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A Computational Method For Solving Singularly Perturbed Initial Value Problems

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ABSTRACT

A computational method is presented for solving singularly perturbed initial value problems with an initial layer on the left end point of the interval. It is designed for the practicing engineer or applied mathematician who needs a practical tool for these problems easy to use, modest problem preparation and ready computer implementation. The zeroth order asymptotic expansion is used to obtain the terminal boundary condition. Then, a initial layer region and a non-initial layer region are created. And so, the given problem is split into two initial value problems. All these problems are separately solved by an uniform and optimal exponentially fitted finite difference scheme. Error estimates for the computational method is derived using maximum principle. Numerical results are given in this paper to demonstrate the applicability of the computational method.

Keywords: singular perturbation problems, exponentially fitted, uniformly convergent, asymptotic expansion, finite difference schemes.

AMS (MOS) subject classification: 65F05, 65N30, 65N3[illegible], 6[illegible]Y05.

1. Introduction

The numerical treatment for singular perturbation problems have always been far from trivial, because of the initial and boundary layer behavior of the solution. These problems occur frequently in fluid mechanics, elasticity and other branches of applied mathematics, science and engineering. A few notable problems are boundary layer problems, WKB problems, convective heat transport problems with large Peclet number, etc. The area of singular perturbations is a field of increasing interest to applied mathematicians. To be specific, we consider the following singular perturbation problem(SPP):

$$Lu(x) \equiv \varepsilon u'(x) + a(x)\, u(x) = f(x), \qquad (1a)$$

for $0 < x < 1$ with

$$u(0) = \varphi \qquad , \qquad (1b)$$

where ε is a small parameter ($0 < \varepsilon << 1$), φ is a given constant, a(x) and f(x) are assumed to be sufficiently continuously differentiable functions in [0, 1], and $a(x) \geq \alpha > 0$ on [0, 1], where α is some positive constant. Under these assumptions SPP(1a,b) has a unique solution u(x) which, in general, displays a boundary layer of width O(ε) at x = 0 for small values of ε [3, 15].

Uniformly convergent finite difference schemes for the SPP(1ab) have been examined by various authors [3, 15]. All these schemes use constant mesh size and it is impractical if one wants to find local behavior of the solution in the neighborhood of ε, where ε is small.

Pearson[12] was perhaps the first to attempt something like net adjustments in finite difference schemes for the boundary value problem with first derivative term . Roberts[13] proposed a boundary value technique and introduced the idea of inner and outer region problems for the domain [0, 1]. Such type of technique is also discussed in [5-7]. Other works include Bender[1], Neyfeh]8] and O'Malley[10].

The objective of the paper is to present a new approach [16] for solving the SPP(1a,b) . It is based on the asymptotic behavior of the solution of the SPP(1a,b). The method consists of the following ste3ps:

(i) the original problem is divided into initial layer region problem and non-initial layer region problem

(ii) terminal boundary condition is obtained from the zeroth order asymptotic expansion for the solution of the SPP(1a,b)

(iii) then, the new initial layer region problem is created and solved numerically using uniform and optimal exponentially fitted schemes with variable mesh

(iv) in urn , the non-initial layer region problem is created and solved numerically using uniform and optimal exponentially fitted schemes which solve the problem exactly for small values of ε with constant mesh

(v) finally, we combine the solutions of the initial layer region problem and the non-initial layer region problem.

The process is to be repeated for different choices of the terminal points until the profiles stabilize in the initial layer region and the non-initial layer region.

In section 2 the description of the computational method is given. The error estimate of the solutions of two region problems are derived in section 3. The error estimate of the numerical solution of two region problems are derived in section 4 . The error estimate of the solutions of the computational method with respect to the solutions of two region problems are derived in section 5 . The numerical experimental results are presented in section 6.

Throughout this paper , we shall use the following notations

$a_0 = a(0)$,

$x_p = t_p \varepsilon$,

$x_q = (1 - x_p)\varepsilon$,

$D_+ u_i = (u_{i+1} - u_i) / h$,

$\rho_1 = h_1 / \varepsilon$,

$\rho_2 = h_2 / \varepsilon$,

$\sigma(\rho_1) = \sigma(-\rho_1 a(x_i))$,

$\sigma(\rho_2) = \sigma(-\rho_2 a(x_i))$

where $\sigma(-x)$ is a Bernoulli's generating function defined as

$$\sigma(-x) = x / [1 - \exp(-x)]$$

for $x>0$ and C is independent of I, h_1, h_2 cad ε.

