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STABILITY ANALYSIS OF A CLASS OF BOUNDARY VALUE PROBLEMS

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Abstract

In this paper, we have discussed the regions of absolute stability of fourth order boundary value problems. The method is applied to solve a fourth-order boundary value problem. Numerical results are given to illustrate the efficiency of our method and compared with exact solution.

AMS Subject Classification: 65L05, 65L06

Keywords: Numerical differentiation; Initial Value Problem; Boundary Value Problem; Absolute Stability, Multistep methods.

1. Introduction

In recent past much progress has been made in the theory and computation of the boundary value problems. Numerical solution and stability analysis of boundary value problems is a field of increasing interest in the applied mathematics. In contrast to single step methods which only uses one previous value of the numerical solution to determine the subsequent value, multistep methods utilize more than one previous value to approximate the next value. Accordingly, multistep methods achieve greater accuracy than one-step methods. Finite difference methods for boundary value problems are discussed in [1]. Multi step methods of second order differential equations are discussed in [2]. The methods based on numerical differentiation for first-order differential equations have been shown to be stiffly stable by Gear [3]. A detailed study of the single step and multistep methods has been carried out by Gear [3], Gragg and Statter [4] and Henrici [5]. Gear [3] and Peter Henrici [5] have derived special multistep methods based on numerical integration and numerical differentiation for solving first-order differential equations. Jain [6] has considered high order stiffly stable methods. Further information can be had from [7] and [8]. Special multistep methods based on numerical differentiation for solving the initial value problem have been derived in Rama Chandra Rao [9]. The motivation for the work carried out in this paper arises from the methods based on numerical differentiation for the first-order differential equations, Special multistep methods based on numerical integration for the solution of the special second-order differential equations by Henrici [5] and Special multistep methods based on numerical differentiation for solving the initial value problem by Rama Chandra Rao [9]. In Henrici [5] methods based on Numerical Integration have been derived by integrating y'' = f (x, y) twice and replacing the function f(x, y) by an interpolating polynomial. Special multistep methods have been derived by replacing y(x) on the left hand side of $y^{iv}(x) = f(x,y)$ by an interpolating polynomial and differentiating it four times. We have investigated a class of implicit methods. It is found that the implicit methods have order (k - 3). Some local truncation errors are provided. The regions of absolute stability of the methods are derived. Numerical tests of the performance of the method has been established by solving differential equation and compared with the exact solution. The numerical results reported show the validity of our method.

2. General linear multistep methods for special fourth-order differential equations

The special fourth order differential equation

$$y^{iv} = f(x,y),\ \ y(0) = y_{0,}\ y'(0) = y_0',\ y''(0) = y_0'',\ y'''(0) = y_0''' \tag{1}$$

occurs frequently in many number of problems of science and engineering.

A general linear multistep method of step number k for the numerical solution of equation (1) is given by $y_{n+1} = \sum_{j=1}^{k} a_j y_{n+1-j} + h^4 \sum_{j=0}^{k} b_j y_{n+1-j}$ (2)

where a_j, b_j are constants and 'h' is the step length.

Introducing the polynomials

$$\rho(\xi) = \xi^k - \sum_{j=1}^{k} a_j \xi^{k-1} \text{ and} \sigma(\xi) = \sum_{j=1}^{k} b_j \xi^{k-1} \tag{3}$$

Equation (2) can be written as

$$\rho(E)\, y_{n-k+1} - h^4\, \sigma(E) y_{n-k+1}^{iv} = 0 \tag{4}$$

In equation (4), 'E' is the shift operator defined by $E(y_n) = y_{n+1}$

Applying (4) to $y^{iv} = \lambda y$, we get the characteristic equation

$$\rho(\xi) - \bar{h}\,\sigma(\xi) = 0, \qquad \text{where } \bar{h} = \lambda h^4 \tag{5}$$

The roots ξ_i of the characteristic equation (5) and $\bar{h}$ are in general, complex and the region of absolute stability is defined to be the region of the complex $\bar{h}$ - plane such that the roots of the characteristic equation (5) lie within the unit circle whenever $\bar{h}$ lies in the interior of the region. Denoting the region of absolute stability of R and its boundary by ∂R, the locus of ∂R is given by

$$\bar{h}(\theta) = \rho(e^{i\theta})/\sigma(e^{i\theta}),\ 0 \leq \theta \leq 2\pi \qquad (6)$$

