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STABILITY ANALYSIS OF HOST-MORTAL COMMENSAL ECO-SYSTEM WITH A CONSTANT HARVESTING OF BOTH THE COMMENSAL AND THE HOST

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Abstract

In this paper we present a two species commensal interaction with limited resources and both the species are harvested at a constant rate where as the commensal species has a Mortality rate. The growth rate equations of this model are characterized by first order non-linear ordinary differential equations. In all, nine equilibrium states are identified. Further, solutions for the linearized perturbed (over the equilibrium states) equations have been obtained and results illustrated. Further, some threshold results are stated followed by the identification of threshold regions through illustrations. Also, global stability is discussed by constructing a suitable Liapunov's function.

1. Introduction

An ecosystem is a complex set of relationships among living resources, habitats and residents of a region. And Ecology is the scientific study of the processes influencing the distribution and abundance of organisms, the interactions among organisms, and the interactions between organisms and the transformation and flux of energy and matter. The ecological interactions can be broadly classified as prey-predation, competition, commensalism, Ammensalism, Neutralism and so on. Research in theoretical Ecology was initiated by Lotka [11] and by Volterra [17]. Since then many Mathematicians and Ecologists contributed to the growth of this area of knowledge as reported in the treatises of Meyer [12], Paul Colinvaux [13], Kupur [6, 7], Svirezhev and Logofet [16], Freedman [5], Kushing [8]. N.C.Srinivas [15] studied the Competitive Ecosystems of two species and three species with limited and unlimited resources. Later Lakshminarayana and Pattabhi Ramacharyulu [9, 10] investigated Prey-Predator Ecological Models with a partial cover for the Prey and alternative food for the Predator and Prey-Predator model with cover for Prey and alternate food for the Predator and time delay. Stability analysis of Competitive species was carried out by Archana Reddy, Pattabhi Ramacharyulu and Gandhi [1, 2], by Bhaskara Rama Sarma and Pattabhi Ramacharyulu [3, 4], while the Mutualism between two species was examined by Ravindra Reddy [14].

The present investigation is on a two species commensalism model with Mortality rate for the Commensal species and both the Commensal and the Host are harvested at a constant rate. The Mathematical Model is characterized by a couple of first order non-linear ordinary differential equations. All possible nine existing equilibrium points of the model are identified and stability criterion for these is discussed. Solutions for the linearized perturbed equations are found and results are presented. Further, some threshold results are stated followed by the identification of threshold regions through illustrations. Also, global stability is discussed by constructing a suitable Liapunov's function.

2. Basic Equations

Notation Adopted:

$N_1(t)$ **:** The population of the Commensal Species (S_1).

$N_2(t)$ **:** The population of the Host Species (S_2).

$e_1(= d_1 / a_{11})$ **:** The extinction coefficient of S_1.

$c(= a_{12} / a_{11})$ **:** The coefficient of Commensalism.

$k_2(= a_2 / a_{22})$**:** The carrying capacity of S_2.

H_1, H_2 **:** The constant harvesting rates of S_1 and S_2.

Further both the variables $N_1(t)$ and $N_2(t)$ are non-negative for all t and all the model parameters $d_1, a_2, a_{11}, a_{22}, a_{12}, H_1$ and H_2 are assumed to be non-negative constants.

Employing the above terminology, the model equations for a two species Commensal-Host Ecological model are given by the following system of non-linear coupled ordinary differential equations.

(i). Equation for the growth rate of the Commensal species (S_1) is

$$\frac{dN_1}{dt} = a_{11}\left[-e_1 N_1 - N_1^2 + cN_1 N_2 - H_1\right] \tag{2.1}$$

(ii). Equation for the growth rate of the Host species (S_2) is

$$\frac{dN_2}{dt} = a_{22}\left[k_2 N_2 - N_2^2 - H_2\right] \tag{2.2}$$

3. Equilibrium States

The system under investigation has the following nine equilibrium states given by $\frac{dN_1}{dt} = 0\,;\frac{dN_2}{dt} = 0$. These states are classified into two categories **A** and **B**.

(A) When the harvesting rates are interdependent

(A.1) When $k_2^2 > 4H_2\ ;\ \left(c\left(\frac{k_2 + \sqrt{k_2^2 - 4H_2}}{2}\right) - e_1\right)^2 = 4H_1$ **(A.1)**

$$E_1:\ \overline{N_1} = \frac{1}{2}\left[c\left(\frac{k_2 + \sqrt{k_2^2 - 4H_2}}{2}\right) - e_1\right]\ ;\quad \overline{N_2} = \frac{k_2 + \sqrt{k_2^2 - 4H_2}}{2} \tag{3.1}$$

(A.2) When $k_2^2 > 4H_2\ ;\ \left(c\left(\frac{k_2 + \sqrt{k_2^2 - 4H_2}}{2}\right) - e_1\right)^2 > 4H_1$ **(A.2)**

$$E_2: \; \overline{N_1} = \frac{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)+\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}}{2} \; ; \; \overline{N_2} = \frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2} \tag{3.2}$$

$$E_3: \; \overline{N_1} = \frac{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)-\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}}{2} \; ; \; \overline{N_2} = \frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2} \tag{3.3}$$

(A.3) When $k_2^{\,2} > 4H_2 \; ; \; \left(c\left(\frac{k_2-\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2 = 4H_1$ **(A.3)**

$$E_4: \; \overline{N_1} = \frac{1}{2}\left[c\left(\frac{k_2-\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right] \; ; \; \overline{N_2} = \frac{k_2-\sqrt{k_2^{\,2}-4H_2}}{2} \tag{3.4}$$

