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A Numerical technique for Solving Singularly Perturbed Two Point Boundary Value Problems

N. Raji Reddy[1], J. Sucharitha[2]
[1]Dept of Mathematics Jyothishmathi Institute of Technology & Science, Karimnagar-505481, Andhra Pradesh, India narahari.rajireddy@gmail.com
[2]Dept of Mathematics, Osmania University, Hyderabad- 500007, Andhra Pradesh, India

Abstract

This paper discusses the solving of singularly perturbed two-point boundary value problems. It consists of replacing the original differential equation by a pair of initial value problems. Then first initial value problem (IVP) is solved in general way and then using the solution of this, solve second initial value problem with introducing exponential fitted finite difference scheme. Several linear and non-linear singular perturbation problems have been solved and the numerical results are presented to support the theory. It is observed that the present method approximates the exact solution very well.

Keywords:
Ordinary differential equations; Singular perturbations; Two Point Boundary value problems; Initial value methods; Boundary layer, exponentially fitted finite difference scheme.

1. Introduction:

The singular perturbation problems arise frequently in many areas of science and engineering such as heat transfer problem with large Peclet numbers, Navier–Stokes flows with large Reynolds numbers, chemical reactor theory, aerodynamics, reaction–diffusion process, quantum mechanics, optimal control, etc. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts and varies slowly in some other parts. The use of classical numerical methods on uniform mesh for solving singularly perturbed problems may give rise to difficulties when the singular perturbation parameter is sufficiently small. A detailed discussion on the analytical and numerical treatment of SPBVPs is given in the books of O'Malley [1], Doolan et al. [2], Roos et al. [3], and Miller et el. [4].

The numerical treatment of singularly perturbed problems presents some major computational difficulties, and in recent years a large number of special purpose methods have been proposed to provide accurate numerical solutions. In the last three decades, many numerical methods have appeared in the literature. Most notable among these are order reduction method and exponentially fitted method. Reddy and Chakravarthy [5] presented a method for finding the solution of the given singularly perturbed boundary value problem by solving a pair of initial value problems, which are deduced from the original problem, using classical fourth order Runge–Kutta method. Awoke [6] presented a numerical method for singular perturbation problems arising in chemical reactor theory, wherein the original second order differential equation is replaced by an approximate first order differential equation with a small deviating argument. By using trapezoidal formula integration in the forward direction with left-layer boundary problems and in backward direction with right layer boundary problems, and both formulas for interior or two boundary layers, a three term recurrence relation is obtained and then solved by Thomas Algorithm.

In this paper, a numerical method is presented to solve general singularly perturbed two-point boundary value problem. In this method, the original second order differential equation is replaced by a pair of initial value problems. The first initial value problem is solved in general way and then the second initial value problem is solved by using an exponential fitted finite difference scheme. To demonstrate the applicability of the method, we have solved six linear (four left and two right end boundary layer) and three nonlinear problems. From the results, it is observed that the present method approximates the exact or the asymptotic expansion solution very well.

2. Description of the Method:

To describe the method, first we consider a linear singularly perturbed two-point boundary value problem of the form

$$Ly(x) \equiv \varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \; p \le x \le q \qquad (2.1)$$

with the boundary conditions

$$y(p) = \alpha, \; y(q) = \beta \qquad (2.2)$$

where ε is a small positive parameter such that $0 < \varepsilon << 1$, α and β are given non-negative constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in $[p, q]$. Moreover, we assume that $a(x) \ge M > 0$ thoughout the interval $[p, q]$, where M is a positive constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = p$. Under these assumptions equation

(2.1) - (2.2) has a unique solution.

Following [5], we obtain the first order differential equation by substituting ε = 0 in Eq. (3.1). Let $y_0(x)$ be the solution of the reduced equation, then

$$a(x)y_0' + b(x)y_0(x) = f(x) \qquad (2.3)$$

with

$$y_0(q) = \beta. \qquad (2.4)$$

Let

$$\varepsilon y'(x) + a(x)y(x) = z(x) \qquad (2.5)$$

Substituting (2.5) in (2.1), we get

$$z'(x) + [b(x) - a'(x)]y(x) = f(x) \qquad (2.6)$$

The initial condition for (2.6) is obtained by using $y_0(x)$, the solution of the reduced problem, in Eq. (2.5) at $x = q$. Hence, the initial condition for (2.6) is given by