3. Derivation of the methods

Let p(x) be the backward difference interpolating polynomial of y(x) at (k+1) abscissas x_{n+1}, x_n, - - - - - - - - - $x_{n-k+1.}$ Then p(x) is given by

$$p(x) = \sum_{m=0}^{k} (-1)^m \binom{-s}{m} \nabla^m y_{n+1}, \qquad s = \frac{(x - x_{n+1})}{h} \qquad (7)$$

Differentiating (7) four times with respect to x, we get

$$p^{iv}(x) = \left(\frac{1}{h^4}\right) \sum_{m=0}^{k} \frac{d^4}{ds^4} \left[(-1^m)\binom{-s}{m}\right] \nabla^m y_{n+1}.$$

Replacing $y^{iv}(x)$ by $p^{iv}(x)$ in equation (1) and putting x = $x_{n+1-r,}$ i.e. s = -r,

we get , $$\sum_{m=0}^{k} \delta_{r,\ m}\ \nabla^m y_{n+1} = h^4 f_{n+1-r} \qquad (8)$$

where $$\delta_{r,m} = \frac{d^4}{ds^4}[(-1)^m \binom{-s}{m}] \qquad (9)$$

Taking r = 1/2 in (8), a class of method can be attained which are given by

$$\sum_{m=0}^{k} \delta_{\frac{1}{2},m} \nabla^m y_{n+1} = h^4 f_{n+\frac{1}{2}} \qquad (10)$$

The coefficients $\delta_{0,m}$ are shown in table 1.

Differences in (10) are expressed in terms of function values. After simplification, the equation (10) will turn out into the form $\sum_{j=0}^{k} a_j y_{n+1-j} = h^4 f_{n+\frac{1}{2}}$ (11)

The coefficients a_j are shown in table 2.

The local truncation error of the formula (11) is given by

$$LTE = \delta_{0,k+1} h^{k+1} y^{k+1}(\eta) \quad (12)$$

Table 1

Coefficients of $\delta_{\frac{1}{2},\ m}$; $m = 0\ (1)\ 9$

m	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
$\boldsymbol{\delta_{\frac{1}{2},\ m}}$	0	0	0	0	1	$\frac{3}{2}$	$\frac{41}{24}$	$\frac{85}{48}$	$\frac{3861}{1920}$	$\frac{6587}{3840}$

Table 2

Coefficients of a_j ; $j = 0\,(1)\,k,\ k = 4\,(1)\,9$

K	**J** 0	1	2	3	4	5	6	7	8	9
4	1	-4	6	-4	1					
5	$\frac{5}{2}$	$\frac{-23}{2}$	$\frac{42}{2}$	$\frac{-38}{2}$	$\frac{17}{2}$	$\frac{-3}{2}$				
6	$\frac{101}{24}$	$\frac{-522}{24}$	$\frac{1119}{24}$	$\frac{-1276}{24}$	$\frac{819}{24}$	$\frac{-282}{24}$	$\frac{41}{24}$			
7	$\frac{287}{48}$	$\frac{-1639}{48}$	$\frac{4023}{48}$	$\frac{-5527}{48}$	$\frac{4613}{48}$	$\frac{-2349}{48}$	$\frac{677}{48}$	$\frac{-85}{48}$		
8	$\frac{15341}{1920}$	$\frac{-96448}{1920}$	$\frac{269028}{1920}$	$\frac{-437296}{1920}$	$\frac{454790}{1920}$	$\frac{-310176}{1920}$	$\frac{135188}{1920}$	$\frac{-34288}{1920}$	$\frac{3861}{1920}$	
9	$\frac{37269}{3840}$	$\frac{-252179}{3840}$	$\frac{775188}{3840}$	$\frac{-1427900}{3840}$	$\frac{1739542}{3840}$	$\frac{-1450314}{3840}$	$\frac{823684}{3840}$	$\frac{-305708}{3840}$	$\frac{67005}{3840}$	$\frac{-6587}{3840}$

It follows that the k-step method (14) has the order k-3.

