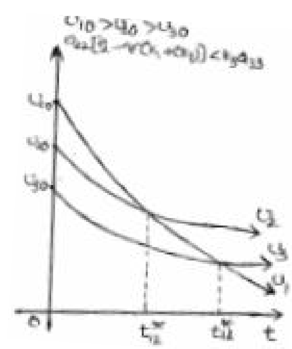

Figure: 16

Case: **(5.4.B (ii))** $U_{10} > U_{20} > U_{30}$; $a_{22}\left[e_2 - q(k_1 + ck_3)\right] > k_3 a_{33}$

The prey and predator dominates the host till the time instants

$$t_{12}^{\ *} = \frac{1}{(k_3 a_{33} - a_{22}[e_2 - q(k_1 + ck_3)])} \log\left(\frac{\beta_1 - U_{10}}{U_{10} - U_{20} - \beta_2}\right) \text{ and}$$

$$t_{13}^{\ *} = \frac{1}{(k_3 a_{33} - a_{22}[q(k_1 + ck_3) - e_2])} \log\left(\frac{U_{30} - U_{10} + \beta_1}{U_{10} - \beta_2}\right)$$ respectively after which

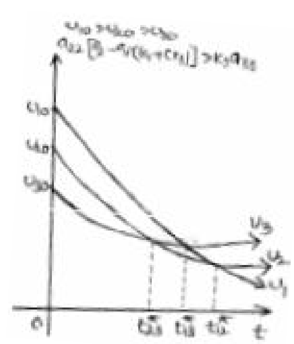

Figure: 17

the host dominates both the prey and the predator. Here the prey dominates over the predator till the time instant $t_{23}^{*} = \frac{1}{\left(k_3 a_{33} - a_{22}[q(k_1 + ck_3) - e_2]\right)} \log\left(\frac{U_{30}}{U_{20}}\right)$ after which the predator dominates over the prey. In this case the three species converge asymptotically to equilibrium point.

Case: C When $q(k_1 + ck_3) = e_2$, one of three roots would be zero so that the state is **unstable**.

When $q(k_1 + ck_3) = e_2$ and $U_{10} = \beta_1 + \beta_2$ then (5.4.2) becomes

$$U_1 = U_{10} e^{-k_3 a_{33} t} + \beta_1 (1 - e^{-k_3 a_{33} t}) \; ; \; U_2 = U_{20} \; ; \; U_3 = U_{30} e^{-k_3 a_{33} t} \quad \text{………......……. (5.4.5)}$$

Trajectories of Perturbations:

In this case, the trajectories of (5.4.5) in the $U_1 - U_2$ plane, $U_2 - U_3$ plane and $U_3 - U_1$ plane are given by

$$x = N + (1-N)\, z \quad ; \quad \frac{U_2}{U_{20}} = 1 \qquad \text{………………………. (5.4.C)}$$

where $x = \frac{U_1}{U_{10}}, z = \frac{U_3}{U_{30}}, N = \frac{\beta_1}{U_{10}}$

Some solution curves of (5.4.5) are illustrated in the following figures and the conclusions are presented

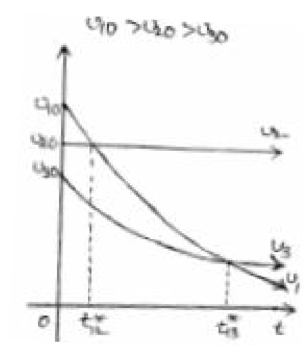

Figure: 18

Case: **(5.4.C (i))** $U_{10} > U_{20} > U_{30}$

The prey out-number both the predator and the host till the time instants

$t_{12}^{*} = \frac{1}{k_3 a_{33}} \log\left(\frac{\beta_2}{U_{20} - \beta_1}\right)$ and $t_{13}^{*} = \frac{1}{k_3 a_{33}} \log\left(\frac{U_{30} - \beta_2}{\beta_1}\right)$ respectively after which the predator out-number both the prey and the host. In this case the prey and the host converge asymptotically to the equilibrium point while the predator goes away form the equilibrium point.