$$z(q) = \varepsilon y'(q) + a(q)y(q) \qquad (2.7)$$

The initial condition for (2.5) is $y(p) = \alpha$. Replacing y(x) by $y_0(x)$, we get the first initial value problem as

$$z'(x) = -[b(x) - a'(x)]y_0(x) + f(x) \qquad (2.8)$$

with

$$z(q) = \varepsilon y_0'(q) + a(q)y_0(q)$$

From this we can find $z(x)$ directly. Now the second initial value problem of first order is

$$\varepsilon u'(x) + a(x)u(x) = z(x), \qquad (2.9)$$

with

$$u(p) = \alpha \qquad (2.10)$$

where $a(x)$ and $z(x)$ are sufficiently smooth functions.

To solve the initial value problem (2.9) subject to the condition (2.10), we use the exponential fitted finite difference method. The finite difference corresponding to (2.9) is

$$\varepsilon D^+(u_i) + a(x_i)u_i = z(x_i), \qquad (2.11)$$

with

$$u_0 = \alpha, \qquad (2.12)$$

where

$$D^+(u_i) = \frac{u_{i+1} - u_i}{h}, \; h = \frac{(q-p)}{n}, \; p = x_0 < x_1 < \cdots < x_{n-1} < x_n = q,$$

Now, we introduce a fitting factor $\sigma_i(\rho)$ in (2.11) which is to be determined in such a way that the solution converges uniformly to the solution of (2.9) and (2.10),

$$\varepsilon\, \sigma_i(\rho)\, D^+(u_i) + a(x_i)u_i = z(x_i), \qquad i = 0(1)n-1 \qquad (2.13)$$

with

$$u_0 = \alpha, \qquad (2.14)$$

Following Doolan et al. [2], the fitting factor is given by

$$\sigma_i(\rho) = \rho a(x_i)\left[1 - \exp\left(-\rho a(x_i)\right)\right]^{-1} \qquad (2.15)$$

Equation (2.13) with the boundary condition (2.14) can be easily solved by forward substitution.

3. Error Estimation:

In this section, the error estimate for the solution of the original problem (2.l)-(2.2) and the scheme (2.13)-(2.14) is described.

Let $y(x)$ is the solution of the BVP (2.1)-(2.2), and if u_i^* is the numerical solution of the problem (2.13)-(2.14). Consider the IVPs

$$\varepsilon y'(x) + a(x)y(x) = \theta(x), \quad \theta(p) = \alpha \tag{3.1}$$

$$\varepsilon u'(x) + a(x)u(x) = \emptyset(x), \quad \emptyset(p) = \alpha \tag{3.2}$$

where θ is the solution of the problem

$$\theta' = -[b(x) - a'(x)]y(x) + f(x), x \in [p, q), \quad \theta(q) = \gamma_1, \tag{3.3}$$

and $\emptyset$ is the solution of the problem

$$\emptyset' = -[b(x) - a'(x)]y_0(x) + f(x), x \in [p, q), \quad \emptyset(q) = \gamma_2 \tag{3.4}$$

Here $y_0(x)$ is the solution of the reduced equation by substituting ε = 0 in Eq. (2.1) and $|\theta(x) - \emptyset(x)| \leq C\epsilon, \ x \in [p, q]$ [2]. It may be noted that the equation (3.2) is obtained from the equation. (3.1) by replacing $z(x)$ by $z^*(x)$, that is $y(x)$ by $y_0(x)$.

Let

$$z = y(x) - u(x).$$

then

$$|z(x)| \leq C|\theta(x) - \emptyset(x)|$$

Since $|\theta(x) - \emptyset(x)| \leq C\epsilon, \ x \in [p, q]$, we get

$$|z(x)| \leq C\epsilon,$$

$$\text{i.e. } |y(x) - u(x)| \leq C\varepsilon, \ x \in [p, q] \tag{3.5}$$

But

$$|y(x_i) - u_i^*| \leq |y(x_i) - u(x_i)| + |u(x_i) - u_i^*| \tag{3.6}$$

If $u(x)$ and u_i^* are the solutions of Initial value problems (2.9)-(2.10) and (2.13)-(2.14), respectively, then from [2], we have

$$|u(x_i) - u_i^*| \leq Ch, \ p \leq ih \leq q. \tag{3.7}$$

where C is independent of $i$and h.