For the method (13), we have

$$\rho(\xi) = \sum_{j=0}^{k} a_j \xi^{k-j} \;\; and \; \sigma(\xi) = \;\; \xi^k \tag{13}$$

The regions of absolute stability of the method for k = 4, 5, 6, 7, 8 and 9 are shown in figures 1 to 6 (Taking real part on x-axis and imaginary part on y-axis). The region of absolute stability is the region lying outside the boundary.

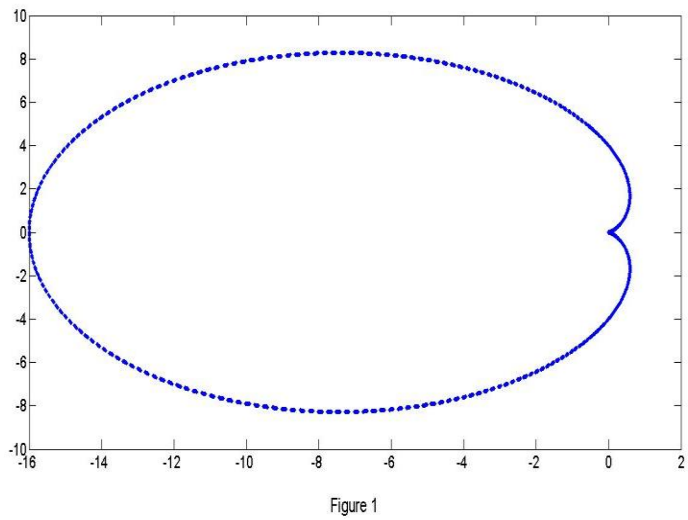

Figure 1: The region of absolute stability of the method (13) for k = 4

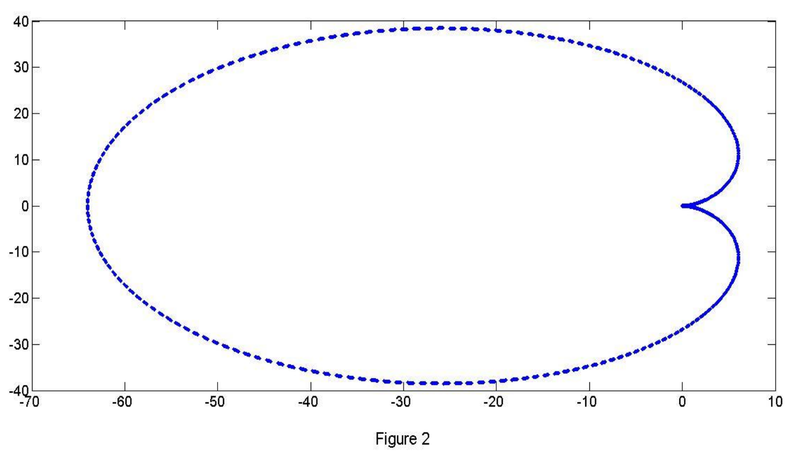

Figure 2: The region of absolute stability of the method (13) for k = 5

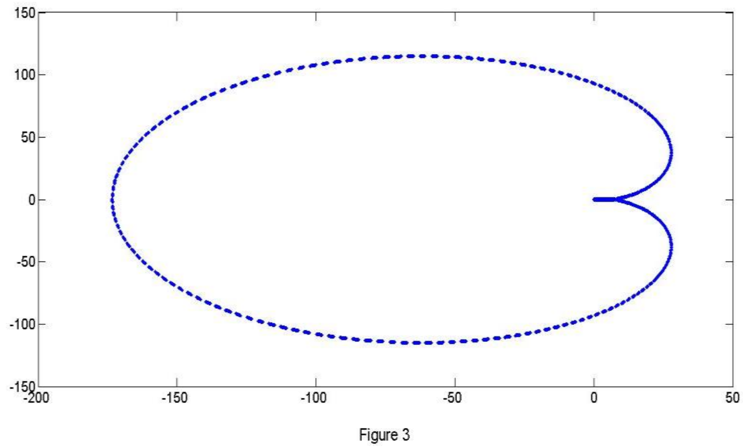

Figure 3: The region of absolute stability of the method (13) for k = 6

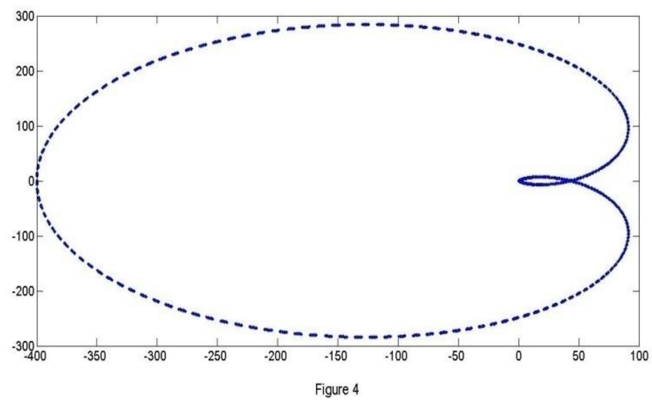

Figure 4: The region of absolute stability of the method (13) for k = 7

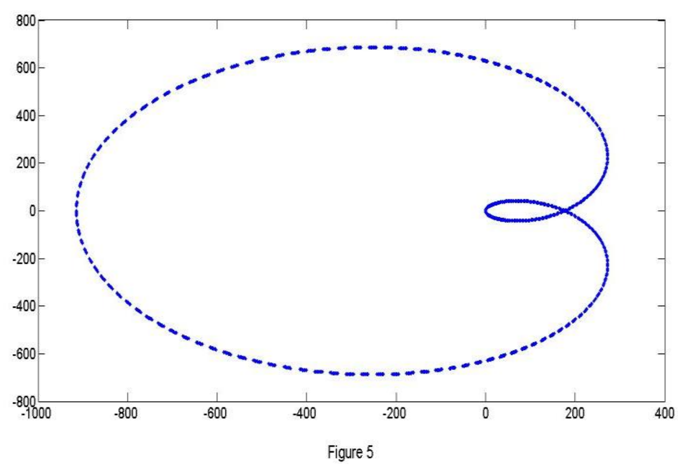

Figure 5: The region of absolute stability of the method (13) for k = 8

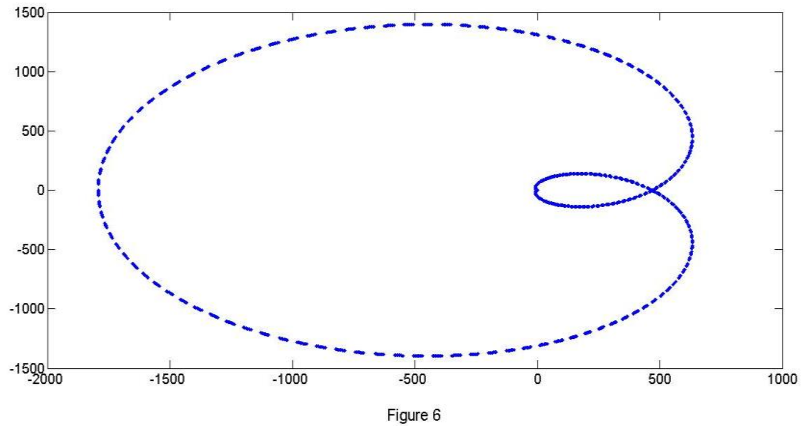

Figure 6: The region of absolute stability of the method (13) for k = 9

4. Numerical example

In this section, we have applied ND methods to solve the differential equation

$$y^{iv} = y + \sin 2x\,, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = -1, \quad y'''(0) = 0 \qquad (14)$$

in the interval [0, 2] with h = 0.01 and h = 0.02 and the results are shown in tables 3 and 4

The fifth order numerical differentiation method derived in this paper for k = 6 is

$$y_{n+1} = \frac{522}{101}y_n - \frac{1119}{101}y_{n-1} + \frac{1276}{101}y_{n-2} - \frac{819}{101}y_{n-3} + \frac{282}{101}y_{n-4} - \frac{41}{101}y_{n-5} + \frac{24}{101}h^4 f_{n+\frac{1}{2}} \qquad (15)$$

The sixth order numerical differentiation method derived in this paper for k = 7 is

$$y_{n+1} = \frac{1639}{287}y_n - \frac{4023}{287}y_{n-1} + \frac{5527}{287}y_{n-2} - \frac{4613}{287}y_{n-3} + \frac{2349}{287}y_{n-4} - \frac{677}{287}y_{n-5} + \frac{85}{287} + \frac{48}{287}h^4 f_{n+\frac{1}{2}} \qquad (16)$$

