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# FITTED VAN VELDHUIZEN FINITE DIFFERENCE METHOD FOR SINGULAR PERTURBATION PROBLEMS WITH LAYER BEHAVIOUR

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## ABSTRACT
In this paper, we present a fitted Van Veldhuizen finite difference method for solving a singular perturbation problem with layer behaviour. In this method, we introduce a fitting factor in Van Veldhuizen finite difference method to reduce the global error. The discrete invariant imbedding algorithm is used to solve the tridiagonal system. This method controls the rapid changes that occur in the boundary layer region and it gives good results in both cases i.e., $h \leq \varepsilon$ and $\varepsilon << h$. We have presented maximum absolute errors for the standard examples chosen from the literature to describe the method.

**Keywords**: Singularly perturbed two point boundary value problem, Boundary layer, Tridiagonal matrix, diagonally dominant, Maximum absolute error.

## 1 INTRODUCTION

Singularly perturbed boundary value problems arise frequently in many areas of science and engineering such as heat transfer problem with large Peclet numbers, Navier–Stokes flows with large Reynolds numbers, chemical reactor theory, aerodynamics, reaction–diffusion process, quantum mechanics, optimal control etc. due to the variation in the width of the layer with respect to the small perturbation parameter $\varepsilon$. Several difficulties are experienced in solving the singular perturbation problems using standard numerical methods with uniform mesh. Equations of this type typically exhibit solutions with layers; that is, the domain of the differential equation contains narrow regions where the solution derivatives are extremely large.

The numerical treatment of singularly perturbed differential equations gives major computational difficulties due to the presence of boundary and/or interior layers. This type of problem was solved asymptotically by O'Malley (1974), Nayfeh (1981), Kevorkian and Cole (1996), Bender and Orszag (1978), Bellman (1964) and numerically by Van Veldhuizen (1979), Kreiss (1982), Miller *et.al*. (1995), Kadalbajoo and Patidar (2001), Reddy (1986, 1990), Roos *et.al*. (2008), Lin and Vancouver (1989), Vulanovic (1991) and Kadalbajoo and Devendra Kumar (2009) etc.

It is well known that standard discretization methods for solving singular perturbation problems are unstable and fail to give accurate results when the perturbation parameter $\varepsilon$ is small. In this paper, we present a special second order method for solving a singular perturbation problem with layer behaviour. In general, if we applied classical second order finite difference method to solve singular perturbation problem, it will give good results only when $h \le \varepsilon$ and when $\varepsilon << h$ it gives oscillatory solution and rapid changes in the boundary layer region. In this paper, we present a fitted Van Veldhuizen finite difference method for solving a singular perturbation problem with layer behaviour. In this method, we introduce a fitting factor in Van Veldhuizen finite difference method to reduce the global error. The discrete invariant imbedding algorithm is used to solve the tridiagonal system. This method controls the rapid changes that occur in the boundary layer region and it gives good results in both cases i.e., $h \le \varepsilon$ and $\varepsilon << h$. We have presented maximum absolute errors for the standard examples chosen from the literature to describe the method.

## 2 DESCRIPTION OF THE METHOD

### 2.1 Left – end boundary layer problems

To describe this method, we consider a linearly singularly perturbed two point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) = f(x), \quad x \in [0,1] \tag{1}$$

with the boundary conditions $y(0) = \alpha$ (2.1)

and $y(1) = \beta$; (2.2)

where $\varepsilon$ is a small positive parameter ($0 < \varepsilon << 1$) and $\alpha, \beta$ are known constants.

We assume that *a(x)* and *f(x)* are sufficiently continuously differentiable functions in [0, 1]. Further more, we assume that $a(x) \ge M > 0$ throughout the interval [0, 1], where M is some positive constant. Under these assumptions, (1) has a unique solution *y(x)* which in general, displays a boundary layer of width O($\varepsilon$) at $x$ =0 for small values of $\varepsilon$.