Substituting (3.5) and (3.7) in (3.6), we get

$$|y(x_i) - u_i^*| \leq C(h + \varepsilon), x_i \in [p, q].$$

4. Numerical experiments and results:

In this section, some examples are presented to illustrate the method presented in this paper. The computational results are tabulated in their tables. The solutions are computed for $h = 10^{-3}$ and $\varepsilon = 10^{-4}, 10^{-5} and\ 10^{-6}$. Non-linear singular perturbation problems were converted as a sequence of linear singular perturbation problems by using quasi-linearization method. We considered three nonlinear singular perturbation problems with left-end boundary layer to demonstrate the applicability of the present method. The numerical results are presented in the tables and the solutions are computed for $h = 10^{-3}$ and $\varepsilon = 10^{-3}, 10^{-4}$.

Table 1: Numerical results of example (1).

x-values	$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-6}$	
	y-values	Exact Values	y-values	Exact Values	y-values	Exact Values
0.000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.020	0.3754111	0.3753479	0.3753211	0.3753148	0.3753121	0.3753115
0.040	0.3829929	0.3829296	0.3829029	0.3828966	0.3828939	0.3828932
0.060	0.3907278	0.3906645	0.3906378	0.3906315	0.3906288	0.3906282
0.080	0.3986190	0.3985557	0.3985291	0.3985227	0.3985201	0.3985194
0.100	0.4066696	0.4066062	0.4065797	0.4065733	0.4065707	0.4065700
0.200	0.4494289	0.4493649	0.4493390	0.4493326	0.4493300	0.4493293
0.300	0.4966853	0.4966201	0.4965953	0.4965888	0.4965863	0.4965856
0.400	0.5489116	0.5488445	0.5488216	0.5488150	0.5488126	0.5488120
0.500	0.6066307	0.6065609	0.6065407	0.6065337	0.6065317	0.6065310
0.600	0.6704200	0.6703469	0.6703300	0.6703228	0.6703210	0.6703203
0.700	0.7409182	0.7408404	0.7408283	0.7408205	0.7408193	0.7408184
0.800	0.8188307	0.8187471	0.8187408	0.8187324	0.8187318	0.8187310
0.900	0.9049374	0.9048465	0.9048474	0.9048384	0.9048384	0.9048375
1.000	1.0001000	1.0000000	1.0000100	1.0000000	1.0000010	1.0000000

Example 1. Consider the following homogeneous singular perturbation problem from ([7], Page 480):

$$\varepsilon y''(x) + y'(x) - y(x) = 0, \quad x \in [0,1]$$

with $y(0) = 1$, $y(1) = 1$.

whose exact solution is $y(x) = \frac{[(e^{b}-1)e^{ax}+(1-e^{a})e^{bx}]}{e^{b}-e^{a}}$,

where $a = \frac{-1+\sqrt{(1+4\varepsilon)}}{2\varepsilon}$, and $b = \frac{-1-\sqrt{(1+4\varepsilon)}}{2\varepsilon}$.

The boundary layer exist at left-end i.e., $x = 0$ of the underlying interval
The numerical results are presented in Table 1.

Example 2. Now consider the following non-homogeneous singular perturbation problem from ([8], Example-2):

$$\varepsilon y''(x) + y'(x) = 1 + 2x., \quad x \in [0,1]$$

with $y(0) = 0$, $y(1) = 1$.

The exact solution is given by $y(x) = x(x + 1 - 2\varepsilon) + \frac{\left[(2\varepsilon-1)(1-e^{\left(-\frac{x}{\varepsilon}\right)})\right]}{(1-e^{\left(-\frac{1}{\varepsilon}\right)})}$.

The problem has left-end boundary later of the underlying of the given interval.
The numerical results are presented in Table 2.

