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ON THE STABILITY OF A FOUR SPECIES: A PREY-PREDATOR-HOST-COMMENSAL-SYN ECO-SYSTEM-V (PREY WASHED OUT STATES)

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ABSTRACT: This paper deals with an investigation on a Four Species Syn-Ecological System (Prey washed out states). The System comprises of a Prey (S_1), a Predator (S_2) that survives upon S_1, two Hosts S_3 and S_4 for which S_1, S_2 are commensal respectively i.e., S_3 and S_4 benefit S_1 and S_2 respectively, without getting effected either positively or adversely. Further S_3 and S_4 are neutral. The model equations of the system constitute a set of four first order non-linear ordinary differential coupled equations. In all, there are sixteen equilibrium points. Criteria for the asymptotic stability of four of the sixteen equilibrium points : the Prey washed out states only are established in this paper. The system would be stable if all the characteristic roots are negative, in case they are real, and have negative real parts, in case they are complex. The linearized equations for the perturbations over the equilibrium points are analyzed to establish the criteria for stability and the trajectories illustrated.

Key words: Asymptotic, Equillibrium point, Host, Prey, Predator, Stability, Neutrally stable.

1. INTRODUCTION:

Mathematical modeling of Eco-System was initiated in 1925 by Lotka [11]. The general concepts of modeling have been presented in the treatises of Meyer[12], Kushing[8], Kapur J.N. [6,7] and several others. The ecological interactions can be broadly classified as Prey-Predator, Commensalism, Competition, Neutralism, Mutualism and so on. N.C. Srinivas [14] studied competitive eco-systems of two species and three species with limited and unlimited resources. Later Lakshminarayan [9], Lakshminarayan and Pattabhi Ramacharyulu [10] studied Prey-Preadtor ecological models with partial cover for the Prey and alternate food for the Predator. Recently, Archana Reddy [1] and Bhaskara

Rama Sharma [2] investigated diverse problems related to two species competitive systems with time delay, employing analytical and numerical techniques. Further Phani Kumar, Seshagiri Rao and Pattabhi Ramacharyulu [13] studied the stability of a Host-A flourishing commensal species pair with limited resources. The present authors Hari Prasad B and Pattabhi Ramacharyulu.N.Ch studied the stability of the fully washed out state [3], co-existent state [5], Prey and Predator washed out states [4]. Continuation of this criteria for the stability of the Prey only washed out states of the system are presented in this paper.

2. BASIC EQUATIONS OF THE MODEL:

Notation Adopted:

S_1 : Prey for S_2 and commensal for S_3.

S_2 : Predator surviving upon S_1 and commonsal for S_4.

S_3 : Host for the commonsal – Prey (S_1).

S_4 : Host of the commonsal – Predator (S_2)

$N_1(t)$: The Population of the Prey (S_1)

$N_2(t)$: The Population of the Predator (S_2)

$N_3(t)$: The Population of the Host (S_3) of the Prey (S_1)

$N_4(t)$: The Population of the Host (S_4) of the Predator (S_2)

t : Time instant

a_1,a_2,a_3,a_4 : Natural growth rates of S_1, S_2, S_3, S_4

$a_{11},a_{22},a_{33},a_{44}$: Self inhibition coefficients of S_1, S_2, S_3, S_4

a_{12},a_{21} : Interaction (Prey-Predator) coefficients of S_1 due to S_2 and S_2 due to S_1

a_{13} : Coefficient for commensal for S_1 due to the Host S_3

a_{24} : Coefficient for commensal for S_2 due to the Host S_4

$k_1 = \frac{a_1}{a_{11}}, k_2 = \frac{a_2}{a_{22}}, k_3 = \frac{a_3}{a_{33}}, k_4 = \frac{a_4}{a_{44}}$: Carrying capacities of S_1, S_2, S_3, S_4

Further the variables N_1, N_2, N_3, N_4 are non-negative and the model parameters a_1, a_2, a_3, a_4; a_{11}, a_{22}, a_{33}, a_{44}; a_{12}, a_{21}, a_{13}, a_{24} are assumed to be non-negative constants.

The model equations for the growth rates of S_1, S_2, S_3, S_4 are

$$\frac{dN_1}{dt} = a_1N_1 - a_{11}N_1^2 - a_{12}N_1N_2 + a_{13}N_1N_3 \qquad \text{....} \qquad (2.1)$$

$$\frac{dN_2}{dt} = a_2N_2 - a_{22}N_2^2 + a_{21}N_1N_2 + a_{24}N_2N_4 \qquad \text{....} \qquad (2.2)$$

$$\frac{dN_3}{dt} = a_3 N_3 - a_{33} N_3^{\ 2} \qquad \ldots. \qquad (2.3)$$

$$\frac{dN_4}{dt} = a_4 N_4 - a_{44} N_4^{\ 2} \qquad \ldots. \qquad (2.4)$$

3 EQUILIBRIUM STATES:

The system under investigation has sixteen equilibrium states defined by

$$\frac{dN_i}{dt} = 0,\ i = 1, 2, 3, 4 \qquad \ldots\ldots\ldots\ldots (3.1)$$

are given in the following table.

S.No.	Equilibrium State	Equilibrium Point
1	Fully Washed out state	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$
2	Only the Host (S_4)of S_2 survives	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$
3	Only the Host (S_3)of S_1 survives	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$
4	Only the Predator S_2 survives	$\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = 0, \overline{N_4} = 0$
5	Only the Prey S_1 survives	$\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$
6	Prey (S_1) and Predator (S_2) washed out	$\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = \frac{a_4}{a_{44}}$
7	Prey (S_1) and Host (S_3) of S_1 washed out	$\overline{N_1} = 0, \overline{N_2} = \frac{a_2 a_{44} + a_4 a_{24}}{a_{22} a_{44}}, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$
8	Prey (S_1) and Host (S_4) of S_2 washed out	$\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$
9	Predator (S_2) and Host (S_3) of S_1 washed out	$\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$
10	Predator (S_2) and Host (S_4) of S_2 washed out	$\overline{N_1} = \frac{a_1 a_{33} + a_3 a_{13}}{a_{11} a_{13}}, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$