From the theory of singular perturbations O'Malley (1974), it is known that the solution of (1)-(2) is of the form

$$y(x) = y_0(x) + \frac{a(0)}{a(x)}(\alpha - y_0(0))e^{\int_0^x \left(\frac{a(x)}{\varepsilon}\right)dx} + O(\varepsilon) \tag{3}$$

Where $y_0(x)$ is the solution of

$$a(x)y_0'(x) = f(x), \quad y_0(1) = \beta \tag{4}$$

By taking Taylor's series expansion for *a(x)* about the point '0' and restricting to their first terms, (3) becomes,

$$y(x) = y_0(x) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon}\right)x} + O(\varepsilon) \tag{5}$$

Now we divide the interval [0, 1] into N equal parts with constant mesh length h. Let 0=$x_1, x_2, \ldots\ldots. x_N$ =1 be the mesh points. Then we have $x_i = ih : i = 0, 1, 2, \ldots, N$. Let us denote the exact solution *y(x)* at the grid points $x_i$ by $y_i$, similarly $y_i' = y'(x_i)$.

From (5), we have

$$y(x_i) = y_0(x_i) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon}\right)x_i} + O(\varepsilon)$$

i.e., $y(ih) = y_0(ih) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon}\right)ih} + O(\varepsilon)$

therefore

$$\lim_{h\to 0} y(ih) = y_0(0) + (\alpha - y_0(0))e^{-(a(0))i\rho} \qquad (6)$$

where $\rho = \frac{h}{\varepsilon}$

Now consider the difference scheme given by M. Van Veldhuizen (1979)

$$\left(\frac{\varepsilon}{h} - \frac{a_i}{2} + \frac{a_i^2 h}{12\varepsilon}\right)y_{i-1} - 2\left(\frac{\varepsilon}{h} + \frac{a_i^2 h}{12\varepsilon}\right)y_i + \left(\frac{\varepsilon}{h} + \frac{a_i}{2} + \frac{a_i^2 h}{12\varepsilon}\right)y_{i+1} = \frac{h}{3}\left(f_i + f_{i+\frac{1}{2}} + f_{i-\frac{1}{2}}\right) + \frac{h^2 a_i}{12\varepsilon}\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right)$$

which can be rewritten as

$$\varepsilon\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + a_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + \frac{a_i^2 h}{12\varepsilon}\left(y_{i-1} - 2y_i + y_{i+1}\right) = \frac{h}{3}\left(f_i + f_{i+\frac{1}{2}} + f_{i-\frac{1}{2}}\right) + \frac{h^2 a_i}{12\varepsilon}\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right) \qquad (7)$$

Now, we introduce a fitting factor $\sigma(\rho)$ in the above scheme as follows

$$\varepsilon\sigma(\rho)\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + a_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + \frac{a_i^2 h}{12\varepsilon}\left(y_{i-1} - 2y_i + y_{i+1}\right) = \frac{h}{3}\left(f_i + f_{i+\frac{1}{2}} + f_{i-\frac{1}{2}}\right) + \frac{h^2 a_i}{12\varepsilon}\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right) \qquad (8)$$

The fitting factor $\sigma(\rho)$ is to be determined in such a way that the solution of (8) converges uniformly to the solution of (1)-(2).

Multiplying (8) by h and taking the limit as $h \to 0$ ; we get

$$\lim_{h\to 0}\left[\left(\frac{\sigma}{\rho} + \frac{(a(ih))^2\rho}{12}\right)\left(y(ih-h) - 2y(ih) + y(ih+h)\right) + a(ih)\left(\frac{y(ih+h) - y(ih-h)}{2}\right)\right] = 0 \qquad (9)$$

if the right hand side function of equation of (8) is bounded.

Therefore,

Substituting (6) in (9) and simplifying, we get

$$\sigma = \rho\left(\frac{a(0)}{2}\coth\left(\frac{a(0)\rho}{2}\right) - \frac{\rho a^2(0)}{12}\right) \qquad (10)$$

which is a constant fitting factor.