(A.4) When $k_2^{\,2} > 4H_2 \; ; \; \left(c\left(\frac{k_2-\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2 > 4H_1$ **(A.4)**

$$E_5: \; \overline{N_1} = \frac{\left(c\left(\frac{k_2-\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)+\sqrt{\left(c\left(\frac{k_2-\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}}{2} \; ; \; \overline{N_2} = \frac{k_2-\sqrt{k_2^{\,2}-4H_2}}{2} \tag{3.5}$$

$$E_6: \; \overline{N_1} = \frac{\left(c\left(\frac{k_2-\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)-\sqrt{\left(c\left(\frac{k_2-\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}}{2} \; ; \; \overline{N_2} = \frac{k_2-\sqrt{k_2^{\,2}-4H_2}}{2} \tag{3.6}$$

(B) When the harvesting rates are not interdependent

(B.1) When $k_2^{\,2} = 4H_2 \; ; \; \left(\frac{ck_2}{2}-e_1\right)^2 = 4H_1$ **(B.1)**

$$E_7: \; \overline{N_1} = \frac{1}{2}\left[\frac{ck_2}{2}-e_1\right] \; ; \; \overline{N_2} = \frac{k_2}{2} \tag{3.7}$$

(B.2) When $k_2^{\,2} = 4H_2 \; ; \; \left(\frac{ck_2}{2}-e_1\right)^2 > 4H_1$ **(B.2)**

$$E_8: \; \overline{N_1} = \frac{\left(\frac{ck_2}{2}-e_1\right)+\sqrt{\left(\frac{ck_2}{2}-e_1\right)^2-4H_1}}{2} \; ; \; \overline{N_2} = \frac{k_2}{2} \tag{3.8}$$

$$E_9: \; \overline{N_1} = \frac{\left(\frac{ck_2}{2}-e_1\right)-\sqrt{\left(\frac{ck_2}{2}-e_1\right)^2-4H_1}}{2} \; ; \; \overline{N_2} = \frac{k_2}{2} \tag{3.9}$$

4. Stability of the Equilibrium States

Let N = $(N_1, N_2) = \overline{N} + \bar{u}$ (4.1)

where $\bar{u} = (u_1, u_2)$. The perturbations u_1, u_2 over the equilibrium state $\overline{N} = (\overline{N_1}, \overline{N_2})$ are so small that their second and higher powers and products are negligible. The basic equations (2.1) and (2.2) are linearized to obtain the equations for the perturbed state.

$$\frac{dU}{dt} = AU \tag{4.2}$$

where $$A = \begin{bmatrix} -e_1 a_{11} - 2a_{11}\overline{N_1} + ca_{11}\overline{N_2} & ca_{11}\overline{N_1} \\ 0 & k_2 a_{22} - 2a_{22}\overline{N_2} \end{bmatrix} \tag{4.3}$$

The characteristic equation for the system is

$$\det[A - \lambda I] = 0 \tag{4.4}$$

The equilibrium state is stable, only if both the roots of the equation (4.4) are negative real.

4.1 Stability of the Equilibrium State E_1:

The corresponding linearized perturbed equations are

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{ca_{11}}{2}\left[c\left(\frac{k_2 + \sqrt{k_2^{\,2} - 4H_2}}{2}\right) - e_1\right] \\ 0 & -a_{22}\sqrt{k_2^{\,2} - 4H_2} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.5}$$

The corresponding characteristic equation of (4.5) is

$$\lambda\left(\lambda + a_{22}\sqrt{k_2^{\,2} - 4H_2}\right) = 0 \tag{4.6}$$

The roots of this equations are $\lambda_1 = 0$ and $\lambda_2 = -a_{22}\sqrt{k_2^{\,2} - 4H_2} < 0$. Since one of the two roots is zero, the steady state is **unstable**.

The solutions of the linearized perturbed equations (4.5) are given by

$$u_1 = [u_{10} + L_1] - L_1 e^{-\left(a_{22}\sqrt{k_2^{\,2} - 4H_2}\right)t} \tag{4.7}$$

where $$L_1 = \frac{u_{20} ca_{11}\left[c\left(\frac{k_2 + \sqrt{k_2^{\,2} - 4H_2}}{2}\right) - e_1\right]}{2\left(a_{22}\sqrt{k_2^{\,2} - 4H_2}\right)} \tag{4.7.1}$$

$$u_2 = u_{20} e^{-\left(a_{22}\sqrt{k_2^{\,2} - 4H_2}\right)t} \tag{4.8}$$

The equilibrium state $E_1 \to (u_{10} + L_1, 0)$ as $t \to \infty$.

The solution curves of (4.7) and (4.8) are illustrated in Figures 1 & 2.

Case (i): When $u_{10} > u_{20}$

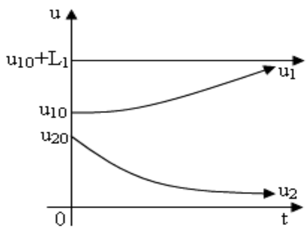


Fig. 1

The initial population strength of the commensal is greater than that of the host. In this case the commensal out - numbers the host all the time. Further, the commensal species is observed to diverge away from the equilibrium point while the host is asymptotic to the equilibrium point as shown in Fig.1.

Case (ii): When $u_{10} < u_{20}$

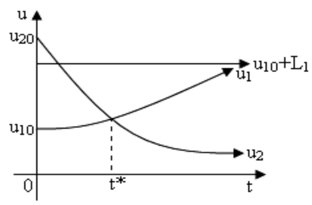
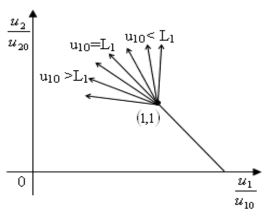


Fig. 2

In this case the host out-numbers the commensal up to the time $t = t^* = \frac{1}{a_{11}\sqrt{k_2^2 - 4H_2}}\log\left(\frac{u_{20}+L_1}{u_{10}+L_1}\right)$ after which the dominance is reversed. This is the time of dominance reversal as shown in Fig.2.

4.1 (a) Trajectories of the Perturbed Species



Eliminating't' between the equations (4.7) and (4.8), we obtain

$$\frac{u_1}{u_{10}} = \left(1+\frac{L_1}{u_{10}}\right) - \left(\frac{L_1}{u_{10}}\right)\left(\frac{u_2}{u_{20}}\right) \qquad (4.9)$$

and the resulting curves are shown in Fig.3.