11	Prey (S_1) and Predator (S_2)survives	$\overline{N_1} = \dfrac{a_1 a_{22} - a_2 a_{12}}{a_{11} a_{22} + a_{12} a_{21}}, \overline{N_2} = \dfrac{a_1 a_{21} + a_2 a_{11}}{a_{11} a_{22} + a_{12} a_{21}}, \overline{N_3} = 0, \overline{N_4} = 0$
12	Only the Prey (S_1) washed out	$\overline{N_1} = 0, \overline{N_2} = \dfrac{a_2 a_{44} + a_4 a_{24}}{a_{22} a_{44}}, \overline{N_3} = \dfrac{a_3}{a_{33}}, \overline{N_4} = \dfrac{a_4}{a_{44}}$
13	Only the predator (S_2) washed out	$\overline{N_1} = \dfrac{a_1 a_{23} + a_3 a_{13}}{a_{11} a_{13}}, \overline{N_2} = 0, \overline{N_3} = \dfrac{a_3}{a_{33}}, \overline{N_4} = \dfrac{a_4}{a_{44}}$
14	Only the Host (S_3) of S_1 washed out	$\overline{N_1} = \dfrac{\delta_2}{\delta_1}, \overline{N_2} = \dfrac{\delta_3}{\delta_1}, \overline{N_3} = 0, \overline{N_4} = \dfrac{a_4}{a_{44}}$ *where* $\delta_1 = a_{44}(a_{11} a_{22} + a_{12} a_{21}) > 0$ $\delta_2 = a_1 a_{22} a_{44} - a_{12}(a_2 a_{44} + a_4 a_{24})$ $\delta_3 = a_1 a_{21} a_{44} - a_{11}(a_2 a_{44} + a_4 a_{24})$
15	Only the Host (S_4) of S_2 washed out	$\overline{N_1} = \dfrac{\sigma_2}{\sigma_1}, \overline{N_2} = \dfrac{\sigma_3}{\sigma_1}, \overline{N_3} = \dfrac{a_3}{a_{33}}, \overline{N_4} = 0$ *where* $\sigma_1 = a_{33}(a_{11} a_{22} + a_{12} a_{21}) > 0$ $\sigma_2 = a_{22}(a_1 a_{33} + a_3 a_{13}) - a_2 a_{12} a_{33}$ $\sigma_3 = a_{21}(a_1 a_{33} + a_3 a_{13}) + a_2 a_{11} a_{33} > 0$
16	The co-existent state (or) Normal steady state	$\overline{N_1} = \dfrac{a_{22} a_{44} \psi_1 - a_{12} a_{33} \psi_2}{\psi_3}, \overline{N_2} = \dfrac{a_{21} a_{44} \psi_1 + a_{11} a_{33} \psi_2}{\psi_3},$ $\overline{N_3} = \dfrac{a_3}{a_{33}}, \overline{N_4} = \dfrac{a_4}{a_{44}}$ *where* $\psi_1 = a_1 a_{33} + a_3 a_{13} > 0$ $\psi_2 = a_2 a_{44} + a_4 a_{24} > 0$ $\psi_3 = a_{33} a_{44}(a_{11} a_{22} + a_{12} a_{21}) > 0$

The present paper deals with the Prey washed out states only. The stability of the other equilibrium states will be presented in the forth coming communications.

4. Stability of the Prey washed out equilibrium states only :

(Sl. Nos. 4, 7, 8, 12 in the above table)

The Equilibrium point $\bar{N}_1 = 0$, $\bar{N}_2 = \frac{a_2}{a_{22}}$, $\bar{N}_3 = 0$, $\bar{N}_4 = 0$ (Sl.No. 4)

was already discussed in the paper "On the Stability of a Four Species: A Prey – Predator – Host – Commensal – Syn Eco-System-IV" communicated to "Internation eJournal of Mathematics and Engineering".

4.1. Equilibrium point $\bar{N}_1 = 0$, $\bar{N}_2 = \frac{a_2 a_{44} + a_4 a_{24}}{a_{22} a_{44}}$, $\bar{N}_3 = 0$, $\bar{N}_4 = \frac{a_4}{a_{44}}$:

Let us consider small deviations from the steady state

i.e. $N_i(t) = \bar{N}_i + u_i(t)$, $i = 1, 2, 3, 4$ .... (4.1.1)

where $u_i(t)$ is a small perturbations in the species S_i.

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4.

We get

$\frac{du_1}{dt} = r_1 u_1$ .... (4.1.2), $\frac{du_2}{dt} = a_{21} p u_1 - a_{22} p u_2 + a_{24} p u_4$ .... (4.1.3)

$\frac{du_3}{dt} = a_3 u_3$ .... (4.1.4), $\frac{du_4}{dt} = -a_4 u_4$ .... (4.1.5)

Here $p = \frac{a_2 a_{44} + a_4 a_{24}}{a_{22} a_{44}} > 0$ and $r_1 = a_1 - a_{12} p$ .... (4.1.6)

The characteristic equation of which is

$(\lambda - r_1)(\lambda + a_{22} p)(\lambda - a_3)(\lambda + a_4) = 0$ .... (4.1.7)

The roots $-a_{22}p$, $-a_4$ are negative and a_3 is positive.

Case (A): If $\frac{a_1}{a_{12}} < p$ (i.e. $r_1 < 0$)

The root r_1 is negative. Hence steady state is **unstable** and the solutions of the equations (4.1.2), (4.1.3), (4.1.4), (4.1.5) are

$u_1 = u_{10} e^{r_1 t}$ .... (4.1.8), $u_2 = m e^{r_1 t} + (u_{20} - m - n) e^{-a_{22} p t} + n e^{-a_4 t}$ .... (4.1.9)

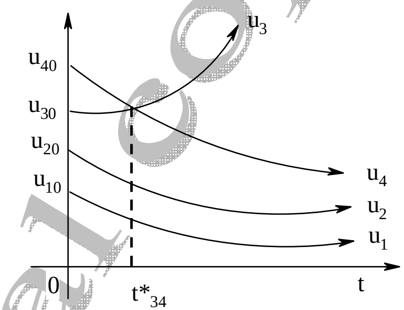
$u_3 = u_{30} e^{a_3 t}$ .... (4.1.10), $u_4 = u_4 e^{-a_4 t}$ .... (4.1.11)

Here $m = \frac{u_{20} a_{21} p}{a_1 + p(a_{22} - a_{12})}$, $n = \frac{u_{40} a_{24} p}{a_{22} p - a_4}$ .... (4.1.12)

and $u_{10}, u_{20}, u_{30}, u_{40}$ are the initial values of u_1, u_2, u_3, u_4 respectively.