2. Computational method

Consider the SPP(1a,b) as the original problem. We split the original problem into two initial value problems(IVP), namely, the initial layer region problem and the non-initial layer region problem.

First, we compute the terminal conditions as follows.

2.1. Terminal condition

Let x_p be the terminal point or common point or width or thickness of the initial layer region. To find the terminal condition we use the solution of the reduced problem

$$a(x)\, u_0(x) = f(x),\ x \in (0, 1) \qquad (2)$$

and the transformed equation

$$d\, v_0(\tau)\, / d\tau + a(0)\, v_0(\tau) = 0, \; \tau \in (0,\infty) , \qquad (3a)$$

$$v_0(0) = \varphi - u_0(0) \qquad (3b)$$

The reduced problem (2) is got by setting ε = 0 in the SPP(1a,b). And the initial value problems (3a,b) is obtained by the Taylor's expansion of the coefficient a(x) about x = 0 making a change of variable

$$X \rightarrow \tau = x / \varepsilon$$

and equating powers of ε.

The $zero^{th}$ order asymptotic expansion for the solution of the SPP(1a, b) is given by

$$U = u_0 + v_0 \qquad (4)$$

where

$$u_0(x) = f(x) \; / \; a(x) , \qquad (5)$$

$$v_0(x) = p^* \exp(-a_0 \; x) \qquad (6)$$

and

$$p^* = [\, \varphi - u_0(0) \,] \qquad (7)$$

It can be observed that [3, 15], if u is the solution of (1a,b) and U is given by (4)

$$\left| \; u(x - U)(x) \; \right| \le C\, \varepsilon , \text{ for } 0 \le x \le 1 \qquad (8)$$

for sufficiently smooth function a(x) and f(x) .

From (4), the l terminal condition is taken as

$$u(x_p) = u_0(x_p) + v_0(x_p) = \varphi_1 \; . \qquad (9)$$

Note that the terminal point x_p will be of the form

$x_p = t_p \; \varepsilon$ where t_p = 1, 10, 20, 30, … .

Now using the terminal condition we split the SPP(1a,b) into two initial value problems.

2.2. Initial layer region problem

The terminal point x_p is common to both the initial layer region and the non-initial layer region. We have the initial layer region problem as follows:

$$\varepsilon\, 'u(x) + a(x)\, u(x) = f(x), \qquad (10a)$$

for $0 < x < x_p$ with

$$u(0) = \varphi \tag{10b}$$

2..3 Non-initial layer region problem

Using the terminal point x_p and the right end point we have the non-initial layer region problem as follows:

$$\varepsilon u'(x) + a(x)\, u(x) = f(x), \tag{11a}$$

for $x_p < x < 1$ with

$$u(x_p) = \varphi_1 \tag{11b}$$

2.4 Solution of the original problem

After solving the above two problems, we combine the solutions of these two problems to obtain an approximate solution to the original problem over the interval [0, 1].

We repeat the process for various choices of terminal points x_p until the solution profiles do not differ much from iteration to iteration. For a computational point of view, we use an absolute error estimate of the form

$$\left| U(x)^{m+1} - U(x)^{m} \right| \leq \delta \ ,\ 0 \leq x \leq x_p \tag{12}$$

where $U(x)^m$ is the m^{th} iteration of the initial layer region solution and δ is the prescribed tolerance bound.

2.5. Numerical method

We use an explicit uniform and optimal finite difference scheme for the numerical solution of the SPP(1a,b) to solve the above two initial value problems which is presented in [3, 15] and it is defined as follows:

$$L^h u_i \equiv \varepsilon\, \sigma(-\rho)\, D_+ u_i + a(x_{i+1})\, u_i = f(x_{i+1}),\ 0 < I < N \ , \tag{13a}$$

$$u_0 = \varphi \tag{13b}$$

where

$$\sigma(-\rho) = \sigma(-\rho\, a(x_{i+1})),\ \rho = h / \varepsilon. \tag{13c}$$

It is proved that in [3, 15],

$$\left| u(x_i) - u_i \right| \leq C \min(h, \varepsilon)$$

where u(x) and u_i are the solutions of SPP(1a.b) and (13a-c) respectively.

3. Error estimates-regions wise

Using maximum principle we derive error estimates for the solutions of the initial layer region problem and the non-initial layer region problem in Theorem 2 and 3 respectively. The maximum principle is stated as follows in Theorem 1 [3, 15]:

Theorem 1.

Let v be any smooth function and L be the operator defined as in (1a).