From (8), we have

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i; \quad i = 1,2,....\text{N-1} \qquad (11)$$

where

$$E_j = \left( \frac{\varepsilon\sigma}{h^2} - \frac{a_i}{2h} + \frac{a_i^{\,2}}{12\varepsilon} \right)$$

$$F_j = \left( \frac{2\varepsilon\sigma}{h^2} + \frac{2a_i^{\,2}}{12\varepsilon} \right)$$

$$G_j = \left( \frac{\varepsilon\sigma}{h^2} + \frac{a_i}{2h} + \frac{a_i^{\,2}}{12\varepsilon} \right)$$

$$H_j = \left( f_i + f_{i+1/2} + f_{i-1/2} \right) + \frac{ha_i}{12\varepsilon}\left( f_{i+1/2} - f_{i-1/2} \right)$$

where $\sigma$ is given by (10). This gives us a tridiagonal system which can be solved by the discrete invariant imbedding algorithm.

## 2.2. Right-End Layer Problems

We discuss our method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval. To be specific, we consider a class of singular perturbation problem of the form (1) with (2.1) and (2.2), where $\varepsilon$ is a small positive parameter ($0 < \varepsilon << 1$) and $\alpha, \beta$ are known constants. We assume that *a(x)* and *f(x)* are sufficiently continuously differentiable functions in [0, 1]. Further more, we assume that *a(x)* $\leq$M <0 throughout the interval [0, 1], where M is some negative constant. Under these assumptions, (1) has a unique solution *y(x)* which in general, displays a boundary layer of width O($\varepsilon$) at $x$ =1 for small values of $\varepsilon$.

From the theory of singular perturbation s it is known that the solution of (1)-(2) is of the form

$$y(x) = y_0(x) + \frac{a(1)}{a(x)}(\beta - y_0(1))e^{\int_x^1 \left(\frac{a(x)}{\varepsilon}\right)dx} + O(\varepsilon) \tag{12}$$

where $y_0(x)$ is the solution of

$$a(x)y_0'(x) = f(x), \quad y_0(0) = \alpha \tag{13}$$

By taking Taylor's series expansion for *a(x)* and *b(x)* about the point '1' and restricting to their first terms, (13) becomes,

$$y(x) = y_0(x) + (\beta - y_0(1))e^{-\left(\frac{a(1)}{\varepsilon}\right)(1-x)} + O(\varepsilon) \tag{14}$$

Now we divide the interval [0, 1] into N equal parts with constant mesh length h. Let $0 = x_1, x_2, .......x_N = 1$ be the mesh points. Then we have $x_i = ih \ for \ i = 0, 1, 2,$ N.

From (14), we have

i.e., $y(ih) = y_0(ih) + (\beta - y_0(1))e^{-\left(\frac{a(1)}{\varepsilon}\right)(1-ih)} + O(\varepsilon)$

therefore

$$\lim_{h\to 0} y(ih) = y_0(0) + (\beta - y_0(1))e^{-a(1)(\frac{1}{\varepsilon} - i\rho)} \tag{15}$$

where $\rho = \dfrac{h}{\varepsilon}$

Now proceeding the similar steps as in left-end boundary layer problem, we get the fitting factor as

$$\sigma = \rho\left(\frac{a(0)}{2}\coth\left(\frac{a(1)\rho}{2}\right) - \frac{\rho a^2(0)}{12}\right) \tag{16}$$

which is a constant fitting factor.
From (8), we have

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i; \quad i = 1,2,....\text{N-1} \tag{17}$$

where

$$E_j = \left(\frac{\varepsilon\sigma}{h^2} - \frac{a_i}{2h} + \frac{a_i^{\,2}}{12\varepsilon}\right)$$

$$F_j = \left(\frac{2\varepsilon\sigma}{h^2} + \frac{2a_i^{\,2}}{12\varepsilon}\right)$$

$$G_j = \left(\frac{\varepsilon\sigma}{h^2} + \frac{a_i}{2h} + \frac{a_i^{\,2}}{12\varepsilon}\right)$$

$$H_j = \left(f_i + f_{i+1/2} + f_{i-1/2}\right) + \frac{ha_i}{12\varepsilon}\left(f_{i+1/2} - f_{i-1/2}\right)$$

where $\sigma$ is given by (16). This gives us a tridiagonal system which can be solved by Thomas Algorithm.