In the three equilibrium states, there would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates a_1, a_2, a_3, a_4 and the initial values of the perturbations $u_{10}(t), u_{20}(t), u_{30}(t), u_{40}(t)$ of the species S_1, S_2, S_3, S_4. Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations.

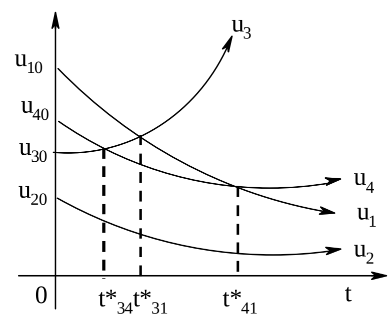
Case (i): If $u_{10} < u_{20} < u_{30} < u_{40}$ and $r_1 < a_2 < a_3 < a_4$



In this case the Prey (S_1) has the least natural birth rate. Initially the Host (S_4) of S_2 dominates over the Host (S_3) of S_1 till the instant t^*_{34} and thereafter the dominance is reversed.

Here $t^*_{34} = \dfrac{1}{a_3 + a_4} \log\left(\dfrac{u_{40}}{u_{30}}\right)$ ……….. (4.1.13)

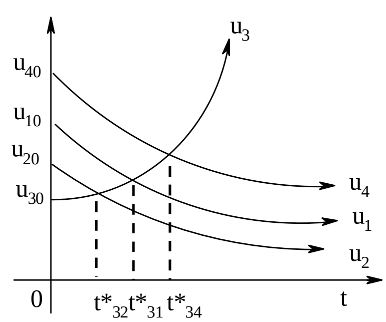
Case (ii): If $u_{20} < u_{30} < u_{40} < u_{10}$ and $a_2 < r_1 < a_4 < a_3$



In this case the Predator (S_2) has the least natural birth rate. Initially the Host (S_4) of S_2 dominates over the Host (S_3) of S_1 till the time instant t^*_{34} and there after the dominance is reversed. Also the Prey (S_1) dominates its Host (S_3), the Host (S_4) of S_2 till the time instant t^*_{31}, t^*_{41} respectively and the dominance gets reversed there after.
Here

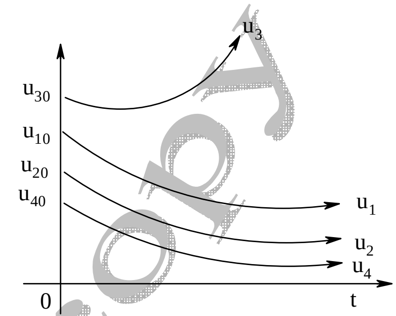
$$t^*_{31} = \frac{1}{r_1 + a_3} \log\left(\frac{u_{30}}{u_{10}}\right),\; t^*_{41} = \frac{1}{r_1 - a_4} \log\left(\frac{u_{40}}{u_{10}}\right) \quad .. (4.1.14)$$

Case (iii): If $u_{30} < u_{20} < u_{10} < u_{40}$ and $a_2 < a_3 < r_1 < a_4$



In this case the Predator (S_2) has the least natural birth rate. Initially it is dominated over by the Host (S_3) of S_1 till the time instant t^*_{32} and there after the dominance is reversed. Also the Prey (S_1), Host (S_4) of S_2 dominates over the Host (S_3) of S_1 till the time instant t^*_{31}, t^*_{34} respectively and the dominance gets reversed there after.

Case (iv): If $u_{40} < u_{20} < u_{10} < u_{30}$ and $a_4 < a_2 < a_3 < r_1$

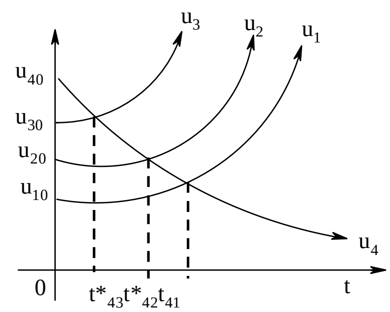


In this case the Host (S_4) of S_2 has the least natural birth rate and the Host (S_3) of S_1 dominates the Prey (S_1), Predator (S_2), Host (S_4) of S_2 in natural growth rate as well as in its population strength.

Case B: If $\frac{a_1}{a_{12}} > p$ $\left(\text{i.e. } r_1 > 0\right)$

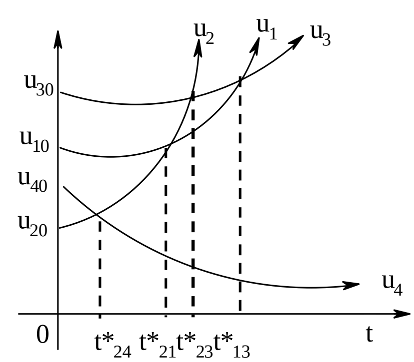
The root r_1 is positive. Hence the steady state is **unstable** and the solutions in this case are same as in case (A).

Case (i): If $u_{10} < u_{20} < u_{30} < u_{40}$ and $r_1 < a_2 < a_3 < a_4$



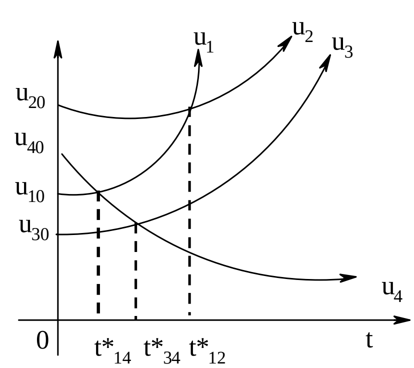
In this case the Host (S_4) of S_2 has the least natural birth rate. Initially it is dominated over by the Host (S_3) of S_1, Predator (S_2), Prey (S_1) till the time instant $t^*_{43}, t^*_{42}, t^*_{41}$ respectively and thereafter the dominance is reversed.