(i) if $v(0) \geq 0$ and $L\,v(x) \geq 0$ for $x \in (0, 1)$, then we have

$v(x) \geq 0$, for all $x \in [0, 1]$,

(ii) for all $x \in [0, 1]$, we have

$|v(x)| \leq C\ [\ |v(0)| + (1/\alpha)\max |Lv(y)|\ ]$, $y \in [0, 1]$ and $C > 0$.

Proof : See Doolan et al., [3, 15] for proof of Theorem 1.

Theorem 2.

Let u and u^1 be the solutions of the SPP(1a,b) and (10a,b) respectively. Then , for all $x \in [0, x_p,]$,

$$|u(x) - u^1(x)| = 0 \qquad (14)$$

where C is independent of ε.

Proof. For all $0 < x < x_p$, we have

$L\,[u(x) - u^1(x)] = L\,u(x) - L\,u^1(x) = f(x) - f(x) = 0.$

For x=0, $u(0) - u^1(0) = \varphi - \varphi = 0.)$.

Using maximum principle, for all $x \in [0, x_p,]$, we have

$|u(x) - u^1(x)| = 0$.

Theorem 3.

Let u and u^2 be the solutions of the SPP(1a,b) and (11a,b) respectively. Then , for all $x \in [x_p, 1]$,

$$| u(x) - u^2 (x) | \le C \varepsilon . \qquad (15)$$

where C is independent of ε.

Proof. For $x_p < x < 1$, we have

$$L [u(x) - u^2 (x)] = L u(x) - L u^2 (x) = f(x) - f(x) = 0.$$

And for $x = x_p$, $u(x_p) - u^2 (x_p) = u(x_p) - \varphi_1$

$$= u(x_p) – [u(x_p) + O(\varepsilon)]$$

$$= O(\varepsilon).$$

Using maximum principle, for all $x \in [x_p, 1]$, we have

$$| u(x) - u^2 (x) | \le | u(x_p) - u^2 (x_p) | \le C \varepsilon .$$

4. Error estimates-Numerical solutions

Using the discrete maximum principle we derive error estimates for the numerical solutions of the initial layer region problem and non-initial layer region problem in Theorem 5 and 6 respectively using the numerical method (13a-c). The discrete maximum principle is stated as follows in Theorem 4 [3, 15]:

Theorem 4.

Let v_i be a mesh function and L^h be the operator defined as in (13a)

(i) if $v_0 \ge 0$ and $L^h v_i \ge 0$, for all $1 \le i \le N$, then we have

$v_i \ge 0$, for all $0 \le i \le N$,

(ii) for all $0 \le i \le N$, we have

$$| v(x) | \le C (| v_0 | + (1/\alpha) \max | L^h v_j |) , \text{ for all } 0 \le j \le N$$

and $C > 0$.

Proof: See Doolan et al., [3, 15] for proof of Theorem 4.

Theorem 5.

Let u^1 and u^1_i be the solution of the IVP(10a,b) and the numerical solution of the IVP (10a,b) using the scheme(13a -c) respectively. Then , for all $x \in [0, x_p,], 0 \le i \le N$

$$\left| u^1(x_i) - u^1_i \right| \le C \min(h_1, \varepsilon) \quad (16)$$

where C is independent of i, h_1 and ε.

Proof. See [3, 15].

Theorem 6.

Let u^2 and u^2_i be the solution of the IVP(11a,b) and the numerical solution of the IVP (11a,b) using the scheme (13a-c) respectively. Then , for all $x \in [x_p , 1], 0 \le i \le N$

$$\left| u^2 (x_i) - u^2_i \right| \le C \min(h_2, \varepsilon) \quad (17)$$

where C is independent of i h_2 and ε.

***Proof*:.** See [3, 15].

5. Error estimate- Computational method

Using the maximum principle, we derive error estimates between the solution of the original SPP(1a,b) and the numerical solutions of the initial layer region problem and the non-initial layer region problem in Theorem 7 and 8 respectively.

Theorem 7.

Let u and u^1_i be the solution of the SPP (1a,b) and the numerical solution of the problem(10a,b) using the scheme(13a-c) respectively. Then , for all $x \in [0, x_p ,]$ and for $0 \le i \le N$

$$\left| u (x_i) - u^1_i \right| \le C \min(h_1, \varepsilon) \quad (18)$$

where C is independent of i, $h_1$and ε.