Fig. 3

4.2 Stability of the Equilibrium State E_2 **:**

As before, the linearized perturbed equations for the perturbations u_1 and u_2 over the steady state $\left(\overline{N_1}, \overline{N_2}\right)$ are given by the system.

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -a_{11}\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)^2-4H_1} & \frac{ca_{11}}{2}\left[\left(c\left(\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)+\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)^2-4H_1}\right] \\ 0 & -a_{22}\sqrt{k_2^2-4H_2} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(4.10)

The characteristic equation for the system (4.10) is

$$\left(\lambda + a_{11}\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)^2-4H_1}\right)\left(\lambda + a_{22}\sqrt{k_2^2-4H_2}\right) = 0 \qquad (4.11)$$

The roots of the equation (4.11) are

$\lambda_1 = -a_{11}\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)^2-4H_1} < 0$ and $\lambda_2 = -a_{22}\sqrt{k_2^2-4H_2} < 0$, both are negative and hence this **co-existence** state is **stable**.

The solutions of the equations in (4.10) are given by

$$u_1 = [u_{10} - L_2]e^{-\left(a_{11}\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)^2-4H_1}\right)t} + L_2 e^{-\left(a_{22}\sqrt{k_2^2-4H_2}\right)t} \quad (4.12)$$

where

$$L_2 = \frac{ca_{11}u_{20}\left[\left(c\left(\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)+\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)^2-4H_1}\right]}{2\left[a_{11}\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)^2-4H_1}-\left(a_{22}\sqrt{k_2^2-4H_2}\right)\right]} \quad (4.12.1)$$

$$u_2 = u_{20}e^{-\left(a_{22}\sqrt{k_2^2-4H_2}\right)t} \quad (4.13)$$

It is to be noted that $\left(c\left(\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)^2 > 4H_1$ and also noticed that $(u_1, u_2) \to 0$ as $t \to \infty$.

There arise the following **two** cases:

CASE 2A : $u_{10} = L_2$; **CASE 2B :** $u_{10} \neq L_2$

Case 2A: When $u_{10} = L_2$, then the equations (4.12) and (4.13) become

$$u_1 = u_{10}e^{-\left(a_{22}\sqrt{k_2^2-4H_2}\right)t} \quad ; \quad u_2 = u_{20}e^{-\left(a_{22}\sqrt{k_2^2-4H_2}\right)t} \quad (4.14); (4.15)$$

Here both u_1 and u_2 are exponentially decay with the same characteristic time $1/a_{22}\sqrt{k_2^2-4H_2}$, the initial values (u_{10} and u_{20}) may however be different. Hence the equilibrium point is **stable**.

The solution curves are illustrated as follows.

Case 2A.1: When $u_{10} > u_{20}$

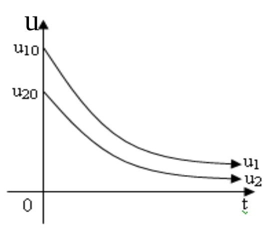
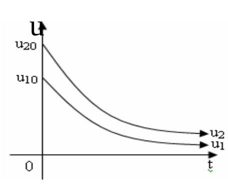


Fig. 4

In this case the commensal species always out- number the host species in natural growth rate as well as in its initial population strength. It is noted that both the commensal and the host converge asymptotically to the equilibrium point as shown in Fig.4.

Case 2A.2: When $u_{10} < u_{20}$



The host species dominates over the commensal species in its initial population strength. Also both the species move towards to the equilibrium point as seen in Fig.5.

Fig. 5

4.2 (a) Trajectories of the Perturbed Species

Eliminating ‘t’ between the equations (4.14) and (4.15), we obtain

$$\frac{u_1}{u_{10}} = \frac{u_2}{u_{20}} \tag{4.16}$$

and the corresponding trajectory is a straight line is shown in Fig.6.

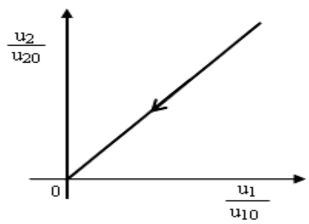
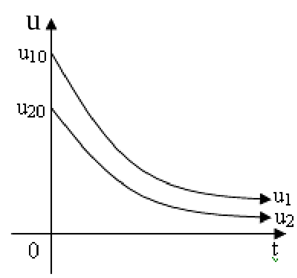
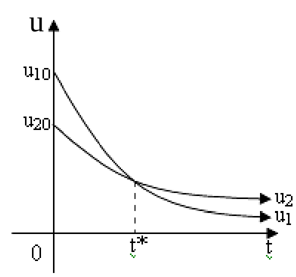
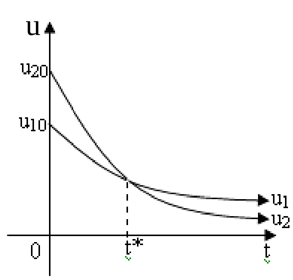
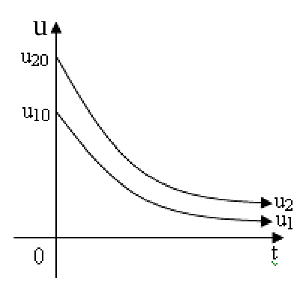


Fig. 6

Case 2B: $u_{10} \neq L_2$

The solution curves in this case are illustrated below from Figures 7 to 10.

	$a_{11}\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)^2-4H_1} < a_{22}\sqrt{k_2^2-4H_2}$	$a_{11}\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)^2-4H_1} > a_{22}\sqrt{k_2^2-4H_2}$
$u_{10} > u_{20}$	**Case 2B.1**  **Fig. 7**	**Case 2B.2**  **Fig. 8**
$u_{10} < u_{20}$	**Case 2B.3**  **Fig. 9**	**Case 2B.4**  **Fig. 10**

Observations:

Case 2B.1: The initial population strength of the commensal is greater than that of the host i.e., $u_{10} > u_{20}$. In this case the commensal dominates over the host all the time as shown in Fig.7.