Case (ii): If $u_{20} < u_{40} < u_{10} < u_{30}$ and $a_3 < r_1 < a_4 < a_2$



In this case the Host (S_4) of S_2 has the least natural birth rate. Initially it is dominated over by the Predator (S_2) till the time instant t^*_{24} and thereafter the dominance is reversed. Also the Prey (S_1) dominates over the Predator (S_2) till the time instant t^*_{21} and thereafter the dominance is reversed. Similarly the Host (S_3) of S_1 dominates over the Predator (S_2), Prey (S_1) till the time instant t^*_{23}, t^*_{13} respectively and the dominance gets reversed thereafter.

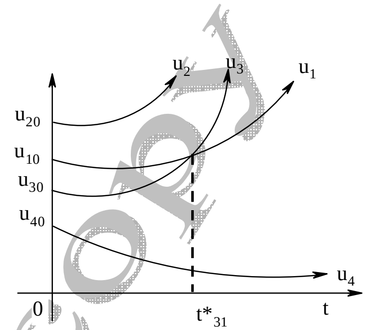
Case (iii): If $u_{30} < u_{10} < u_{40} < u_{20}$ and $a_3 < a_2 < r_1 < a_4$



In this case the Host (S_4) of S_2 has the least natural birth rate. Initially it is dominated over by the Prey (S_1), Host (S_3) of S_1 till the time instant t^*_{14}, t^*_{34} respectively and thereafter the dominance is reversed. Also the Predator (S_2) dominates over the Prey (S_1) till the time instant t^*_{12} and the dominance gets reversed thereafter.

Case (iv): If $u_{40} < u_{30} < u_{10} < u_{20}$ and $r_1 < a_3 < a_4 < a_2$

In this case the Host (S_4) of S_2 has the least natural birth rate. Initially the Prey (S_1) dominates its Host (S_3) till the time instant t^*_{31} and thereafter the dominance is reversed.



Trajectories of perturbations:

The trajectories in the $u_1 - u_3$, $u_1 - u_4$, $u_3 - u_4$, $u_1 - u_2$, $u_2 - u_3$, $u_2 - u_4$ planes are

$$\left(\frac{u_1}{u_{10}}\right)^{a_3} = \left(\frac{u_3}{u_{30}}\right)^{r_1}, \quad \left(\frac{u_1}{u_{10}}\right)^{-a_4} = \left(\frac{u_4}{u_{40}}\right)^{r_1}, \left(\frac{u_3}{u_{30}}\right)^{-a_4} = \left(\frac{u_4}{u_{40}}\right)^{a_3} \quad \ldots\ldots\ldots \quad (4.1.15)$$

$$y = Ax + Bx^{\frac{-a_{22}P}{r_1}} + Cx^{\frac{-a_4}{r_1}} \quad \ldots\ldots\ldots \quad (4.1.16)$$

$$y = Ax_1^{\frac{r_1}{a_3}} + Bx_1^{\frac{-a_{22}P}{a_3}} + Cx_1^{\frac{-a_4}{a_3}} \quad \ldots\ldots\ldots \quad (4.1.17)$$

$$y = Ax_2^{\frac{-r_1}{a_4}} + Bx_2^{\frac{a_{22}P}{a_4}} + Cx_2 \quad \ldots\ldots\ldots \quad (4.1.18)$$

respectively

where $x = \frac{u_1}{u_{10}}$, $y = \frac{u_2}{u_{20}}$, $x_1 = \frac{u_3}{u_{30}}$, $x_2 = \frac{u_4}{u_{40}}$ ……… (4.1.19)

$$A = \frac{m}{u_{20}},\ B = \frac{u_{20} - m - n}{u_{20}},\ C = \frac{n}{u_{20}} \quad \ldots\ldots\ldots \quad (4.1.20)$$

4.2 Equilibrium point $\bar{N}_1 = 0$, $\bar{N}_2 = \frac{a_2}{a_{22}}$, $\bar{N}_3 = \frac{a_3}{a_{33}}$, $\bar{N}_4 = 0$:

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher power of u_1, u_2, u_3, u_4

We get

$$\frac{du_1}{dt} = b_1 u_1 + a_{13} u_3 \quad \ldots\ldots\ldots \quad (4.2.1)$$

$$\frac{du_2}{dt} = \frac{a_2 a_{21}}{a_{22}} u_1 - a_2 u_2 + \frac{a_2 a_{24}}{a_{22}} u_4 \quad \ldots\ldots\ldots \quad (4.2.2)$$

$$\frac{du_3}{dt} = -a_3 u_3 \quad \ldots\ldots\ldots \ (4.2.3), \qquad \frac{du_4}{dt} = a_4 u_4 \quad \ldots\ldots\ldots \ (4.2.4)$$

Here $b_1 = a_1 - \frac{a_2 a_{12}}{a_{22}} + \frac{a_3 a_{13}}{a_{33}}$ ……… (4.2.5)

The characteristic equation of which is

$$(\lambda - b_1)(\lambda + a_2)(\lambda + a_3)(\lambda - a_4) = 0 \quad \ldots\ldots\ldots \quad (4.2.6)$$

Case A: If $b_1 < 0$ $\left(\text{i.e. } a_1 < \frac{a_2 a_{12}}{a_{22}} + \frac{a_3 a_{13}}{a_{33}}\right)$

The roots $b_1, -a_2, -a_3$ are negative and a_4 is positive. Hence the steady state is **unstable** and the solutions of the equations (4.2.1), (4.2.2), (4.2.3), (4.2.4) are

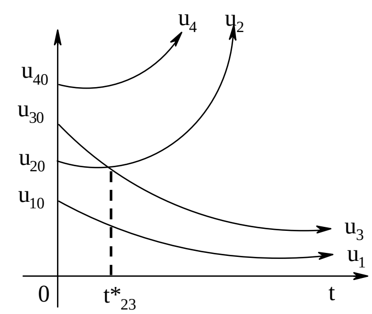
$$u_1 = (u_{10} + A_1)e^{b_1 t} - A_1 e^{-a_3 t} \quad \text{.........} \quad (4.2.7)$$
$$u_2 = (u_{20} - A_2 - A_3)e^{-a_2 t} + A_2 e^{b_1 t} + A_3 e^{a_4 t} \quad \text{.........} \quad (4.2.8)$$

$$u_3 = u_{30} e^{-a_3 t} \quad \text{.........} \quad (4.2.9), \qquad u_4 = u_{40} e^{a_4 t} \quad \text{.........} \quad (4.2.10)$$