Proof. Using triangle inequality, we have

$$\left| u (x_i) - u^1_i \right| \le \left| u (x_i) - u^1(x_i) \right| + \left| u^1(x_i) - u^1_i \right|.$$

From the estimate(14) and (16)

$\left| u(x_i) - u^1_i \right| \leq C \min(h_1, \varepsilon)$.

Theorem 8..

Let u and u^2_i be the solution of the SPP(1a,b) and the numerical solution of the problem(11a,b) using the scheme (13a-c) respectively . Then , for all $x \in [x_q, 1]$, and for $0 \leq i \leq N$

$$\left| u(x_i) - u^2_i \right| \leq C(\varepsilon + \min(h_2, \varepsilon)) \quad (19)$$

where C is independent of i, h_2 and ε .

***Proof*:.** Using triangle inequality, we have

$$\left| u(x_i) - u^2_i \right| \leq \left| u(x_i) - u^2(x_i) \right| + \left| u^2(x_i) - u^2_i \right|.$$

From the estimate(15) and (17)

$$\left| u(x_i) - u^2_i \right| \leq C(\varepsilon + \min(h_2, \varepsilon)) .$$

Remark. It is to be noted that the mesh size h_1 used in the initial layer region. And the mesh size h_2 used in the non-initial layer region is not equal to h_1 .

6. Numerical experiment

To demonstrate the applicability of the computational method , we have implemented it on two SPPs. Computed results are tabulated in Tables. From the Tables, the underlined value indicates that it is a terminal condition obtained from (9) and the corresponding x value denotes terminal points x_p.

In the last column of the Tables , we have given the absolute error of the numerical solution at $x_p = 30\varepsilon$ and $x_q = 1 - 30\varepsilon$ to the exact solution. The mesh size used in Tables 1A and 2A are $h_2 = 0.1$ and

$h_1 = 10^{-4}, 10^{-3}, 2 \times 10^{-3}, 3 \times 10^{-3}$, … for the intervals got for t_p = 1, 10, 20, 30, … respectively. The mesh size used in Tables 1B and 2B are $h_2 = 0.1$ and $h_1 = 10^{-5}, 10^{-4}, 2 \times 10^{-4}, 3 \times 10^{-4}$, … for the intervals got for t_p = 1, 10, 20, 30, … respectively.

Example 1.

Consider the following SPP[3,15]

$$u'(x) = -\eta[\ u(x) - \exp(-x)\ ] - \exp(-x)\ , \quad 0 < x \le 1,$$
$$u(0) = 0,\ -\eta = 1/\varepsilon\ .$$

The numerical results are presented in Tables 1A and 1B, for $\varepsilon = 10^{-6}$ and 10^{-8} .

Example 2.

Consider the following non-homogeneous SPP from Doolan et. al.[3]

$$u'(x) = -\eta[\ u(x) - g(x)\ ] + g'(x)\ , \quad 0 < x \le 1,$$
$$u(0) = 10,\ -\eta = 1/\varepsilon\ ,$$

where g(x) -= 10 – (10 + x) exp(-x) .

The numerical results are presented in Tables 2A and 2B, for $\varepsilon = 10^{-6}$ and 10^{-8}.

Discussion and conclusion

We have presented a practical method, exactly implemented on a computer to solve singularly perturbed initial vale problem. It is observed that the computational method approximates the exact solution well, with two examples.

The present method gives more mesh points inside the initial layer with an a priori chosen accuracy, even though this method is not uniformly convergent. Uniform and optimal schemes with constant mesh become impractical if one wants to find the local behavior of the solution in the neighborhood of ε when ε is small. The present method is practical in such situation.

It can be observed from Tables that the present method approximates the exact solution very well.

All computations were performed in Pascal single precision on a Micro Vax II computer at Bharathidasan University, Tiruchirapalli-620 024, Tamil Nadu, India.