Since Eq. (11) or (17) holds for i=1, 2, ……, N-1, we have N-1 linear equations in the N-1 unknowns $y_1, y_2, ..........., y_{N-1}$. The matrix of this set of linear equations will be called $A_N$.

**Lemma:** for all $\varepsilon > 0$ and all h=1/N the matrix $A_N$ is an irreducible and diagonally dominant matrix.

**Proof.** Clearly, $A_N$ is a tridiagonal matrix. Hence, $A_N$ is irreducible if its codiagonals contain non-zero elements only. The codiagonals contain $E_i, G_i$. It is easily seen that these two expressions do not vanish for all $\varepsilon > 0$, $h > 0$ and $a_i \in R$. Hence $A_N$ is irreducible.

Since $E_i, G_i$ do not vanish for all $\varepsilon > 0$, $h > 0$ and $a_i \in R$ these expressions are of constant sign. Then obviously, $E_i > 0,\ G_i > 0.$

Now in each row of $A_N$, the sum of the two off-diagonal elements less than or equal to the modulus of the diagonal element. This proves the diagonal dominant of $A_N$.

Under these conditions the discrete invariant imbedding algorithm is stable (Kadalbazoo and Reddy 1986).

**3. Numerical Examples:** To demonstrate the applicability of the method we have applied it to two linear singular perturbation problems with left-end boundary layer and three linear problems with right-end boundary layer. These examples have been chosen because they have been widely discussed in literature and because exact solutions are available for comparison. The approximate solution is compared with the exact solution.

**Example 1.** Consider the following non-homogeneous singular perturbation problem from fluid dynamics for fluid of small viscosity

$$\varepsilon y''(x) + y'(x) = 1 + 2x \quad ; \ x \in [0, 1]$$

with $y$ (0) =0 and $y$ (1) =1.

The exact solution is given by $y(x)=x(x+1-2\varepsilon)+\dfrac{(2\varepsilon-1)(1-e^{-x/\varepsilon})}{(1-e^{-1/\varepsilon})}$

The numerical results are given in tables 1 and 2 for different values of ε with and without fitting factor respectively.

**Example 2.** Consider the following singular perturbation problem

$\varepsilon y''(x)+y'(x)=2$; x $\in[0,1]$

with $y(0)=0$ and $y(1)=1$.

The exact solution is given by $y(x)=2x+\dfrac{1-e^{-\left(x/\varepsilon\right)}}{e^{-\left(1/\varepsilon\right)}-1}$.

The maximum absolute errors with fitting factor are presented in tables 3 for different values of $\varepsilon$ and the maximum absolute errors without fitting factor are presented in table 4 for comparison.

**Example 3.** Consider the following singular perturbation problem

$\varepsilon y''(x)-y'(x)=0$; x $\in[0, 1]$

with $y(0)=1$ and $y(1)=0$.

Clearly, this problem has a boundary layer at x =1. i.e., at the right end of the underlying interval.

The exact solution is given by $y(x)=\dfrac{\left(e^{(x-1)/\varepsilon}-1\right)}{\left(e^{-1/\varepsilon}-1\right)}$

The maximum absolute errors with fitting factor are presented in tables 5 for different values of $\varepsilon$ and the maximum absolute errors without fitting factor are presented in table 6 for comparison.

**Example 4.** Consider the following singular perturbation problem

$\varepsilon y''(x)-y'(x)=-1$; x $\in[0, 1]$

with $y(0)=0$ and $y(1)=0$.

Clearly, this problem has a boundary layer at x =1. i.e., at the right end of the underlying interval.

The exact solution is given by $y(x)=x-\dfrac{\left(e^{(x-1)/\varepsilon}-e^{(-1/\varepsilon)}\right)}{\left(1-e^{-1/\varepsilon}\right)}$

The maximum absolute errors with fitting factor are presented in tables 7 for different values of $\varepsilon$ and the maximum absolute errors without fitting factor are presented in table 8 for comparison.

**Example 5.** Consider the following singular perturbation problem

$\varepsilon y''(x)-y'(x)=1$; x $\in[0, 1]$

with $y(0)=0$ and $y(1)=1$.