Here $A_1 = \frac{a_{13} u_{30}}{a_3 b_1}$, $A_2 = \frac{a_2 a_{21}(u_{10} + A_1)}{a_{22}(b_1 + a_2)}$, $A_3 = \frac{a_2 a_{24} u_{40}}{a_{22}(a_2 + a_4)}$ .... (4.2.11)

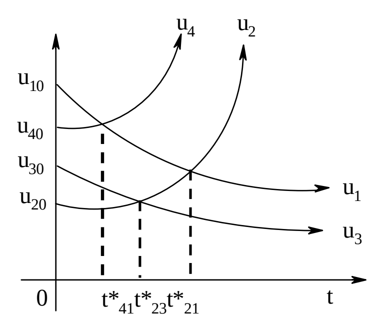
Case (i): If $u_{10} < u_{20} < u_{30} < u_{40}$ and $b_1 < a_2 < a_3 < a_4$

In this case the Prey (S_1) has the least natural birth rate. Initially the Host (S_3) of S_1 dominates over the Predator (S_2) till the time instant t^*_{23} and thereafter the dominance is reversed.



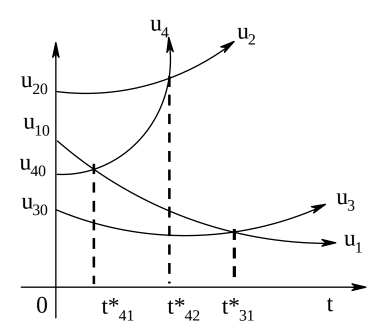
Case (ii): If $u_{20} < u_{30} < u_{40} < u_{10}$ and $a_3 < a_2 < a_4 < b_1$

In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominated over by the Predator (S_2) till the time instant t^*_{23} and there after the dominance is reversed. Also the Prey (S_1) dominates over the Host (S_4) of (S_2), Predator (S_2) till the time instant t^*_{41}, t^*_{21} respectively and the dominance gets reversed thereafter.

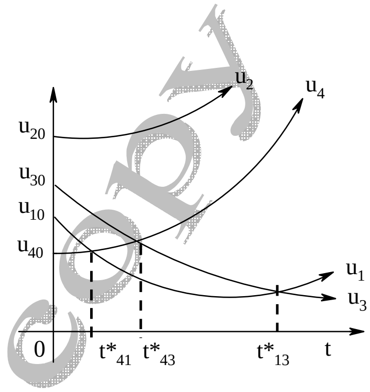


Case (iii): If $u_{30} < u_{40} < u_{10} < u_{20}$ and $a_2 < b_1 < a_3 < a_4$

In this case the Prey (S_1) has the least natural birth rate. Initially it is dominated over by the Host (S_4) of S_2, Host (S_3) of S_1 till the time instant t^*_{41}, t^*_{31} respectively and thereafter the dominance is reversed. Also the Predator (S_2) dominates its Host (S_4) till the time instant t^*_{42} and the dominance gets reversed thereafter.



Case (iv): If $u_{40} < u_{10} < u_{30} < u_{20}$ and $a_4 < a_3 < b_1 < a_2$

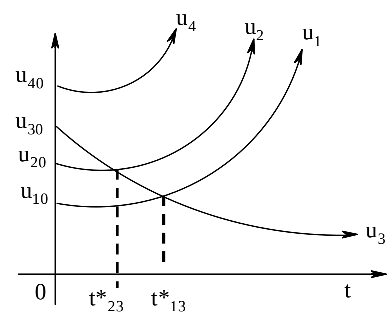


In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominated over by the Host (S_4) of S_2, Prey (S_1) till the time instant t^*_{43}, t^*_{13} respectively and thereafter the dominance is reversed. Also the Prey (S_1) dominates over the Host (S_4) of S_2 till the time instant t^*_{41} and the dominance gets reversed thereafter.

Case B: If $b_1 > 0$ $\left(\text{i.e.,} a_1 > \frac{a_2 a_{12}}{a_{22}} + \frac{a_3 a_{13}}{a_{33}}\right)$

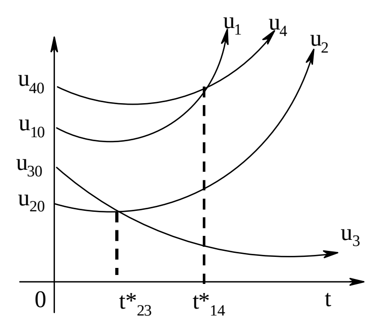
The roots $-a_2, -a_3$ are negative and b_1, a_4 are positive. Hence the steady state is **unstable** and the solutions in this case are same as in case (A).

Case (i): If $u_{10} < u_{20} < u_{30} < u_{40}$ and $b_1 < a_2 < a_3 < a_4$



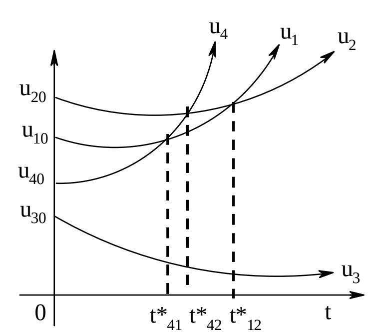
In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominated over by the Predator (S_2), Prey (S_1) till the time instant t^*_{23}, t^*_{13} respectively and thereafter the dominance is reversed.

Case (ii): If $u_{20} < u_{30} < u_{10} < u_{40}$ and $a_2 < a_4 < b_1 < a_3$



In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominated over by the Predator (S_2) till the time instant t^*_{23} and there after the dominance is reversed. Also the Host (S_4) of S_2 dominates over the Prey (S_1) till the time instant t^*_{14} and the dominance gets reversed thereafter.

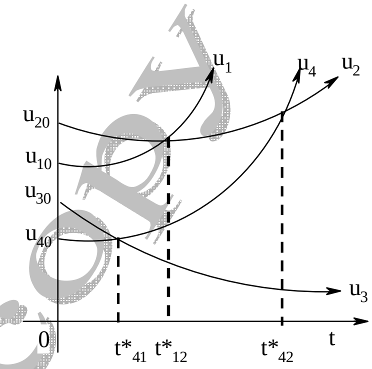
Case (iii): If $u_{30} < u_{40} < u_{10} < u_{20}$ and $a_2 < b_1 < a_3 < a_4$



In this case the Host (S_3) of S_1 has the least natural birth rate. Initially the Predator (S_2) dominates over the Prey (S_1), Host (S_4) of S_2 till the time instant t^*_{42}, t^*_{12} respectively and thereafter the dominance is reversed. Also the Prey (S_1) dominates over the Host (S_4) of S_2 till the time instant t^*_{41} and the dominance gets reversed thereafter.