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TABLE 1 A

NUMERICAL RESULTS FOR EXAMPLE 1 , ε = 1.00000E -04

t_p	1	10	20	30	EXACT	ERROR
x	U(x)	U(x)	U(x)	U(x)	SOLUTION	AT t = 30
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.00000E-05	9.51531E-02	9.51531E-02	9.51531E-02	9.51531E-02	9.51526E-02	4.99189E-07
2.00000E-05	1.81250E-01	1.81250E-01	1.81250E-01	1.81250E-01	1.81249E-01	9.53674E-07
3.00000E-05	2.59153E-01	2.59153E-01	2.59153E-01	2.59153E-01	2.59152E-01	1.25170E-07
4.00000E-05	3.29642E-01	3.29642E-01	3.29642E-01	3.29642E-01	3.29640E-01	1.63913E-06
5.00000E-05	3.93421E-01	3.93421E-01	3.93421E-01	3.93421E-01	3.93419E-01	1.99676E-06
6.00000E-05	4.51131E-01	4.51131E-01	4.51131E-01	4.51131E-01	4.51128E-01	2.26498E-06
7.00000E-05	5.03347E-01	5.03347E-01	5.03347E-01	5.03347E-01	5.03345E-01	2.44379E-06
8.00000E-05	5.50594E-01	5.50594E-01	5.50594E-01	5.50594E-01	5.50591E-01	2.74181E-06
9.00000E-05	5.93343E-01	5.93343E-01	5.93343E-01	5.93343E-01	5.93340E-01	2.98023E-06
1.00000E-04	6.32024E-01	6.32024E-01	6.32024E-01	6.32024E-01	6.32021E-01	3.21865E-06
2.00000E-04		8.64469E-01	8.64469E-01	8.64469E-01	8.64465E-01	4.41074E-06
4.00000E-04		9.81290E-01	9.81290E-01	9.81290E-01	9.81284E-01	5.06639E-06
6.00000E-04		9.96927E-01	9.96927E-01	9.96927E-01	9.96921E-01	5.12600E-06
8.00000E-04		9.98870E-01	9.98870E-01	9.98870E-01	9.98865E-01	5.12600E-06
1.00000E-03		9.98960E-01	9.98960E-01	9.98960E-01	9.98955E-01	5.18560E-06
1.20000E-03			9.98800E-01	9.98800E-01	9.98795E-01	5.18560E-06
1.40000E-03			9.98605E-01	9.98605E-01	9.98600E-01	5.18560E-06
1.60000E-03			9.98406E-01	9.98406E-01	9.98401E-01	5.12600E-06
1.80000E-03			9.98207E-01	9.98207E-01	9.98202E-01	4.94719E-06
2.00000E-03			9.98007E-01	9.98007E-01	9.98002E-01	4.88758E-06
2.20000E-03				9.97808E-01	9.97802E-01	5.06639E-06
2.40000E-03				9.97608E-01	9.97603E-01	4.88758E-06
2.60000E-03				9.97408E-01	9.97403E-01	5.06639E-06
2.80000E-03				9.97209E-01	9.97203E-01	4.94719E-06
3.00000E-03				9.97010E-01	9.97005E-01	5.06639E-06
4.00000E-03					9.96008E-01	
1.00000E-01	9.04837E-01	9.04837E-01	9.04837E-01	9.04837E-01	9.04837E-01	0.00000E+00
2.00000E-01	8.18649E-01	8.18649E-01	8.18649E-01	8.18649E-01	8.18731E-01	8.18968E-05
3.00000E-01	7.40744E-01	7.40744E-01	7.40744E-01	7.40744E-01	7.40818E-01	7.40886E-05
4.00000E-01	6.70253E-01	6.70253E-01	6.70253E-01	6.70253E-01	6.70320E-01	6.70552E-05
5.00000E-01	6.06470E-01	6.06470E-01	6.06470E-01	6.06470E-01	6.06531E-01	6.06775E-05
6.00000E-01	5.48757E-01	5.48757E-01	5.48757E-01	5.48757E-01	5.48812E-01	5.48959E-05
7.00000E-01	4.96536E-01	4.96536E-01	4.96536E-01	4.96536E-01	4.96585E-01	4.96805E-05
8.00000E-01	4.49284E-01	4.49284E-01	4.49284E-01	4.49284E-01	4.49329E-01	4.49717E-05
9.00000E-01	4.06529E-01	4.06529E-01	4.06529E-01	4.06529E-01	4.06570E-01	4.06802E-05
1.00000E+00	3.67843E-01	3.67843E-01	3.67843E-01	3.67843E-01	3.67879E-01	3.68655E-05