Case 2B.2: The initial population strength of the commensal is greater than that of the host i.e., $u_{10} > u_{20}$. In this case the commensal dominates over the host till the time instant

$$t = t^* = \frac{1}{a_{11}\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1} - a_{22}\sqrt{k_2^{\,2}-4H_2}}\log\left(\frac{u_{10}-L_2}{u_{20}-L_2}\right)$$

after which the host dominates. This is the dominance reversal time as shown in Fig.8.

Case 2B.3: The initial population strength of the host is greater than that of the commensal i.e., $u_{10} < u_{20}$. In this case initially, the host out-numbers the commensal and this continues up to the time instant

$$t = t^* = \frac{1}{a_{22}\sqrt{k_2^{\,2}-4H_2} - a_{11}\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}}\log\left(\frac{u_{20}-L_2^{\,1}}{u_{10}-L_2^{\,1}}\right)$$

where $L_2^{\,1} = -L_2$ after which, the dominance is reversed. The dominance reversal time is shown in Fig.9.

Case 2B.4: The initial population strength of the host is greater than that of the commensal i.e., $u_{10} < u_{20}$. In this case the host continues to out-number the commensal as shown in Fig.10.

4.2 (b) Trajectories of the Perturbed Species

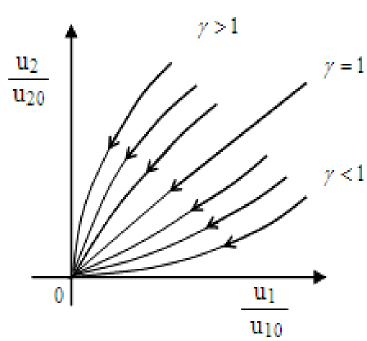


Fig. 11

Eliminating 't' between the equations (4.12) and (4.13), we obtain

$$\frac{u_1}{u_{10}} = \left(\frac{L_2}{u_{10}}\right)\left(\frac{u_2}{u_{20}}\right) + \left(1-\frac{L_2}{u_{10}}\right)\left(\frac{u_2}{u_{20}}\right)^{\gamma} \qquad (4.17)$$

where $\gamma = \frac{a_{11}\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}}{a_{22}\sqrt{k_2^{\,2}-4H_2}}$ and the resulting curves are parabolic type and are shown in Fig.11.This figure exhibits the stability of the equilibrium state.

4.3 Stability of the Equilibrium State E_3:

In this case the corresponding characteristic matrix is

$$A=\begin{bmatrix} a_{11}\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1} & \frac{ca_{11}}{2}\left[\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)-\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}\right] \\ 0 & -a_{22}\sqrt{k_2^{\,2}-4H_2} \end{bmatrix} \quad (4.18)$$

The corresponding characteristic roots of which are

$$\lambda_1=a_{11}\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}>0 \text{ and } \lambda_2=-a_{22}\sqrt{k_2^{\,2}-4H_2}<0.$$

Since one of the two roots is positive, hence the steady state is **unstable**.

The solutions of the linearized perturbed equations in this state are given by

$$u_1=[u_{10}+L_3]e^{\left(a_{11}\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}\right)t}-L_3e^{-\left(a_{22}\sqrt{k_2^{\,2}-4H_2}\right)t} \quad (4.19)$$

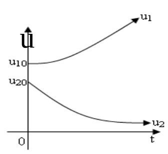
Where

$$L_3=\frac{ca_{11}u_{20}\left[\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)-\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}\right]}{2\left[a_{11}\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}+\left(a_{22}\sqrt{k_2^{\,2}-4H_2}\right)\right]} \quad (4.19.1)$$

$$u_2=u_{20}e^{-\left(a_{22}\sqrt{k_2^{\,2}-4H_2}\right)t} \quad (4.20)$$

In this case the solution curves are discussed as below.

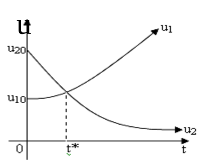
Case (i): When $u_{10}>u_{20}$



The commensal species always out-number the host species in natural growth rate as well as in its initial population strength where as the host declines further is shown in Fig.12.

Fig. 12

Case (ii): When $u_{10}<u_{20}$



The commensal dominates over the host in its natural growth rate but its initial strength is less than that of the host i.e., $u_{10}<u_{20}$. In this case, the host out-numbers the commensal till the time instant

Fig.13

$$t^*=\frac{1}{a_{11}\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}+a_{22}\sqrt{k_2^{\,2}-4H_2}}\log\left(\frac{u_{20}+L_3}{u_{10}+L_3}\right)$$

and there after the commensal out-numbers the host. This is seen in Fig.13.

4.3 (a) Trajectories of the Perturbed Species

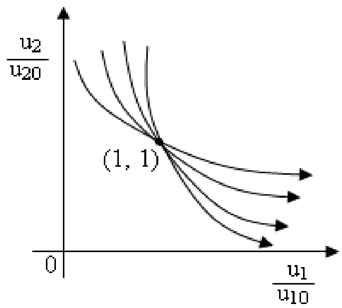


Fig. 14

Eliminating 't' between the equations (4.19) and (4.20), we obtain

$$\frac{u_1}{u_{10}} = \left(1+\frac{L_3}{u_{10}}\right)\left(\frac{u_2}{u_{20}}\right)^{-\frac{a_{11}\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}}{a_{22}\sqrt{k_2^{\,2}-4H_2}}} - \left(\frac{L_3}{u_{10}}\right)\left(\frac{u_2}{u_{20}}\right) \qquad (4.21)$$

and the trajectories are hyperbolic type as shown in Fig.14.