Case (iv): If $u_{40} < u_{30} < u_{10} < u_{20}$ and $a_2 < a_3 < a_4 < b_1$

In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominated over the Host (S_4) of S_2 till the time instant t^*_{43} and thereafter the dominance is reversed. Also the Predator (S_2) dominates over the Prey (S_1), Host (S_4) of S_2 till the time instant t^*_{12}, t^*_{42} respectively and the dominance gets reversed thereafter.



Trajectories of perturbations:

The trajectories in the $u_3 - u_4$, $u_1 - u_3$, $u_1 - u_4$, $u_2 - u_3$, $u_2 - u_4$ planes are

$$\left(\frac{u_3}{u_{30}}\right)^{-a_4} = \left(\frac{u_4}{u_{40}}\right)^{a_3} \qquad \text{……..} \qquad (4.3.12)$$

$$x = P_1 x_1^{\frac{-b1}{a3}} - \bar{A}_1 x_1 \quad , \quad x = P_1 x_2^{\frac{r_1}{a_4}} - \bar{A}_1 x_2^{\frac{-a_3}{a_4}} \qquad \text{……..} \qquad (4.2.13)$$

$$y = P^1 x_1^{\frac{a_2}{a_3}} + \bar{A}_2 x_1^{\frac{-b_1}{a_3}} + \bar{A}_3 x_1^{\frac{-a_4}{a_3}} \quad , \quad y = P^1 x_2^{\frac{-a_2}{a_4}} + \bar{A}_2 x_2^{\frac{b_1}{a_4}} + \bar{A}_3 x_2 \qquad \text{……..} \quad (4.2.14)$$

where $P_1 = \dfrac{u_{10} + A_1}{u_{10}}$, $P^1 = \dfrac{u_{20} - A_2 - A_3}{u_{20}}$ …….. (4.2.15)

$$\bar{A}_1 = \frac{A_1}{u_{10}},\ \bar{A}_2 = \frac{A_2}{u_{20}}, \bar{A}_3 = \frac{A_3}{u_{20}} \qquad \text{……..} \qquad (4.2.16)$$

4.3 Equilibrium point $\bar{N}_1 = 0$, $\bar{N}_2 = \dfrac{a_2 a_{44} + a_4 a_{24}}{a_{22} a_{44}}$, $\bar{N}_3 = \dfrac{a_3}{a_{33}}$, $\bar{N}_4 = \dfrac{a_4}{a_{44}}$:

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher power of u_1, u_2, u_3, u_4

We get

$$\frac{du_1}{dt} = \mu_1 u_1 \qquad \text{……..} \qquad (4.3.1)$$

$$\frac{du_2}{dt} = a_{21}\mu_1 u_1 - \mu_2 u_2 + a_{24}\mu_1 u_4 \qquad \text{……..} \qquad (4.3.2)$$

$$\frac{du_3}{dt} = -a_3 u_3 \quad \text{……..} \ (4.3.3)\ , \qquad \frac{du_4}{dt} = -a_4 u_4 \qquad \text{……..} \qquad (4.3.4)$$

Here $\mu_1 = a_1 - a_{12}\mu + k_3 a_{13}$, $\mu = \dfrac{a_2 a_{44} + a_4 a_{24}}{a_{22} a_{44}}$, $\mu_2 = a_2 + k_4 a_{24} > 0$ …….. (4.3.5)

The characteristic equation of which is

$$(\lambda - \mu_1)(\lambda + \mu_2)(\lambda + a_3)(\lambda + a_4) = 0 \qquad \text{……..} \qquad (4.3.6)$$

Case A: If $\mu_1 < 0$ $\left(\text{i.e. } a_1 < a_{12}\mu + k_3 a_{13}\right)$

The roots $\mu_1, -\mu_2, -a_3, -a_4$ are negative. Hence the steady state is **stable** and the solutions of the equations (4.3.1), (4.3.2), (4.3.3), (4.3.4) are

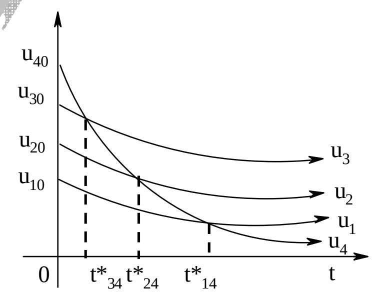
$$u_1 = u_{10} e^{\mu_1 t} \quad \ldots\ldots\ldots \quad (4.3.7)$$

$$u_2 = \left(u_{20} + M - N\right) e^{-\mu_2 t} + N e^{\mu_1 t} - M e^{-a_4 t} \quad \ldots\ldots\ldots \quad (4.3.8)$$

$$u_3 = u_{30} e^{-a_3 t} \quad \ldots\ldots\ldots \ (4.3.9), \qquad u_4 = u_{40} e^{-a_4 t} \quad \ldots\ldots\ldots \ (4.3.10)$$

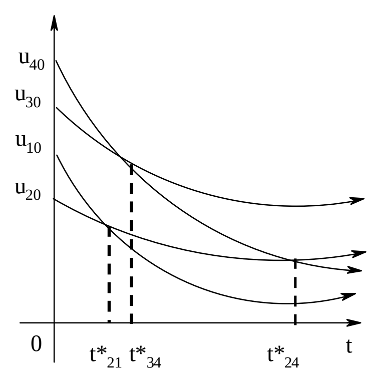
Here $M = \dfrac{a_{24}\mu_1 u_{40}}{a_4 - \mu_2}$, $N = \dfrac{a_{21}\mu_1 u_{10}}{\mu_1 + \mu_2}$, ……... (4.3.11)