Case: **(5.4.C (ii))** $U_{10} > U_{30} > U_{20}$

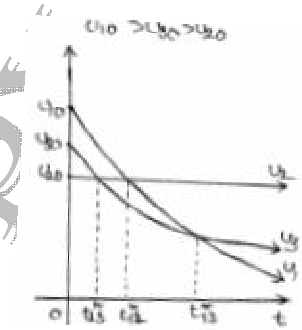
Figure: 19

The prey and the host dominate the predator till the time instants

$t_{12}^{*} = \frac{1}{k_3 a_{33}} \log\left(\frac{\beta_2}{U_{20} - \beta_1}\right)$ and $t_{13}^{*} = \frac{1}{k_3 a_{33}} \log\left(\frac{U_{30} - \beta_2}{\beta_1}\right)$ respectively after

which the predator dominates both the prey and the host. Here the prey

out-numbers the host till the time instant $t_{23}^{*} = \frac{1}{k_3 a_{33}} \log\left(\frac{U_{30}}{U_{20}}\right)$ and then the host

out-numbers the prey and then both declines.

5.5 HOST WASHED OUT EQUILIBRIUM STATE:

In this case, we get

$$\frac{dU_1}{dt} = -\frac{a_{11}(k_1 + pe_2)}{1+pq} U_1 - \frac{a_{11} p(k_1 + pe_2)}{1+pq} U_2 + \frac{a_{11} c(k_1 + pe_2)}{1+pq} U_3 \;;$$

$$\frac{dU_2}{dt} = \frac{a_{22} q(qk_1 - e_2)}{1+pq} U_1 + \frac{a_{22}(e_2 - qk_1)}{1+pq} U_2 \;; \quad \frac{dU_3}{dt} = k_3 a_{33} U_3 \qquad \text{…………..… ……….. (5.5.1)}$$

the characteristic equation is $\left(\lambda^2 + (\alpha + \beta)\lambda + (1 + pq)\alpha\beta\right)(\lambda - k_3 a_{33})$=0 and whose roots

$(\lambda_1, \lambda_2, \lambda_3)$ are $\frac{-(\alpha + \beta) \pm \sqrt{(\alpha + \beta)^2 - 4\alpha\beta(1 + pq)}}{2}$, $k_3 a_{33}$. λ_1, λ_2 are real or complex

according as $(\alpha + \beta)^2 \geq 4\alpha\beta(1 + pq)$ or $(\alpha + \beta)^2 < 4\alpha\beta(1 + pq)$.In any case, one root of

the three is positive and hence the system is always **unstable**.

The equations (5.5.1) yield the solutions

$$U_1(t) = \frac{(\lambda_1 + \beta)\left[U_{10}(\lambda_1 - k_3 a_{33}) + c\alpha U_{30}\right] - \alpha p U_{20}(\lambda_1 - k_3 a_{33})}{(\lambda_1 - \lambda_2)(\lambda_1 - k_3 a_{33})} e^{\lambda_1 t}$$
$$+ \frac{(\lambda_2 + \beta)\left[U_{10}(\lambda_2 - k_3 a_{33}) + c\alpha U_{30}\right] - \alpha p U_{20}(\lambda_2 - k_3 a_{33})}{(\lambda_2 - \lambda_1)(\lambda_2 - k_3 a_{33})} e^{\lambda_2 t} + \frac{(k_3 a_{33} - \beta) c\alpha U_{30}}{(k_3 a_{33} - \lambda_1)(k_3 a_{33} - \lambda_2)} e^{k_3 a_{33} t}$$

$$U_2(t) = \frac{U_{20}(\lambda_1+\alpha) - \beta q[U_{10}(\lambda_1 - k_3 a_{33}) + c\alpha U_{30}]}{(\lambda_1-\lambda_2)(\lambda_1 - k_3 a_{33})} e^{\lambda_1 t}$$
$$+ \frac{U_{20}(\lambda_2+\alpha) - \beta q[U_{10}(\lambda_2 - k_3 a_{33}) + c\alpha U_{30}]}{(\lambda_2-\lambda_1)(\lambda_2 - k_3 a_{33})} e^{\lambda_2 t} + \frac{c\alpha\beta q U_{30}}{(k_3 a_{33}-\lambda_1)(k_3 a_{33}-\lambda_2)} e^{k_3 a_{33} t}$$
$$U_3 = U_{30} e^{k_3 a_{33} t} \qquad \text{……………………………. (5.5.2)}$$

Some real solution curves of (5.5.2) are illustrated in the following figures and the conclusions are presented

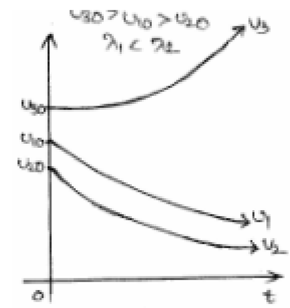

Figure: 20

Case: **(5.5.i)** $U_{30} > U_{10} > U_{20}$; $\lambda_1 < \lambda_2$

Initially the host always out-members both the prey and the predator and the prey always out-members the predator in natural growth rates as well as in its initial population strengths. In this case both the prey and the predator converge asymptotically to the equilibrium point while the host goes away form the equilibrium point.