Table 2: Numerical results of example (2).

x-values	$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-6}$	
	y-values	Exact Values	y-values	Exact Values	y-values	Exact Values
0.000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.020	-0.9793001	-0.9794040	-0.9795700	-0.9795803	-0.9795970	-0.9795980
0.040	-0.9581001	-0.9582080	-0.9583700	-0.9583808	-0.9583970	-0.9583980
0.060	-0.9361001	-0.9362120	-0.9363700	-0.9363812	-0.9363970	-0.9363981
0.080	-0.9133000	-0.9134160	-0.9135700	-0.9135816	-0.9135970	-0.9135981
0.100	-0.8897001	-0.8898200	-0.8899700	-0.8899820	-0.8899970	-0.8899982
0.200	-0.7597001	-0.7598400	-0.7599700	-0.7599840	-0.7599970	-0.7599984
0.300	-0.6097000	-0.6098600	-0.6099700	-0.6099859	-0.6099970	-0.6099986
0.400	-0.4397001	-0.4398800	-0.4399700	-0.4399880	-0.4399970	-0.4399988
0.500	-0.2497001	-0.2499000	-0.2499700	-0.2499900	-0.2499970	-0.2499990
0.600	-0.0397000	-0.0399200	-0.0399699	-0.0399919	-0.0399969	-0.0399991
0.700	0.1903000	0.1900601	0.1900301	0.1900061	0.1900031	0.1900007
0.800	0.4402999	0.4400400	0.4400300	0.4400041	0.4400030	0.4400004
0.900	0.7103000	0.7100201	0.7100301	0.7100021	0.7100031	0.7100003
1.000	1.0002999	1.0000000	1.0000300	1.0000000	1.0000030	1.0000000

Example 3: .Now we consider the following variable coefficient singular perturbation problem from ([9], p. 33):.

$$\varepsilon y''(x) + \left(1 - \frac{x}{2}\right) y'(x) - \left(\frac{1}{2}\right) y(x) = 0, \qquad x \in [0,1]$$

with $y(0) = 0$, $y(1) = 1$.
$x = 0$ is the boundary layer the given problem.

The exact solution is given by $y(x) = \frac{1}{2-x} - \left(\frac{1}{2}\right) \exp\left(-\frac{x - \frac{x^2}{4}}{\epsilon}\right)$.
The numerical results are presented in Table 3.

Table 3: Numerical results of example (3).

x-values	$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-6}$	
	y-values	Exact Values	y-values	Exact Values	y-values	Exact Values
0.000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.020	0.5051515	0.5050505	0.5050606	0.5050505	0.5050516	0.5050505
0.040	0.5103061	0.5102041	0.5102143	0.5102041	0.5102051	0.5102041
0.060	0.5155670	0.5154639	0.5154742	0.5154639	0.5154650	0.5154639
0.080	0.5209375	0.5208333	0.5208438	0.5208333	0.5208344	0.5208333
0.100	0.5264211	0.5263158	0.5263264	0.5263158	0.5263169	0.5263158
0.200	0.5556667	0.5555556	0.5555667	0.5555556	0.5555567	0.5555556
0.300	0.5883529	0.5882353	0.5882471	0.5882353	0.5882365	0.5882353
0.400	0.6251249	0.6250000	0.6250125	0.6250000	0.6250013	0.6250000
0.500	0.6667998	0.6666667	0.6666800	0.6666667	0.6666680	0.6666667
0.600	0.7144281	0.7142857	0.7143000	0.7142857	0.7142872	0.7142857
0.700	0.7693838	0.7692308	0.7692462	0.7692308	0.7692323	0.7692308
0.800	0.8334983	0.8333333	0.8333500	0.8333333	0.8333350	0.8333333
0.900	0.9092695	0.9090909	0.9091092	0.9090909	0.9090928	0.9090909
1.000	1.0001932	1.0000000	1.0000200	1.0000000	1.0000020	1.0000000

Example 4: We now consider the following singular perturbation problem from ([10], p.179)

$$\varepsilon y''(x) + y'(x) = 0, \qquad x \in [0,1]$$

with $y(0) = 0,\ y(1) = 1,$

whose exact solution is $y(x) = \frac{(1-\exp\left(-\frac{x}{\varepsilon}\right))}{(1-\exp\left(-\frac{1}{\varepsilon}\right))}$.

The boundary layer exist at left-end of the underlying interval.
The numerical results are presented in Table 4.