Case (i): If $u_{10} < u_{20} < u_{30} < u_{40}$ and $a_4 < \mu_1 < \mu_2 < a_3$



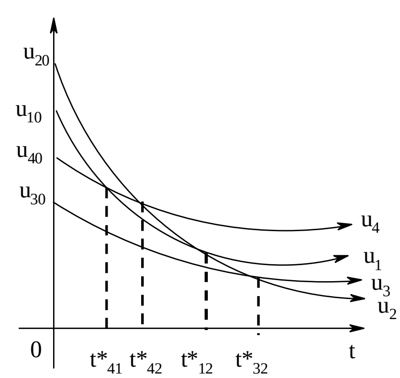
In this case the Host (S_4) of S_2 has the least natural birth rate. Initially it is dominated over by the Prey (S_1), Predator (S_2), Host (S_3) of S_1 till the time instant t^*_{14}, t^*_{24}, t^*_{34} respectively and thereafter the dominance is reversed. And u_1, u_2, u_3, u_4 are converging asymptotically to the equilibrium point. Hence the equilibrium state is **stable.**

Case (ii): If $u_{20} < u_{10} < u_{30} < u_{40}$ and $\mu_1 < a_4 < \mu_2 < a_3$



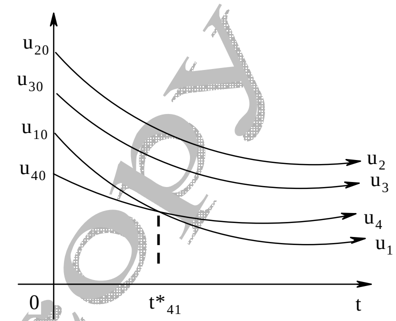
In this case the Prey (S_1) has the least natural birth rate. Initially it is dominated over by the Predator (S_2) till the time instant t^*_{21} and thereafter the dominance is reversed. Also the Host (S_4) of S_2 dominates over the Host (S_3) of S_1, Predator (S_2) till the time instant t^*_{34}, t^*_{24} respectively and the dominance gets reversed thereafter.

Case (iii): If $u_{30} < u_{40} < u_{10} < u_{20}$ and $\mu_2 < a_3 < \mu_1 < a_4$



In this case the Predator (S_2) has the least natural birth rate. Initially it is dominated over by the Host (S_4) of S_2, Prey (S_1), Host (S_3) of S_1 till the time instant t^*_{42}, t^*_{12}, t^*_{32} respectively and thereafter the dominance is reversed. Also the Prey (S_1) dominates over the Host (S_4) of S_2 till the time instant t^*_{41} and the dominance gets reversed thereafter.

Case (iv): If $u_{40} < u_{10} < u_{30} < u_{20}$ and $\mu_1 < a_4 < a_3 < \mu_2$

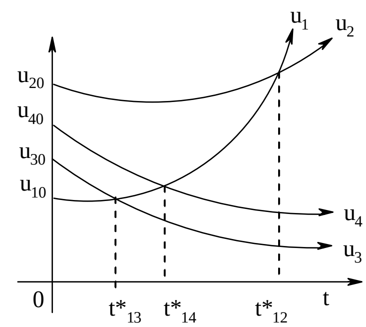


In this case the Prey (S_1) has the least natural birth rate. Initially it is dominated over by the Host (S_4) of S_2 till the time instant t^*_{41} and thereafter the dominance is reversed.

Case B: If $\mu_1 > 0$ $\left(\text{i.e. } a_1 > a_{12}\mu + K_3 a_{13}\right)$

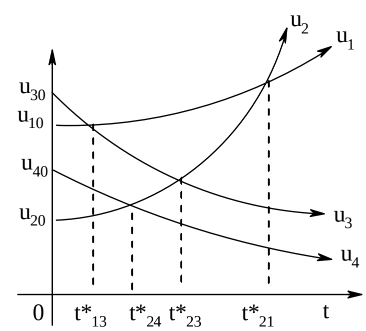
The roots $-\mu_2, -a_3, -a_4$ are negative and μ_1 is positive. Hence the steady state is **unstable** and the solutions in this case are same as in case (A).

Case (i): If $u_{10} < u_{30} < u_{40} < u_{20}$ and $\mu_2 < a_3 < \mu_1 < a_4$



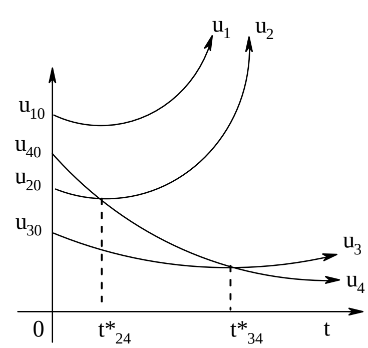
In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominated over by the Prey (S_1) till the time instant t^*_{13} and thereafter the dominance is reversed. Also the Predator (S_2) and its Host (S_4) dominates over the Prey (S_1) till the time instant t^*_{12}, t^*_{14} respectively and the dominance gets reversed thereafter.

Case (ii): If $u_{20} < u_{40} < u_{10} < u_{30}$ and $\mu_1 < a_4 < a_3 < \mu_2$



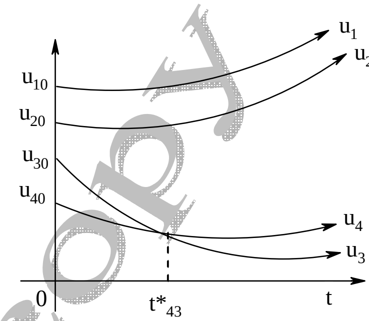
In this case the Host (S_4) of S_2 has the least natural birth rate. Initially the Host (S_3) of S_1, Prey (S_1), Host (S_4) of S_2 dominates over the Predator (S_2) till the time instant $t^*_{23}, t^*_{21}, t^*_{24}$ respectively and thereafter the dominance is reversed. Also the Host (S_3) of S_1 dominates over the Prey (S_1) till the time instant t^*_{13} and the dominance gets reversed thereafter.

Case (iii): If $u_{30} < u_{20} < u_{40} < u_{10}$ and $a_4 < \mu_2 < \mu_1 < a_3$



In this case the Host (S_4) of S_2 has the least natural birth rate. Initially it is dominated over by the Host (S_3) of S_1, Predator (S_2) till the time instant t^*_{34}, t^*_{24} respectively and thereafter the dominance is reversed.

Case (iv): If $u_{40} < u_{30} < u_{20} < u_{10}$ and $\mu_2 < a_3 < \mu_1 < a_4$



In this case the Host (S_3) of S_1 has the least natural birth rate. Initially it is dominated over by the Host (S_4) of S_2 till the time instant t^*_{43} and thereafter the dominance is reversed.