Clearly, this problem has a boundary layer at x =1. i.e., at the right end of the underlying interval.

The asymptotic solution Nayfeh is given by $y(x)=1-x+e^{-(1-x)/\varepsilon}$

The maximum absolute errors with fitting factor are presented in tables 9 for different values of $\varepsilon$ and the maximum absolute errors without fitting factor are presented in table 10 for comparison.

Table 1: The maximum absolute errors in solution of example 1 with fitting factor

| $\varepsilon$ \ $h$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | $2^{-7}$ | $2^{-8}$ | $2^{-9}$ | $2^{-10}$ |
|---|---|---|---|---|---|---|---|---|
| $2^{-3}$ | 2.08(-4) | 1.32(-5) | 8.32(-7) | 5.21(-8) | 3.25(-9) | 2.03(-10) | 1.27(-11) | 7.61(-13) |
| $2^{-4}$ | 1.90(-3) | 1.29(-4) | 8.22(-6) | 5.17(-7) | 3.23(-8) | 2.02(-9) | 1.26(-10) | 8.07(-12) |
| $2^{-5}$ | 1.39(-2) | 1.10(-3) | 7.26(-5) | 4.64(-6) | 2.91(-7) | 1.82(-8) | 1.13(-9) | 7.13(-11) |
| $2^{-6}$ | 6.37(-2) | 7.40(-3) | 5.83(-4) | 3.89(-5) | 2.47(-6) | 1.55(-7) | 9.73(-9) | 6.09(-10) |
| $2^{-10}$ | 2.22 | 5.68(-1) | 1.33(-1) | 2.76(-2) | 4.50(-3) | 5.01(-4) | 3.93(-5) | 2.62(-6) |

Table 2: The maximum absolute errors in solution of example 1 without fitting factor

| $\varepsilon$ \ $h$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | $2^{-7}$ | $2^{-8}$ | $2^{-9}$ | $2^{-10}$ |
|---|---|---|---|---|---|---|---|---|
| $2^{-3}$ | 4.04(-4) | 2.42(-5) | 1.49(-6) | 9.32(-8) | 5.82(-9) | 3.63(-10) | 2.27(-11) | 1.51(-12) |
| $2^{-4}$ | 6.60(-3) | 4.73(-4) | 2.83(-5) | 1.75(-6) | 1.09(-7) | 6.82(-9) | 4.26(-10) | 2.66(-11) |
| $2^{-5}$ | 5.49(-2) | 7.10(-3) | 5.07(-4) | 3.03(-5) | 1.87(-6) | 1.17(-7) | 7.31(-9) | 4.56(-10) |
| $2^{-6}$ | 2.18(-1) | 5.68(-2) | 7.30(-3) | 5.24(-4) | 3.13(-5) | 1.94(-6) | 1.20(-7) | 7.55(-9) |
| $2^{-10}$ | 8.28(-1) | 8.18(-1) | 6.86(-1) | 4.71(-1) | 2.25(-1) | 5.85(-2) | 7.50(-3) | 5.40(-4) |

Table 3: The maximum absolute errors in solution of example 2 with fitting factor

| $\varepsilon$ \ $h$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | $2^{-7}$ | $2^{-8}$ | $2^{-9}$ | $2^{-10}$ |
|---|---|---|---|---|---|---|---|---|
| $2^{-3}$ | 1.11(-16) | 1.66(-16) | 9.99(-16) | 7.77(-16) | 2.94(-15) | 2.24(-14) | 6.73(-14) | 6.19(-14) |
| $2^{-4}$ | 1.11(-16) | 3.88(-16) | 3.33(-16) | 2.66(-15) | 2.66(-15) | 1.50(-14) | 1.97(-14) | 1.41(-13) |
| $2^{-5}$ | 5.55(-17) | 2.22(-16) | 4.44(-16) | 4.44(-16) | 1.66(-15) | 1.18(-14) | 2.06(-14) | 1.96(-13) |
| $2^{-6}$ | 1.11(-16) | 1.11(-16) | 3.33(-16) | 2.44(-15) | 7.00(-15) | 2.81(-14) | 1.13(-13) | 4.27(-13) |
| $2^{-10}$ | 0 | 0 | 2.22(-16) | 0 | 4.44(-16) | 1.11(-16) | 2.22(-16) | 3.78(-14) |