4.4 Stability of the Equilibrium State E_4:

The linearized perturbed equations in this state are

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{ca_{11}}{2}\left[c\left(\frac{k_2-\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right] \\ 0 & a_{22}\sqrt{k_2^{\,2}-4H_2} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (4.22)$$

The characteristic equation for the system (4.22) is

$$\lambda\left(\lambda - a_{22}\sqrt{k_2^{\,2}-4H_2}\right) = 0 \qquad (4.23)$$

The roots for the system (4.23) are $\lambda_1 = 0$ and $\lambda_2 = a_{22}\sqrt{k_2^{\,2}-4H_2} > 0$. Since one of the two roots is zero and other root is positive, the steady state is **unstable**.
The solutions of the system of equations in (4.22) are

$$u_1 = [u_{10} - L_4] + L_4 e^{\left(a_{22}\sqrt{k_2^{\,2}-4H_2}\right)t} \qquad (4.24)$$

where $$L_4 = \frac{\frac{u_{20}ca_{11}}{2}\left[c\left(\frac{k_2-\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right]}{a_{22}\sqrt{k_2^{\,2}-4H_2}} \qquad (4.24.1)$$

$$u_2 = u_{20}e^{\left(a_{22}\sqrt{k_2^{\,2}-4H_2}\right)t} \qquad (4.25)$$

As before here **two** cases would arise.
CASE 4A: $u_{10} = L_4$; **CASE 4B:** $u_{10} \neq L_4$
The solutions in these two cases are illustrated as follows.
CASE 4A: $u_{10} = L_4$

In this case the solutions (4.24) and (4.25) become

$$u_1 = u_{10} e^{\left(a_{22}\sqrt{k_2^{\,2}-4H_2}\right)t} \quad ; \quad u_2 = u_{20} e^{\left(a_{22}\sqrt{k_2^{\,2}-4H_2}\right)t} \qquad (4.26); (4.27)$$

Case 4A.1: When $u_{10} > u_{20}$

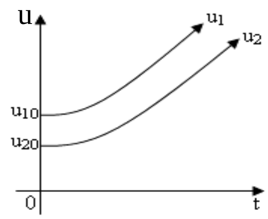


Fig. 15

The Initial population strength of the commensal is greater than that of the host i.e., $u_{10} > u_{20}$. In this case both the species move away from the equilibrium point as shown in Fig.15.

Case 4A.2: When $u_{10} < u_{20}$

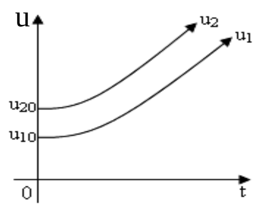


Fig. 16

The initial population strength of the host is greater than that of the commensal i.e., $u_{10} < u_{20}$. In this case the host dominates over the commensal all the time.

4.4 (a) Trajectories of the Perturbed Species

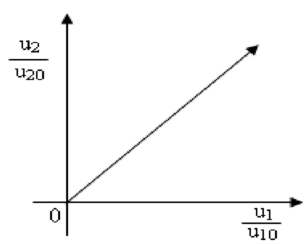


Fig. 17

Eliminating 't' between the equations (4.26) and (4.27), we obtain

$$\frac{u_1}{u_{10}} = \frac{u_2}{u_{20}} \qquad (4.28)$$

and the resulting curve is given in Fig.17.

CASE 4B: $u_{10} \neq L_4$

Case 4B.1: When $u_{10} > u_{20}$

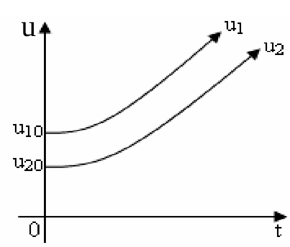
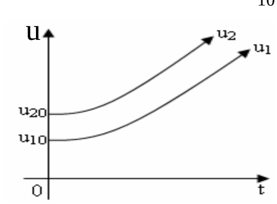


Fig. 18

The commensal species always out - number the host species in natural growth rate as well as in its initial population strength. Here both the species go far away from the equilibrium point as shown in Fig.18.

Case 4B.2: When $u_{10} < u_{20}$



The host species always out-number the commensal species in natural growth rate as well as in its initial population strength. In this case both the species go far away from the equilibrium point. This is illustrated in Fig.19.

Fig. 19 .

4.4 (b) Trajectories of the Perturbed Species

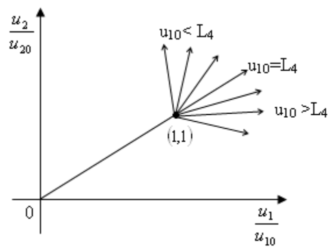


Fig. 20

Eliminating 't' between the equations (4.24) and (4.25), we obtain

$$\frac{u_1}{u_{10}} = \left(1 - \frac{L_4}{u_{10}}\right) + \left(\frac{L_4}{u_{10}}\right)\left(\frac{u_2}{u_{20}}\right) \tag{4.29}$$

and the resulting curves are given in Fig.20.

4.5 Stability of the Equilibrium State E_5:

In this state the corresponding characteristic matrix is

$$A = \begin{bmatrix} -a_{11}\sqrt{\left(c\left(\frac{k_2 - \sqrt{k_2^2 - 4H_2}}{2}\right) - e_1\right)^2 - 4H_1} & \frac{ca_{11}}{2}\left[\left(c\left(\frac{k_2 - \sqrt{k_2^2 - 4H_2}}{2}\right) - e_1\right) + \sqrt{\left(c\left(\frac{k_2 - \sqrt{k_2^2 - 4H_2}}{2}\right) - e_1\right)^2 - 4H_1}\right] \\ 0 & a_{22}\sqrt{k_2^2 - 4H_2} \end{bmatrix} \tag{4.30}$$

The characteristic equation is

$$\left(\lambda + a_{11}\sqrt{\left(c\left(\frac{k_2 - \sqrt{k_2^2 - 4H_2}}{2}\right) - e_1\right)^2 - 4H_1}\right)\left(\lambda - a_{22}\sqrt{k_2^2 - 4H_2}\right) = 0 \tag{4.31}$$

The roots of the equation (4.31) are

$$\lambda_1 = -a_{11}\sqrt{\left(c\left(\frac{k_2 - \sqrt{k_2^2 - 4H_2}}{2}\right) - e_1\right)^2 - 4H_1} < 0 \text{ and } \lambda_2 = a_{22}\sqrt{k_2^2 - 4H_2} > 0.$$

Since one of the two roots is positive, hence this state is **unstable**.