TABLE 1 B

NUMERICAL RESULTS FOR EXAMPLE 1 , ε = 1.00000E -05

t_p	1	10	20	30	EXACT SOLUTION	ERROR AT T=30
x	U(x)	U(x)	U(x)	U(x)		
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.00000E-06	9.51616E-02	9.51616E-02	9.51616E-02	9.51616E-02	9.51616E-02	5.96046E-08
2.00000E-06	1.81267E-01	1.81267E-01	1.81267E-01	1.81267E-01	1.81267E-01	1.19209E-07
3.00000E-06	2.59179E-01	2.59179E-01	2.59179E-01	2.59179E-01	2.59179E-01	5.96046E-08
4.00000E-06	3.29676E-01	3.29676E-01	3.29676E-01	3.29676E-01	3.29676E-01	1.19209E-07
5.00000E-06	3.93465E-01	3.93465E-01	3.93465E-01	3.93465E-01	3.93464E-01	1.78814E-07
6.00000E-06	4.51183E-01	4.51183E-01	4.51183E-01	4.51183E-01	4.51182E-01	1.78814E-07
7.00000E-06	5.03408E-01	5.03408E-01	5.03408E-01	5.03408E-01	5.03408E-01	1.19209E-07
8.00000E-06	5.50663E-01	5.50663E-01	5.50663E-01	5.50663E-01	5.50663E-01	1.78814E-07
9.00000E-06	5.93422E-01	5.93422E-01	5.93422E-01	5.93422E-01	5.93421E-01	2.38419E-07
1.00000E-05	6.32111E-01	6.32111E-01	6.32111E-01	6.32111E-01	6.32111E-01	2.98023E-07
2.00000E-05		8.64645E-01	8.64645E-01	8.64645E-01	8.64645E-01	4.17233E-07
4.00000E-05		9.81645E-01	9.81645E-01	9.81645E-01	9.81644E-01	5.36442E-07
6.00000E-05		9.97462E-01	9.97462E-01	9.97462E-01	9.97461E-01	5.36442E-07
8.00000E-05		9.99585E-01	9.99585E-01	9.99585E-01	9.99585E-01	4.76837E-07
1.00000E-04		9.99855E-01	9.99855E-01	9.99855E-01	9.99855E-01	5.36442E-07
1.20000E-04			9.99874E-01	9.99874E-01	9.99874E-01	5.36442E-07
1.40000E-04			9.99860E-01	9.99860E-01	9.99859E-01	5.96046E-07
1.60000E-04			9.99840E-01	9.99840E-01	9.99840E-01	4.76837E-07
1.80000E-04			9.99821E-01	9.99821E-01	9.99820E-01	5.96046E-07
2.00000E-04			9.99801E-01	9.99801E-01	9.99800E-01	5.36442E-07
2.20000E-04				9.99781E-01	9.99780E-01	5.36442E-07
2.40000E-04				9.99761E-01	9.99760E-01	5.96046E-07
2.60000E-04				9.99741E-01	9.99740E-01	5.96046E-07
2.80000E-04				9.99721E-01	9.99720E-01	5.36442E-07
3.00000E-04				9.99701E-01	9.99700E-01	4.76837E-07
4.00000E-04					9.99600E-01	
1.00000E-01	9.04837E-01	9.04837E-01	9.04837E-01	9.04837E-01	9.04837E-01	0.00000E+00
2.00000E-01	8.18723E-01	8.18723E-01	8.18723E-01	8.18723E-01	8.18731E-01	8.22544E-06
3.00000E-01	7.40811E-01	7.40811E-01	7.40811E-01	7.40811E-01	7.40818E-01	7.39098E-06
4.00000E-01	6.70313E-01	6.70313E-01	6.70313E-01	6.70313E-01	6.70320E-01	6.73532E-06
5.00000E-01	6.06525E-01	6.06525E-01	6.06525E-01	6.06525E-01	6.06531E-01	6.07967E-06
6.00000E-01	5.48806E-01	5.48806E-01	5.48806E-01	5.48806E-01	5.48812E-01	5.48363E-06
7.00000E-01	4.96580E-01	4.96580E-01	4.96580E-01	4.96580E-01	4.96585E-01	4.97699E-06
8.00000E-01	4.49324E-01	4.49324E-01	4.49324E-01	4.49324E-01	4.49329E-01	4.52995E-06
9.00000E-01	4.06566E-01	4.06566E-01	4.06566E-01	4.06566E-01	4.06570E-01	4.11272E-06
1.00000E+00	3.67876E-01	3.67876E-01	3.67876E-01	3.67876E-01	3.67879E-01	3.75509E-06