Table 4: The maximum absolute errors in solution of example 2 without fitting factor

| $\varepsilon \backslash h$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | $2^{-7}$ | $2^{-8}$ | $2^{-9}$ | $2^{-10}$ |
|---|---|---|---|---|---|---|---|---|
| $2^{-3}$ | 5.39(-4) | 3.22(-5) | 1.99(-6) | 1.24(-7) | 7.76(-9) | 4.85(-10) | 3.03(-11) | 2.03(-12) |
| $2^{-4}$ | 7.50(-3) | 5.41(-4) | 3.24(-5) | 2.00(-6) | 1.24(-7) | 7.79(-9) | 4.87(-10) | 3.04(-11) |
| $2^{-5}$ | 5.86(-2) | 7.50(-3) | 5.41(-4) | 3.24(-5) | 2.00(-6) | 1.24(-7) | 7.79(-9) | 4.87(-10) |
| $2^{-6}$ | 2.25(-1) | 5.86(-2) | 7.50(-3) | 5.41(-4) | 3.24(-5) | 2.00(-6) | 1.24(-7) | 7.79(-9) |
| $2^{-10}$ | 8.30(-1) | 8.20(-1) | 6.87(-1) | 4.72(-1) | 2.25(-1) | 5.86(-2) | 7.50(-3) | 5.41(-4) |

Table 5: The maximum absolute errors in solution of example 3 with fitting factor

| $\varepsilon \backslash h$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | $2^{-7}$ | $2^{-8}$ | $2^{-9}$ | $2^{-10}$ |
|---|---|---|---|---|---|---|---|---|
| $2^{-3}$ | 1.11(-16) | 1.33(-15) | 1.33(-15) | 1.77(-15) | 7.66(-15) | 5.12(-14) | 2.03(-13) | 9.51(-14) |
| $2^{-4}$ | 1.11(-16) | 3.33(-16) | 2.55(-15) | 1.33(-15) | 6.99(-15) | 1.73(-14) | 7.03(-14) | 3.37(-13) |
| $2^{-5}$ | 3.33(-16) | 2.22(-16) | 2.22(-16) | 3.66(-16) | 4.55(-15) | 6.88(-15) | 2.77(-14) | 1.06(-13) |
| $2^{-6}$ | 6.66(-16) | 2.22(-16) | 2.22(-16) | 3.33(-16) | 3.55(-15) | 6.72(-14) | 1.87(-13) | 1.40(-12) |
| $2^{-10}$ | 0 | 0 | 2.22(-16) | 0 | 6.66(-16) | 2.22(-16) | 2.22(-16) | 3.33(-16) |

Table 6: The maximum absolute errors in solution of example 3 without fitting factor

| $\varepsilon \backslash h$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | $2^{-7}$ | $2^{-8}$ | $2^{-9}$ | $2^{-10}$ |
|---|---|---|---|---|---|---|---|---|
| $2^{-3}$ | 5.39(-4) | 3.22(-5) | 1.99(-6) | 1.24(-7) | 7.76(-9) | 4.85(-10) | 3.03(-11) | 2.03(-12) |
| $2^{-4}$ | 7.50(-3) | 5.41(-4) | 3.24(-5) | 2.00(-6) | 1.24(-7) | 7.79(-9) | 4.87(-10) | 3.04(-11) |
| $2^{-5}$ | 5.86(-2) | 7.50(-3) | 5.41(-4) | 3.24(-5) | 2.00(-6) | 1.24(-7) | 7.79(-9) | 4.87(-10) |
| $2^{-6}$ | 2.25(-1) | 5.86(-2) | 7.50(-3) | 5.41(-4) | 3.24(-5) | 2.00(-6) | 1.24(-7) | 7.79(-9) |
| $2^{-10}$ | 8.30(-1) | 8.20(-1) | 6.87(-1) | 4.72(-1) | 2.25(-1) | 5.86(-2) | 7.50(-3) | 5.41(-4) |