The solutions of the linearized perturbed equations in this state are

$$u_1 = [u_{10} - L_5]e^{-\left(a_{11}\sqrt{\left(c\left(\frac{k_2 - \sqrt{k_2^2 - 4H_2}}{2}\right) - e_1\right)^2 - 4H_1}\right)t} + L_5 e^{\left(a_{22}\sqrt{k_2^2 - 4H_2}\right)t} \tag{4.32}$$

where
$$L_5 = \frac{c a_{11} u_{20}\left[\left(c\left(\frac{k_2-\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)+\sqrt{\left(c\left(\frac{k_2-\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)^2-4H_1}\right]}{2\left[a_{11}\sqrt{\left(c\left(\frac{k_2-\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)^2-4H_1}+\left(a_{22}\sqrt{k_2^2-4H_2}\right)\right]} \qquad (4.32.1)$$

$$u_2 = u_{20} e^{\left(a_{22}\sqrt{k_2^2-4H_2}\right)t} \qquad (4.33)$$

As before **two** cases would arise.

CASE 5A: $u_{10} = L_5$; **CASE 5B:** $u_{10} \neq L_5$

The solution curves in these **two** cases are illustrated as follows.

CASE 5A: $u_{10} = L_5$

In this case the solutions (4.32) and (4.33) become

$$u_1 = u_{10} e^{\left(a_{22}\sqrt{k_2^2-4H_2}\right)t} \quad ; \; u_2 = u_{20} e^{\left(a_{22}\sqrt{k_2^2-4H_2}\right)t} \qquad (4.34); (4.35)$$

Case 5A.1: When $u_{10} > u_{20}$

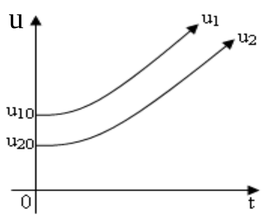


Fig. 21

The initial population strength of the commensal is greater than that of the host i.e., $u_{10} > u_{20}$. In this case both the species move away from the equilibrium point as shown in Fig.21.

Case 5A.2: When $u_{10} < u_{20}$

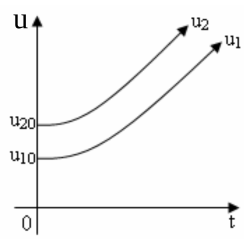


Fig. 22

The initial population strength of the host is greater than that of the commensal i.e., $u_{10} < u_{20}$. In this case the host dominates over the commensal all the time.

4.5 (a) Trajectories of the Perturbed Species

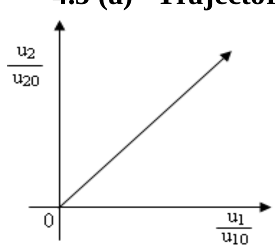


Fig. 23

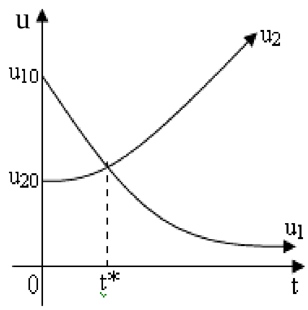
Eliminating 't' between the equations (4.34) and (4.35), we obtain

$$\frac{u_1}{u_{10}} = \frac{u_2}{u_{20}} \qquad (4.36)$$

and the resulting curve is given in Fig.23.

CASE 5B: $u_{10} \neq L_5$

Case 5B.1: When $u_{10} > u_{20}$



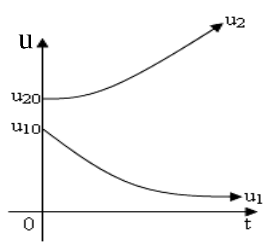
The host dominates over the commensal after the time instant t* but its initial population strength is less than that of the commensal. Here the host dominance time over the commensal is

$$t^* = \frac{1}{a_{11}\sqrt{\left(c\left(\frac{k_2-\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)^2-4H_1}+\left(a_{22}\sqrt{k_2^2-4H_2}\right)}\log\left(\frac{u_{10}-L_5}{u_{20}-L_5}\right)$$

This is illustrated in Fig.24.

Fig. 24

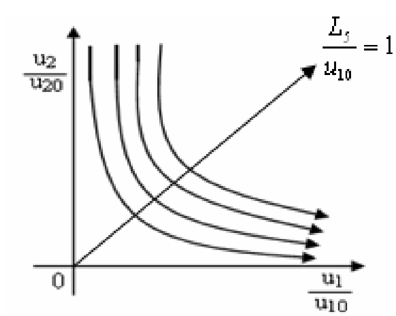
Case 5B.2: When $u_{10} < u_{20}$



In this case the host species always out- number the commensal species. Also it is evident that the host species goes far away from the equilibrium point while the commensal is asymptotic to the equilibrium point as shown in Fig.25.

Fig. 25

4.5 (b) Trajectories of the Perturbed Species



Eliminating 't' between the equations (4.32) and (4.33), we obtain

$$\frac{u_1}{u_{10}} = \left(\frac{u_2}{u_{20}}\right)\left(\frac{L_5}{u_{10}}\right) + \left(1-\frac{L_5}{u_{10}}\right)\left(\frac{u_2}{u_{20}}\right)^{-\gamma} \tag{4.37}$$

where $\gamma = \dfrac{a_{11}\sqrt{\left(c\left(\frac{k_2-\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)^2-4H_1}}{a_{22}\sqrt{k_2^2-4H}}$ and the resulting curves are shown in Fig.26.