TABLE 2A:
NUMERICAL RESULTS FOR EXAMPLE 2 , ε = 1.00000E -04

t_p	1	10	20	30	EXACT SOLUTION	ERROR AT t = 30
X	U(x)	U(x)	U(x)	U(x)		
0.00000E+00	1.00000E+01	1.00000E+01	1.00000E+01	1.00000E+01	1.00000E+01	0.00000E+00
1.00000E-05	9.04846E+00	9.04846E+00	9.04846E+00	9.04846E+00	9.04846E+00	4.76837E-06
2.00000E-05	8.18748E+00	8.18748E+00	8.18748E+00	8.18748E+00	8.18749E+00	7.62939E-06
3.00000E-05	7.40844E+00	7.40844E+00	7.40844E+00	7.40844E+00	7.40845E+00	1.04904E-05
4.00000E-05	6.70355E+00	6.70355E+00	6.70355E+00	6.70355E+00	6.70356E+00	1.33514E-05
5.00000E-05	6.06574E+00	6.06574E+00	6.06574E+00	6.06574E+00	6.06576E+00	1.71661E-05
6.00000E-05	5.48864E+00	5.48864E+00	5.48864E+00	5.48864E+00	5.48866E+00	1.95503E-05
7.00000E-05	4.96646E+00	4.96646E+00	4.96646E+00	4.96646E+00	4.96648E+00	2.24113E-05
8.00000E-05	4.49399E+00	4.49399E+00	4.49399E+00	4.49399E+00	4.49401E+00	2.43187E-05
9.00000E-05	4.06648E+00	4.06648E+00	4.06648E+00	4.06648E+00	4.06651E+00	2.71797E-05
1.00000E-04	3.67967E+00	3.67967E+00	3.67967E+00	3.67967E+00	3.67969E+00	2.86102E-05
2.00000E-04		1.35511E+00	1.35511E+00	1.35511E+00	1.35515E+00	3.85046E-05
4.00000E-04		1.86711E-01	1.86711E-01	1.86711E-01	1.86757E-01	4.54783E-05
6.00000E-04		3.01405E-02	3.01405E-02	3.01405E-02	3.01863E-02	4.57726E-05
8.00000E-04		1.05064E-02	1.05064E-02	1.05064E-02	1.05520E-02	4.56404E-05
1.00000E-03		9.40423E-03	9.40423E-03	9.40423E-03	9.45001E-03	4.57810E-05
1.20000E-03			1.08100E-02	1.08100E-02	1.08100E-02	4.60381E-05
1.40000E-03			1.25547E-02	1.25547E-02	1.26006E-02	4.59086E-05
1.60000E-03			1.43453E-02	1.43453E-02	1.43902E-02	4.48413E-05
1.80000E-03			1.61415E-02	1.61415E-02	1.61878E-02	4.63668E-05
2.00000E-03			1.79383E-02	1.79383E-02	1.79844E-02	4.61172E-05
2.20000E-03				1.97351E-02	1.97802E-02	4.51077E-05
2.40000E-03				2.15312E-02	2.15769E-02	4.56534E-05
2.60000E-03				2.33273E-02	2.33736E-02	4.62662E-05
2.80000E-03				2.51230E-02	2.51684E-02	4.54169E-05
3.00000E-03				2.69185E-02	2.69632E-02	4.47202E-05
4.00000E-03					3.59364E-02	
1.00000E-01	8.61141E-01	8.61141E-01	8.61141E-01	8.61141E-01	8.61141E-01	0.00000E+00
2.00000E-01	1.64970E+00	1.64970E+00	1.64970E+00	1.64970E+00	1.64895E+00	7.53284E-04
3.00000E-01	2.37026E+00	2.37026E+00	2.37026E+00	2.37026E+00	2.36957E+00	6.89030E-04
4.00000E-01	3.02930E+00	3.02930E+00	3.02930E+00	3.02930E+00	3.02867E+00	6.30140E-04
5.00000E-01	3.63200E+00	3.63200E+00	3.63200E+00	3.63200E+00	3.63143E+00	5.76258E-04
6.00000E-01	4.18312E+00	4.18312E+00	4.18312E+00	4.18312E+00	4.18260E+00	5.26905E-04
7.00000E-01	4.68702E+00	4.68702E+00	4.68702E+00	4.68702E+00	4.68654E+00	4.81606E-04
8.00000E-01	5.14729E+00	5.14729E+00	5.14729E+00	5.14729E+00	5.14725E+00	4.40121E-04
9.00000E-01	5.56879E+00	5.56879E+00	5.56879E+00	5.56879E+00	5.56839E+00	4.02451E-04
1.00000E+00	5.95369E+00	5.95369E+00	5.95369E+00	5.95369E+00	5.95333E+00	3.68118E-04