Table 4: Numerical results of example (4).

x-values	$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-6}$	
	y-values	Exact Values	y-values	Exact Values	y-values	Exact Values
0.000	1.3678794	1.3678794	1.3678794	1.3678794	1.3678794	1.3678794
0.020	0.3753486	0.3753111	0.3753149	0.3753111	0.3753115	0.3753111
0.040	0.3829311	0.3828929	0.3828967	0.3828929	0.3828933	0.3828929
0.060	0.3906669	0.3906278	0.3906317	0.3906278	0.3906282	0.3906278
0.080	0.3985589	0.3985190	0.3985230	0.3985190	0.3985195	0.3985190
0.100	0.4066103	0.4065697	0.4065737	0.4065697	0.4065701	0.4065697
0.200	0.4493739	0.4493290	0.4493335	0.4493290	0.4493294	0.4493290
0.300	0.4966349	0.4965853	0.4965903	0.4965853	0.4965858	0.4965853
0.400	0.5488665	0.5488116	0.5488171	0.5488116	0.5488122	0.5488116
0.500	0.6065913	0.6065307	0.6065367	0.6065307	0.6065313	0.6065307
0.600	0.6703870	0.6703200	0.6703268	0.6703200	0.6703207	0.6703200
0.700	0.7408923	0.7408183	0.7408257	0.7408183	0.7408190	0.7408183
0.800	0.8188125	0.8187308	0.8187389	0.8187308	0.8187316	0.8187308
0.900	0.9049278	0.9048374	0.9048465	0.9048374	0.9048384	0.9048374
1.000	1.0001000	1.0000000	1.0000100	1.0000000	1.0000010	1.0000000

Example 5: We now consider the following singular perturbation problem from ([11], p.179)

$$\varepsilon y''(x) + y'(x) - (1+\varepsilon)y(x) = 0, \qquad x \in [0,1]$$

with $y(0) = 1 + \frac{1}{e},\ y(1) = 1 + \exp\left(-\frac{1+\varepsilon}{\varepsilon}\right)$.

The exact solution is $y(x) = \exp\left(-(1+\varepsilon)x/\varepsilon\right) + \exp\left(-(1-x)\right)$.
The boundary layer exist at left-end i.e., $x = 0$ of the underlying interval
The numerical results are presented in Table 5.

Table 5: Numerical results of example (5).

x-values	$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-6}$	
	y-values	Exact Values	y-values	Exact Values	y-values	Exact Values
0.000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.020	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.040	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.060	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.080	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.100	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

0.200	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.300	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.400	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.500	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.600	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.700	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.800	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.900	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
1.000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Example 6:. Consider the following non-linear singular perturbation problem from ([7], Page 480):.

$$\varepsilon y''(x) + 2y'(x) + \exp(y(x)) = 0, \qquad x \in [0,1]$$

with $y(0) = 0,\ y(1) = 0.$

The linear problem concerned to this example is

$$\varepsilon y''(x) + 2y'(x) + \frac{2}{(1+x)} y(x) = \frac{2}{(1+x)}\left[\log_e\left(\frac{2}{1+x}\right) - 1\right].$$

The boundary layer exists at left end i.e., $x = 0$ of the given interval.
We have chosen to use Bender and Orszag's uniformly valid approximation for comparison is

$$y(x) = \log_e\left(\frac{2}{1+x}\right) - (\log_e 2)\exp\left(-\frac{2x}{\epsilon}\right).$$

The numerical results are presented in Table 6.

Table 6: Numerical results of example (6).

x-values	$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-6}$	
	y-values	Exact Values	y-values	Exact Values	y-values	Exact Values
0.000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.020	0.6733196	0.6733446	0.6733420	0.6733446	0.6733443	0.6733446
0.040	0.6539015	0.6539265	0.6539240	0.6539265	0.6539262	0.6539265
0.060	0.6348532	0.6348783	0.6348758	0.6348783	0.6348780	0.6348783
0.080	0.6161611	0.6161861	0.6161836	0.6161861	0.6161859	0.6161861
0.100	0.5978120	0.5978370	0.5978345	0.5978370	0.5978367	0.5978370
0.200	0.5108006	0.5108256	0.5108231	0.5108256	0.5108254	0.5108256
0.300	0.4307579	0.4307829	0.4307804	0.4307829	0.4307826	0.4307829
0.400	0.3566499	0.3566749	0.3566724	0.3566749	0.3566747	0.3566749
0.500	0.2876571	0.2876821	0.2876796	0.2876821	0.2876818	0.2876821
0.600	0.2231185	0.2231435	0.2231410	0.2231435	0.2231433	0.2231435
0.700	0.1624939	0.1625189	0.1625164	0.1625189	0.1625187	0.1625189
0.800	0.1053355	0.1053605	0.1053580	0.1053605	0.1053603	0.1053605
0.900	0.0512683	0.0512933	0.0512908	0.0512933	0.0512930	0.0512933
1.000	-0.000025	0.0000000	-0.000003	0.0000000	-0.0000002	0.0000000