Fig. 26

4.6 Stability of the Equilibrium State E_6 **:**

In this state the corresponding characteristic equation is

$$\left(\lambda - a_{11}\sqrt{\left(c\left(\frac{k_2-\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)^2-4H_1}\right)\left(\lambda - a_{22}\sqrt{k_2^2-4H_2}\right)=0 \tag{4.38}$$

The roots of the equation (4.38) are

$$\lambda_1 = a_{11}\sqrt{\left(c\left(\frac{k_2-\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1} > 0 \quad \text{and } \lambda_2 = a_{22}\sqrt{k_2^{\,2}-4H_2} > 0 .$$

Since both the roots are positive, the steady state is **unstable**.

The solutions of the linearized perturbed equations in this state are

$$u_1 = [u_{10} - L_6]e^{\left(a_{11}\sqrt{\left(c\left(\frac{k_2-\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}\right)t} + L_6 e^{\left(a_{22}\sqrt{k_2^{\,2}-4H_2}\right)t} \qquad (4.39)$$

where

$$L_6 = \frac{ca_{11}u_{20}\left[\left(c\left(\frac{k_2-\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)-\sqrt{\left(c\left(\frac{k_2-\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}\right]}{2\left[a_{11}\sqrt{\left(c\left(\frac{k_2-\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}-\left(a_{22}\sqrt{k_2^{\,2}-4H_2}\right)\right]} \qquad (4.39.1)$$

$$u_2 = u_{20}e^{\left(a_{22}\sqrt{k_2^{\,2}-4H_2}\right)t} \qquad (4.40)$$

It is noticed that $u_1 \to \infty$ and $u_2 \to \infty$ as $t \to \infty$. Hence the state is **unstable**.

There arise the following **two** cases**:**

CASE 6A : $u_{10} = L_6$; **CASE 6B :** $u_{10} \neq L_6$

The solution curves in these two cases are illustrated below.

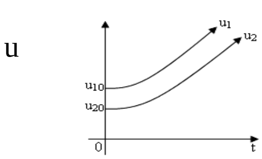
CASE 6A : $u_{10} = L_6$

In this case the solutions (4.39) and (4.40) become

$$u_1 = u_{10}e^{\left(a_{22}\sqrt{k_2^{\,2}-4H_2}\right)t} \qquad (4.41)$$

$$u_2 = u_{20}e^{\left(a_{22}\sqrt{k_2^{\,2}-4H_2}\right)t} \qquad (4.42)$$

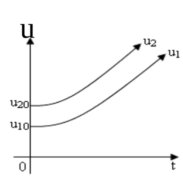
Case 6A.1: When $u_{10} > u_{20}$



The initial population strength of the commensal is greater than that of the host i.e., $u_{10} > u_{20}$. In this case both the specie go far away from the equilibrium point as shown in Fig.27.

Fig. 27

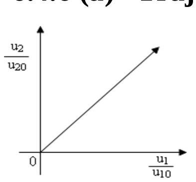
Case 6A.2: When $u_{10} < u_{20}$



The initial population strength of the host is greater than that of the commensal i.e., $u_{10} < u_{20}$. In this case the host species always out-number the commensal species. It is seen in Fig.28.

Fig. 28

6.4.6 (a) Trajectories of the Perturbed Species



Eliminating 't' between the equations (4.41) and (4.42), we obtain

$$\frac{u_1}{u_{10}} = \frac{u_2}{u_{20}} \qquad (4.43)$$

and the resulting curve is a straight line as shown in Fig.29.

Fig. 29

CASE 6B : $u_{10} \neq L_6$

The solution curves in this case are illustrated below from Figures 30 to 33.

<table>
<tr><td></td><td>$a_{11}\sqrt{\left(c\left(\frac{k_2-\sqrt{k_2^{2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}$ $> a_{22}\sqrt{k_2^{2}-4H_2}$</td><td>$a_{11}\sqrt{\left(c\left(\frac{k_2-\sqrt{k_2^{2}-4H_2}}{2}\right)-e_1\right)^2-4H_1}$ $< a_{22}\sqrt{k_2^{2}-4H_2}$</td></tr>
<tr><td>$u_{10} > u_{20}$</td><td>

Case 6B.1

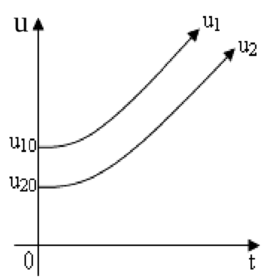


Fig. 30

</td><td>

Case 6B.2

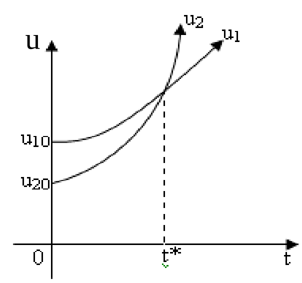


Fig. 31

</td></tr>
<tr><td>$u_{10} < u_{20}$</td><td>

Case 6B.3

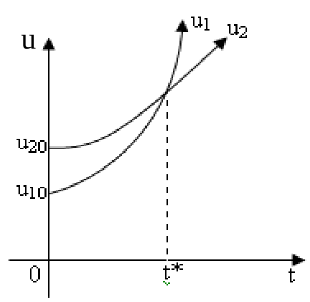


Fig. 32

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Case 6B.4

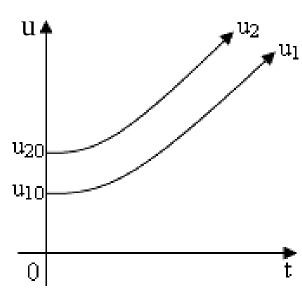

Fig. 33

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</table>

Observations:

Case 6B.1: Initially the first species out-number the second species and it continues to grow. Also we observe that both the species diverge away from the equilibrium point. Hence the equilibrium point is **unstable** as shown in Fig.30.