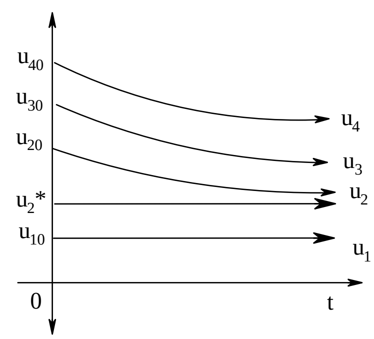
Case C: If $\mu_1 = 0 \quad (\text{i.e. } a_1 = a_{12}\mu + K_3 a_{13})$

The roots $-\mu_2, -a_3, -a_4$ are negative and $\mu_1 = 0$. Hence the steady state is **neutrally stable** and the solutions in this case are :

$$u_1 = u_{10} \text{ ……... (4.4.16) ;} \qquad u_2 = (u_{20} + M - N)e^{-\mu_2 t} + N - Me^{-a_4 t} \text{ ……... (4.4.17)}$$

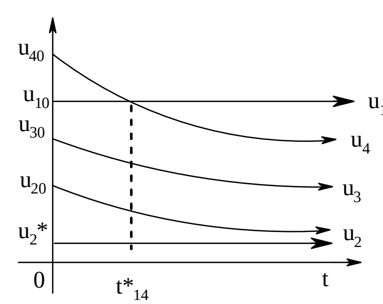
$$u_3 = u_{30}e^{-a_3 t} \text{ ……... (4.4.18) ; } u_4 = u_{40}e^{-a_4 t} \text{ ……...(4.4.19)}$$

Case (i): If $u_{10} < u_{20} < u_{30} < u_{40}$ and $\mu_1 < \mu_2 < a_3 < a_4$



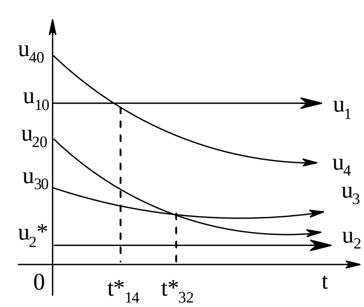
In this case the Prey (S_1) has the least natural birth rate and the Host (S_4) of S_2 dominates the Host (S_3) of S_1, Predator (S_2), Prey (S_1) in natural growth rate as well as in its population strength. And in course of time u_2 is asymptotic to u_2^* which is evident from the equation (4.4.17). Hence the equilibrium state is **neutrally stable.**

Case (ii): If $u_{20} < u_{30} < u_{10} < u_{40}$ and $\mu_2 < a_3 < a_4 < \mu_1$



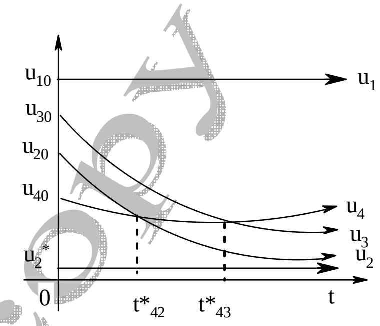
In this case the Predator (S_2) has the least natural birth rate. Initially the Host (S_4) of S_2, dominates over the Prey (S_1) till the time instant t^*_{14} and thereafter the dominance is reversed. Further we notice that u_2 is asymptotic to u_2^* which is evident from the equation (4.4.17). Hence the equilibrium state is **neutrally stable.**

Case (iii): If $u_{30} < u_{20} < u_{10} < u_{40}$ and $\mu_2 < a_3 < a_4 < \mu_1$



In this case the Predator (S_2) has the least natural birth rate. Initially it is dominated over by the Host (S_3) of (S_1) till the time instant t^*_{32} and thereafter the dominance is reversed. Also the Host (S_4) of S_2 dominates over the Prey (S_1) till the time instant t^*_{14} and the dominance gets reversed thereafter. Further we notice that u_2 is asymptotic to u_2^* which is evident from the equation (4.4.17). Hence the equilibrium state is **neutrally stable.**

Case (iv): If $u_{40} < u_{20} < u_{30} < u_{10}$ and $\mu_2 < a_3 < \mu_1 < a_4$



In this case the Predator (S_2) has the least natural birth rate. Initially the Host (S_3) of S_1, Predator (S_2) dominates over the Host (S_4) of S_2 till the time instant t^*_{43}, t^*_{42} respectively and thereafter the dominance is reversed. Further we notice that u_2 is asymptotic to u_2^* which is evident from the equation (4.4.17). Hence the equilibrium state is **neutrally stable.**

Trajectories of perturbations:

The trajectories in the $u_3 - u_4$, $u_1 - u_3$, $u_1 - u_4$, $u_1 - u_2$, $u_2 - u_3$, $u_2 - u_4$ planes are

$$\left(\frac{u_3}{u_{30}}\right)^{a_4} = \left(\frac{u_4}{u_{40}}\right)^{a_3}, \left(\frac{u_1}{u_{10}}\right)^{-a_3} = \left(\frac{u_3}{u_{30}}\right)^{\mu_1}, \left(\frac{u_1}{u_{10}}\right)^{-a_4} = \left(\frac{u_4}{u_{40}}\right)^{\mu_1} \quad \ldots\ldots\ldots (4.3.12)$$

$$y = \bar{M}\, x^{\frac{-\mu_2}{\mu_1}} + N_1 x - M_1 x^{\frac{-a_4}{\mu_1}} \;;\; y = \bar{M}\, x_1^{\frac{\mu_2}{a_3}} + N_1 x_1^{\frac{-\mu_1}{a_3}} - M_1 x_1^{\frac{a_4}{a_3}} \quad \ldots\ldots\ldots (4.3.13)$$

$$y = \bar{M}\, x_2^{\frac{\mu_2}{a_4}} + N_1 x_2^{\frac{-\mu_1}{a_4}} - M_1 x_2 \quad \text{respectively} \quad \ldots\ldots\ldots (4.3.14)$$

Here $$\bar{M} = \frac{u_{20} + M - N}{u_{20}},\; N_1 = \frac{N}{u_{20}}, M_1 = \frac{M}{u_{20}} \quad \ldots\ldots\ldots (4.3.15)$$

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