Table 3
Solution by fifth order ND for k = 6 with h = 0.01

X	Exact Solution	Numerical Solution by fifth order ND	Absolute Error
0	0.0000000000E+00	-3.3008700374E-14	3.3008700374E-14
0.1	9.5000248452E-02	9.5000248452E-02	3.2335245592E-14
0.2	1.8000789084E-01	1.8000789084E-01	3.2029934260E-14
0.3	2.5505939225E-01	2.5505939225E-01	2.7755575616E-14
0.4	3.2024773559E-01	3.2024773559E-01	2.2926105459E-14
0.5	3.7574733428E-01	3.7574733428E-01	1.7708057243E-14
0.6	4.2183572380E-01	4.2183572380E-01	1.0380585280E-14
0.7	4.5891144532E-01	4.5891144532E-01	4.6074255522E-15
0.8	4.8750765774E-01	4.8750765774E-01	4.3853809473E-15
0.9	5.0830115974E-01	5.0830115974E-01	1.1768364061E-14
1	5.2211666367E-01	5.2211666367E-01	1.8429702209E-14
1.1	5.2992633315E-01	5.2992633315E-01	2.2759572005E-14
1.2	5.3284476873E-01	5.3284476873E-01	2.8532731733E-14
1.3	5.3211979428E-01	5.3211979428E-01	3.4861002973E-14
1.4	5.2911955478E-01	5.2911955478E-01	3.7636560535E-14
1.5	5.2531657675E-01	5.2531657675E-01	3.8191672047E-14
1.6	5.2226956144E-01	5.2226956144E-01	4.1189274214E-14
1.7	5.2160377253E-01	5.2160377253E-01	4.2410519541E-14
1.8	5.2499094176E-01	5.2499094176E-01	4.0523140399E-14
1.9	5.3412964448E-01	5.3412964448E-01	3.7414515930E-14
2	5.5072709313E-01	5.5072709313E-01	3.2196467714E-14

Table 4
Solution by fifth order ND for k = 6 with h = 0.02

X	Exact Solution	Numerical Solution by fifth order ND	Absolute Error
0	0.0000000000E+00	1.0667583069E-09	1.0667583069E-09
0.1	9.5000248452E-02	9.5000239003E-02	9.4491035968E-09
0.2	1.8000789084E-01	1.8000787149E-01	1.9354154918E-08
0.3	2.5505939225E-01	2.5505936387E-01	2.8376869077E-08
0.4	3.2024773559E-01	3.2024769932E-01	3.6268672343E-08
0.5	3.7574733428E-01	3.7574729146E-01	4.2814860823E-08
0.6	4.2183572380E-01	4.2183567596E-01	4.7844171902E-08
0.7	4.5891144532E-01	4.5891139408E-01	5.1236667353E-08
0.8	4.8750765774E-01	4.8750760481E-01	5.2929533867E-08
0.9	5.0830115974E-01	5.0830110682E-01	5.2920682281E-08
1	5.2211666367E-01	5.2211661240E-01	5.1269945178E-08
1.1	5.2992633315E-01	5.2992628506E-01	4.8097817951E-08
1.2	5.3284476873E-01	5.3284472515E-01	4.3581887432E-08
1.3	5.3211979428E-01	5.3211975633E-01	3.7950970588E-08
1.4	5.2911955478E-01	5.2911952330E-01	3.1477311202E-08
1.5	5.2531657675E-01	5.2531655228E-01	2.4467117221E-08
1.6	5.2226956144E-01	5.2226954419E-01	1.7249739859E-08
1.7	5.2160377253E-01	5.2160376237E-01	1.0166110065E-08
1.8	5.2499094176E-01	5.2499093820E-01	3.5566819490E-09
1.9	5.3412964448E-01	5.3412964673E-01	2.2504385022E-09
2	5.5072709313E-01	5.5072710008E-01	6.9508315770E-09

5. Discussion and Conclusion

The methods based on numerical differentiation are found to be absolutely stable outside some closed boundaries. It has been noticed from the tables that the solution obtained by numerical differentiation method derived in this paper is more accurate and is very near to the exact solution.

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