Table 7: The maximum absolute errors in solution of example 4 fitting factor

| $\varepsilon \backslash h$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | $2^{-7}$ | $2^{-8}$ | $2^{-9}$ | $2^{-10}$ |
|---|---|---|---|---|---|---|---|---|
| $2^{-3}$ | 5.55(-17) | 4.44(-16) | 5.55(-16) | 1.11(-15) | 4.88(-15) | 3.30(-14) | 8.74(-14) | 8.32(-14) |
| $2^{-4}$ | 1.11(-16) | 1.11(-16) | 6.66(-16) | 5.55(-16) | 2.99(-15) | 1.27(-14) | 3.65(-14) | 1.07(-13) |
| $2^{-5}$ | 1.11(-16) | 1.11(-16) | 6.66(-16) | 9.99(-16) | 3.33(-15) | 4.10(-15) | 1.54(-14) | 1.19(-13) |
| $2^{-6}$ | 2.22(-16) | 1.11(-16) | 2.22(-16) | 3.33(-16) | 6.43(-15) | 4.85(-14) | 1.37(-13) | 1.02(-12) |
| $2^{-10}$ | 0 | 0 | 2.22(-16) | 1.11(-16) | 1.11(-16) | 3.33(-16) | 3.55(-15) | 6.87(-14) |

Table 8: The maximum absolute errors in solution of example 4 without fitting factor

| $\varepsilon \backslash h$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | $2^{-7}$ | $2^{-8}$ | $2^{-9}$ | $2^{-10}$ |
|---|---|---|---|---|---|---|---|---|
| $2^{-3}$ | 5.39(-4) | 3.22(-5) | 1.99(-6) | 1.24(-7) | 7.76(-9) | 4.85(-10) | 3.03(-11) | 1.94(-12) |
| $2^{-4}$ | 7.50(-3) | 5.41(-4) | 3.24(-5) | 2.00(-6) | 1.24(-7) | 7.79(-9) | 4.87(-10) | 3.05(-11) |
| $2^{-5}$ | 5.86(-2) | 7.50(-3) | 5.41(-4) | 3.24(-5) | 2.00(-6) | 1.24(-7) | 7.79(-9) | 4.87(-10) |
| $2^{-6}$ | 2.25(-1) | 5.86(-2) | 7.50(-3) | 5.41(-4) | 3.24(-5) | 2.00(-6) | 1.24(-7) | 7.79(-9) |
| $2^{-10}$ | 8.30(-1) | 8.20(-1) | 6.87(-1) | 4.72(-1) | 2.25(-1) | 5.86(-2) | 7.50(-3) | 5.41(-4) |

Table 9: The maximum absolute errors in solution of example 5 fitting factor

| $\varepsilon \backslash h$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | $2^{-7}$ | $2^{-8}$ | $2^{-9}$ | $2^{-10}$ |
|---|---|---|---|---|---|---|---|---|
| $2^{-3}$ | 3.35(-4) | 3.35(-4) | 3.35(-4) | 3.35(-4) | 3.35(-4) | 3.35(-4) | 3.35(-4) | 3.35(-4) |
| $2^{-4}$ | 1.12(-7) | 1.12(-7) | 1.12(-7) | 1.12(-7) | 1.12(-7) | 1.12(-7) | 1.12(-7) | 1.12(-7) |
| $2^{-5}$ | 1.26(-14) | 1.26(-14) | 1.30(-14) | 1.26(-14) | 1.42(-14) | 1.79(-14) | 2.00(-14) | 4.38(-14) |
| $2^{-6}$ | 5.82(-16) | 1.11(-16) | 1.11(-16) | 1.58(-15) | 3.33(-15) | 1.97(-14) | 5.14(-14) | 3.90(-13) |
| $2^{-10}$ | 0 | 0 | 1.45(-16) | 1.11(-16) | 5.55(-16) | 2.22(-16) | 3.74(-15) | 7.62(-14) |