TABLE 2B: NUMERICAL RESULTS FOR EXAMPLE 2, ε = 1.00000E -05

t_p	1	10	20	30	EXACT SOLUTION	ERROR AT t = 30
X	U(x)	U(x)	U(x)	U(x)		
0.00000E+00	1.00000E+01	1.00000E+01	1.00000E+01	1.00000E+01	1.00000E+01	0.00000E+00
1.00000E-06	9.04838E+00	9.04838E+00	9.04838E+00	9.04838E+00	9.04838E+00	9.53674E-07
2.00000E-06	8.18732E+00	8.18732E+00	8.18732E+00	8.18732E+00	8.18733E+00	9.53674E-07
3.00000E-06	7.40821E+00	7.40821E+00	7.40821E+00	7.40821E+00	7.40821E+00	4.76837E-07
4.00000E-06	6.70324E+00	6.70324E+00	6.70324E+00	6.70324E+00	6.70324E+00	1.43051E-06
5.00000E-06	6.06535E+00	6.06535E+00	6.06535E+00	6.06535E+00	6.06535E+00	2.38419E-06
6.00000E-06	5.48817E+00	5.48817E+00	5.48817E+00	5.48817E+00	5.48817E+00	1.90735E-06
7.00000E-06	4.96591E+00	4.96591E+00	4.96591E+00	4.96591E+00	4.96592E+00	1.43051E-06
8.00000E-06	4.49336E+00	4.49336E+00	4.49336E+00	4.49336E+00	4.49336E+00	1.90735E-06
9.00000E-06	4.06578E+00	4.06578E+00	4.06578E+00	4.06578E+00	4.06578E+00	2.38419E-06
1.00000E-05	3.67888E+00	3.67888E+00	3.67888E+00	3.67888E+00	3.67889E+00	3.33786E-06
2.00000E-05		1.35353E+00	1.35353E+00	1.35353E+00	1.35353E+00	4.05312E-06
4.00000E-05		1.83512E-01	1.83512E-01	1.83512E-01	1.83516E-01	4.05312E-06
6.00000E-05		2.53229E-02	2.53229E-02	2.53229E-02	2.53273E-02	4.36045E-06
8.00000E-05		4.07002E-03	4.07002E-03	4.07002E-03	4.07465E-03	4.63519E-06
1.00000E-04		1.34937E-03	1.34937E-03	1.34937E-03	1.35427E-03	4.89433E-06
1.20000E-04			1.13685E-03	1.13685E-03	1.14100E-03	4.15498E-06
1.40000E-04			1.26364E-03	1.26364E-03	1.26812E-03	4.47419E-06
1.60000E-04			1.43642E-03	1.43642E-03	1.44117E-03	4.75347E-06
1.80000E-04			1.61542E-03	1.61542E-03	1.62044E-03	5.02600E-06
2.00000E-04			1.79523E-03	1.79523E-03	1.79960E-03	4.36907E-06
2.20000E-04				1.97523E-03	1.97983E-03	4.60027E-06
2.40000E-04				2.155[illegible]E-03	2.15912E-03	3.96115E-06
2.60000E-04				2.335[illegible]E-03	2.33936E-03	4.14159E-06
2.80000E-04				2.51508E-03	2.51961E-03	4.52925E-06
3.00000E-04				2.69512E-03	2.69890E-03	3.77442E-06
4.00000E-04					3.6[illegible]012E-03	--------------
1.00000E-01	8.61141E-01	8.61141E-01	8.61141E-01	8.61141E-01	8.61141E-01	0.00000E+00
2.00000E-01	1.64902E+00	1.64902E+00	1.64902E+00	1.64902E+00	1.64895E+00	7.53403E-05
3.00000E-01	2.36964E+00	2.36964E+00	2.36964E+00	2.36964E+00	2.36957E+00	6.89030E-05
4.00000E-01	3.02873E+00	3.02873E+00	3.02873E+00	3.02873E+00	3.02867E+00	6.29425E-05
5.00000E-01	3.63149E+00	3.63149E+00	3.63149E+00	3.63149E+00	3.63143E+00	5.76973E-05
6.00000E-01	4.18265E+00	4.18265E+00	4.18265E+00	4.18265E+00	4.18260E+00	5.24521E-05
7.00000E-01	4.68659E+00	4.68659E+00	4.68659E+00	4.68659E+00	4.68654E+00	4.81606E-05
8.00000E-01	5.14729E+00	5.14729E+00	5.14729E+00	5.14729E+00	5.14725E+00	4.38690E-05
9.00000E-01	5.56843E+00	5.56843E+00	5.56843E+00	5.56843E+00	5.56839E+00	4.00543E-05
1.00000E+00	5.95336E+00	5.95336E+00	5.95336E+00	5.95336E+00	5.95333E+00	3.71933E-05