Example 7: Finally we consider the following singular perturbation problem from ([9], p. 56):

$$\varepsilon y''(x) + y(x)y'(x) - y(x) = 0, \qquad x \in [0,1]$$

$$\text{with } y(0) = -1, \; y(1) = 3.9995.$$

The linear problem concerned to this example is

$$\varepsilon y''(x) + (x + 2.9995)y'(x) = x + 2.9995.$$

We have chosen to use the Kevorkian and Cole's uniformly valid approximation and for comparison,

$$y(x) = x + C_1 \tanh\left[\left(\frac{C_1}{2}\right)\left(\frac{x}{\epsilon} + C_2\right)\right].$$

$$\text{where } C_1 = 2.9995, \; C_2 = \left(\frac{1}{C_1}\right)\log_e\left[\frac{(C_1 - 1)}{(C_1 + 1)}\right].$$

$x = 0$ is the boundary layer of the given problem.
The numerical results are presented in Table 7.

Table 7: Numerical results of example (7).

x-values	$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-6}$	
	y-values	Exact Values	y-values	Exact Values	y-values	Exact Values
0.000	-1.000000	-1.000000	-1.000000	-1.000000	-1.0000000	-1.000000
0.020	3.0195332	3.0195000	3.0195031	3.0195000	3.0195003	3.0195000
0.040	3.0395327	3.0395000	3.0395033	3.0395000	3.0395002	3.0395000
0.060	3.0595324	3.0595000	3.0595031	3.0595000	3.0595005	3.0595000
0.080	3.0795326	3.0795000	3.0795031	3.0795000	3.0795004	3.0795000
0.100	3.0995324	3.0994999	3.0995030	3.0994999	3.0995004	3.0994999
0.200	3.1995311	3.1995001	3.1995029	3.1995001	3.1995001	3.1995001
0.300	3.2995303	3.2995000	3.2995031	3.2995000	3.2995002	3.2995000
0.400	3.3995295	3.3995001	3.3995030	3.3995001	3.3995001	3.3995001
0.500	3.4995284	3.4995000	3.4995027	3.4995000	3.4995003	3.4995000
0.600	3.5995278	3.5995002	3.5995028	3.5995002	3.5995002	3.5995002
0.700	3.6995270	3.6995001	3.6995027	3.6995001	3.6995001	3.6995001
0.800	3.7995265	3.7995000	3.7995026	3.7995000	3.7995002	3.7995000
0.900	3.8995256	3.8995001	3.8995023	3.8995001	3.8995001	3.8995001
1.000	3.9995251	3.9995000	3.9995024	3.9995000	3.9995003	3.9995000

Example 8:. Now we consider the following singular perturbation problem from ([1], p:9-10):

$$\varepsilon y''(x) - y(x)y'(x) = 0, \qquad x \in [-1,1]$$

$$\text{with } y(-1) = 0, \; y(1) = -1.$$

The linear problem concerned to this example is

$$\varepsilon y''(x) + y'(x) = 0.$$

We have chosen to use O'Malley's approximation solution for comparison,

$$y(x) = -(1 - \exp\left(\frac{-x-1}{\varepsilon}\right))/(1 + exp\left(\frac{-x-1}{\varepsilon}\right)) .$$

The boundary layer exist at left-end.

i.e., $x = 0$ of the underlying interval

The numerical results are presented in Table 8.

Table 8: Numerical results of example (8).

x-values	$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-6}$	
	y-values	Exact Values	y-values	Exact Values	y-values	Exact Values
0.000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.020	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000
0.040	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000
0.060	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000
0.080	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000
0.100	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000
0.200	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000
0.300	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000
0.400	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000
0.500	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000
0.600	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000
0.700	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000
0.800	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000
0.900	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000
1.000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000	-1.0000000

5. Discussion and Conclusions:

In this paper, a method is presented to solve a singularly perturbed two-point boundary value problem by replacing it with a pair of initial value problems. The first initial value problem (IVP) is solved in general way and the second one is solved using exponential fitted finite difference scheme. Numerical experiments are conducted by considering several linear and non-linear singular perturbation problems. It is observed that the present method approximates the exact solution very well.

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