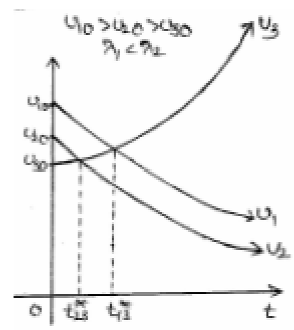

Figure: 21

Case: **(5.5.2ii)** $U_{10} > U_{20} > U_{30}$; $\lambda_1 < \lambda_2$

Both the prey and the predator dominates the host up to some time after which the host dominates both the prey and the predator. In this case both the prey and the predator converge asymptotically to the equilibrium point.

5.6 STABILITY OF CO-EXISTING STATE:

In this case, we have

$$\frac{dU_1}{dt} = -\frac{a_{11}[k_1+pe_2+ck_3]}{1+pq}U_1 - \frac{a_{11}p[k_1+pe_2+ck_3]}{1+pq}U_2 + \frac{cq_{11}[k_1+pe_2+ck_3]}{1+pq}U_3 \;;$$
$$\frac{dU_2}{dt} = \frac{a_{22}q[qk_1+qck_3-e_2]}{1+pq}U_1 - \frac{a_{22}[qk_1+qck_3-e_2]}{1+pq}U_2 \;; \quad \frac{dU_3}{dt} = -k_3 a_{33} U_3 \qquad \text{…...……….. (5.6.1)}$$

the characteristic equation is $\left(\lambda^2 + (\alpha+\beta)\lambda + (1+pq)\alpha\beta\right)(\lambda + k_3 a_{33}) = 0$ and whose roots

$(\lambda_1, \lambda_2, \lambda_3)$ are $\dfrac{-(\alpha+\beta) \pm \sqrt{(\alpha+\beta)^2 - 4\alpha\beta(1+pq)}}{2}$, $-k_3 a_{33}$. λ_1, λ_2 are real or complex

according as $(\alpha+\beta)^2 \ge 4\alpha\beta(1+pq)$ or $(\alpha+\beta)^2 < 4\alpha\beta(1+pq)$. In any case, all the three roots are negative when they are real or has a negative real part when they are complex. Hence the system is always **stable**.

The equation (5.6.1) yield the solutions

$$U_1(t) = \frac{(\lambda_1+\beta)[U_{10}(\lambda_1+k_3a_{33})+c\alpha U_{30}]-\alpha p U_{20}(\lambda_1+k_3a_{33})}{(\lambda_1-\lambda_2)(\lambda_1+k_3a_{33})}e^{\lambda_1 t}$$
$$+\frac{(\lambda_2+\beta)[U_{10}(\lambda_2+k_3a_{33})+c\alpha U_{30}]-\alpha p U_{20}(\lambda_2+k_3a_{33})}{(\lambda_2-\lambda_1)(\lambda_2+k_3a_{33})}e^{\lambda_2 t}+\frac{(\beta-k_3a_{33})c\alpha U_{30}}{(\lambda_1+k_3a_{33})(\lambda_2+k_3a_{33})}e^{-k_3a_{33}t}$$

$$U_2(t) = \frac{U_{20}(\lambda_1+\alpha)(\lambda_1+k_3a_{33})+\beta q[U_{10}(\lambda_1+k_3a_{33})+c\alpha U_{30}]}{(\lambda_1-\lambda_2)(\lambda_1+k_3a_{33})}e^{\lambda_1 t}$$
$$+\frac{U_{20}(\lambda_2+\alpha)(\lambda_2+k_3a_{33})+\beta q[U_{10}(\lambda_2+k_3a_{33})+c\alpha U_{30}]}{(\lambda_2-\lambda_1)(\lambda_2+k_3a_{33})}e^{\lambda_2 t}+\frac{c\alpha\beta q U_{30}}{(\lambda_1+k_3a_{33})(\lambda_2+k_3a_{33})}e^{-k_3a_{33}t}$$

$$U_3 = U_{30}e^{-k_3a_{33}t} \qquad \text{.................................. (5.6.2)}$$

Case A: When $(\alpha+\beta)^2 \geq 4\alpha\beta(1+pq)$ then all the three roots are negative real and hence the equilibrium state is **stable**.