Table 10: The maximum absolute errors in solution of example 5 without fitting factor

| $\varepsilon \backslash h$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | $2^{-7}$ | $2^{-8}$ | $2^{-9}$ | $2^{-10}$ |
|---|---|---|---|---|---|---|---|---|
| $2^{-3}$ | 3.35(-4) | 3.35(-4) | 3.35(-4) | 3.35(-4) | 3.35(-4) | 3.35(-4) | 3.35(-4) | 3.35(-4) |
| $2^{-4}$ | 7.50(-3) | 5.41(-4) | 3.23(-5) | 1.93(-6) | 1.12(-7) | 1.12(-7) | 1.12(-7) | 1.12(-7) |
| $2^{-5}$ | 5.86(-2) | 7.50(-3) | 5.41(-4) | 3.24(-5) | 2.00(-6) | 1.24(-7) | 7.79(-9) | 4.87(-10) |
| $2^{-6}$ | 2.25(-1) | 5.86(-2) | 7.50(-3) | 5.41(-4) | 3.24(-5) | 2.00(-6) | 1.24(-7) | 7.79(-9) |
| $2^{-10}$ | 8.30(-1) | 8.20(-1) | 6.87(-1) | 4.72(-1) | 2.25(-1) | 5.86(-2) | 7.50(-3) | 5.41(-4) |

## References

[1] Nayfeh AH (1981). *Introduction to Perturbation Methods*, John Wiley and Sons, New York.
[2] Bender CM, Orszag SA (1978). *Advanced mathematical methods for scientists and engineers*, Mc. Graw-Hill, New York.
[3] O'Malley RE (1974). *Introduction to Singular Perturbations*, Academic Press, New York.
[4] Kevorkian J, Cole JD (1996). *Multiple Scale and Singular Perturbation Methods*, Springer-Verlag, New York.
[5] Bellman R (1964). *Perturbation Techniques in Mathematics, Physics and Engineering*, Holt, Rinehart & Winston, New York.
[6] Van Veldhuizen M (1979). *Higher order schemes of positive type for singular perturbation problems*, in Hemker PW, Miller JJH, *Numerical analysis of singular perturbation problems*, Academic Press, New York, pp. 361-383.
[7] Kreiss B, Kreiss HO (1982). Numerical methods for singular perturbation problems, *SIAM J. Numer. Anal.* 46, pp. 138–165.
[8] Miller JJH, Riordan EO and Shishkin GI (1995). On piecewise uniform meshes for upwind and central difference operators for solving singularly perturbed problems, *IMA J. Numer. Anal.* 15, pp. 89–99.
[9] Kadalbajoo MK, Patidar KC (2001). Numerical solution of singularly perturbed two point boundary value problems by spline in compression, *Int. J. Comput. Math.* 77, pp. 263–283.
[10] Kadalbajoo MK, Reddy YN (1986). A non asymptotic method for general singular perturbation problems, *Journal of Optimization Theory and Applications*, 55, pp. 256-269.
[11] Reddy YN (1990). A Numerical integration method for solving singular perturbation problems, *Appl. Math. Comput.* 37, pp. 83–95.
[12] Stroud AH(1960).Numerical integration formulas of degree two, *Math. Comput* 14(69), pp.21–26.
[13] Stynes M, Roos HG and Tobiska L (2008). Robust numerical methods for singularly perturbed differential equations , *Springer series in Computational Mathematics*, Springer-Verlag, Berlin, 24.
[14] Lin P, Vancouver (1989). A Numerical solution of quasilinear singularly perturbed ordinary differential equation without turning points, *Appl. Math. Mech.* 10, pp. 1005-1010.
[15] Vulanovic R (1991). A second order numerical method for quasilinear singular perturbation problems without turning points, *Zh. Vychisl.Mat.Mat.Fiz*. 31, pp. 522-532.
[16] Kadalbajoo MK, Devendra Kumar (2009). Initial value technique for singularly perturbed two point boundary value problems using an exponentially fitted finite difference scheme, *Computers and Mathematics with Applications*, 57, pp.1147-1156.