Some solution curves of (5.6.2) are illustrated here under passing some remarks.

Case: (5.6.A (i)) $U_{10} > U_{20} > U_{30}$; $\lambda_1 > \lambda_2 > \lambda_3$

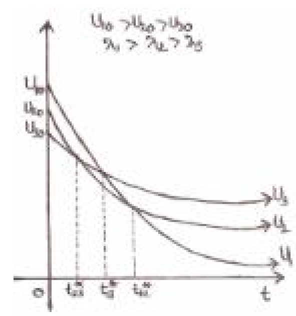

Figure: 22

In this case commensal out-number both the predator and the host for some time after which the dominance is reversed. Here the predator out-numbers the host for some time after which the host out-numbers the predator. However the three species converge asymptotically to the equilibrium point.

Case: (5.6.A (ii)) $U_{10} > U_{20} > U_{30}$; $\lambda_1 > \lambda_3 > \lambda_2$

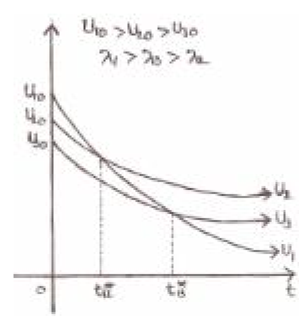

Figure: 23

The prey out-number both the predator and the host for some time after which both the predator and the host out-number the prey-commensal. In this case the three species converge asymptotically to the equilibrium point.

Case: (5.6.A (iii)) $U_{10} > U_{20} > U_{30}$; $\lambda_3 > \lambda_2 > \lambda_1$

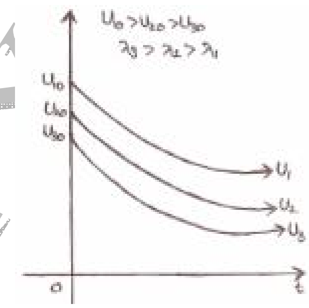

Figure: 24

In this case the prey always out-numbers both the predator and the host and thepredator out-numbers the host in natural growth rates as well as in its initial population strengths. However the three species converges asymptotically to the equilibrium point.

Case B: When $(\alpha+\beta)^2 < 4\alpha\beta(1+pq)$ then the two roots (λ_1, λ_2) of three are complex with negative real part and the third is (λ_3) negative real and hence the equilibrium state is **stable**.

In this case some solution curves of (5.6.2) are illustrated in the following figures and the conclusions are presented.

Case: (5.6.B (i)) $U_{10} > U_{20} > U_{30}$; $|\lambda_1| > |\lambda_2| > |\lambda_3|$

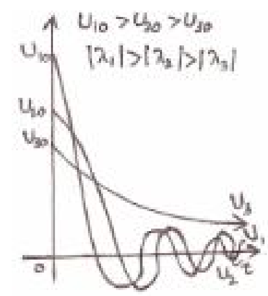

Figure: 25

In this case the prey dominates both the predator and the host in their initial population strengths. Here all the three species converge asymptotically to the equilibrium point.

Case: (5.6.B (ii)) $U_{20} > U_{10} > U_{30}$; $|\lambda_1| > |\lambda_3| > |\lambda_2|$

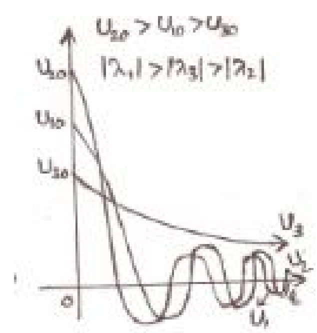

Figure: 26

The predator dominates both the prey and the host and the prey also dominates the host in their initial population strengths. Here the three species converge asymptotically to the equilibrium point.

Case: (5.6.B (iii)) $U_{30} > U_{20} > U_{10}$; $|\lambda_3| > |\lambda_1| > |\lambda_2|$

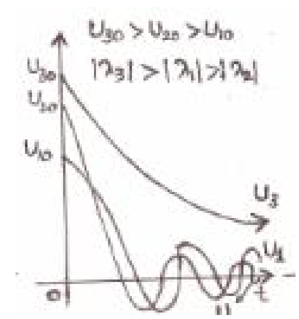

Figure: 27

In this case the host always out-numbers both the prey and the predator in natural growth rates as well as in their initial population strengths. However thethree species converge asymptotically to equilibrium point.

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