Case 6B.2: The initial population strength of the commensal is greater than that of the host i.e., $u_{10} > u_{20}$. Initially the commensal out-numbers the host and this continues up to the time

$$t = t^* = \frac{1}{a_{22}\sqrt{k_2^{\,2} - 4H_2} - a_{11}\sqrt{\left(c\left(\frac{k_2 - \sqrt{k_2^{\,2} - 4H_2}}{2}\right) - e_1\right)^2 - 4H_1}} \log\left(\frac{u_{10} - L_6}{u_{20} - L_6}\right)$$ after which the host out-numbers the commensal.

Case 6B.3: The initial population strength of the host is greater than that of the commensal i.e., $u_{10} < u_{20}$. Initially the host out-numbers the commensal and this continues up to the time

$$t = t^* = \frac{1}{a_{11}\sqrt{\left(c\left(\frac{k_2 - \sqrt{k_2^{\,2} - 4H_2}}{2}\right) - e_1\right)^2 - 4H_1} - a_{22}\sqrt{k_2^{\,2} - 4H_2}} \log\left(\frac{u_{20} - L_6}{u_{10} - L_6}\right)$$ after which, the dominance is reversed.

Case 6B.4: The initial population strength of the host is greater than that of the commensal i.e., $u_{10} < u_{20}$. In this case the second species out-number the first species all the time as shown in Fig.33.

4.6 (b) Trajectories of the Perturbed Species

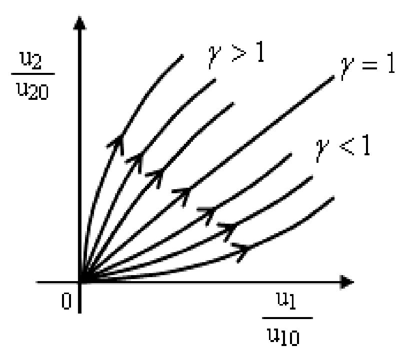


Fig. 34

Eliminating 't' between the equations (4.39) and (4.40), we obtain

$$\frac{u_1}{u_{10}} = \left(\frac{u_2}{u_{20}}\right)\left(\frac{L_6}{u_{10}}\right) + \left(1 - \frac{L_6}{u_{10}}\right)\left(\frac{u_2}{u_{20}}\right)^{\gamma} \qquad (4.44)$$

where $\gamma = \dfrac{a_{11}\sqrt{\left(c\left(\frac{k_2 - \sqrt{k_2^{\,2} - 4H_2}}{2}\right) - e_1\right)^2 - 4H_1}}{a_{22}\sqrt{k_2^{\,2} - 4H_2}}$ and the resulting curves are parabolic type as shown in Fig.34. This exhibits the instability of the equilibrium point.

4.7 Stability of the Equilibrium State E_7:

The corresponding linearized perturbed equations are

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{ca_{11}(ck_2 - 2e_1)}{4} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (4.45)$$

The corresponding characteristic equation is $\lambda^2 = 0$ (4.46)

The roots of the equation (4.46) are $\lambda_1 = 0$ and $\lambda_2 = 0$. Since the roots are zero so that the steady state is **unstable**.

The solutions of the system of equations in (4.45) are

$$u_1 = u_{10} + \frac{ca_{11}(ck_2 - 2e_1)u_{20}}{4} t \tag{4.47}$$

$$u_2 = u_{20} \tag{4.48}$$

The solution curves of (4.47) and (4.48) are illustrated below.

Case (i): When $u_{10} > u_{20}$

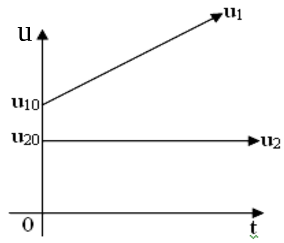


Fig. 35

In this case the commensal continues out-number the host through out its natural growth rate as well as in its initial population strength as shown in Fig.35. Here both the species go far away from the equilibrium point $(\overline{N_1}, \overline{N_2})$.

Case (ii): When $u_{10} < u_{20}$

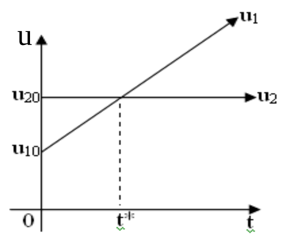


Fig. 36

The host dominates over the commensal till the time instant $t^* = \frac{4(u_{20} - u_{10})}{ca_{11}(ck_2 - 2e_1)u_{20}}$ after which the dominance is reversed. In this case both the species move away from the equilibrium point. This is seen in Fig.36.

4.7 (a) Trajectories of the Perturbed Species

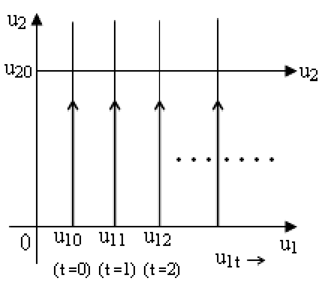


Fig. 37

The instantaneous variation w.r.t.'t' of u_1 (designated by u_{1t}) vrs. u_2 (constant) as per the equations (4.47) and (4.48) are depicted in Fig.37. The functions $u_{1t} \to \infty$ as $t \to \infty$.

4.8 Stability of the Equilibrium State E_8:

The characteristic matrix **A** is

$$A = \begin{bmatrix} -a_{11}\sqrt{\left(\frac{ck_2}{2} - e_1\right)^2 - 4H_1} & \frac{ca_{11}}{2}\left(\left(\frac{ck_2}{2} - e_1\right) + \sqrt{\left(\frac{ck_2}{2} - e_1\right)^2 - 4H_1}\right) \\ 0 & 0 \end{bmatrix} \tag{4.49}$$

The characteristic equation is

$$\lambda\left(\lambda + a_{11}\sqrt{\left(\frac{ck_2}{2} - e_1\right)^2 - 4H_1}\right) = 0 \tag{4.50}$$

The roots of the equation (4.50) are $\lambda_1 = 0$ and $\lambda_2 = -a_{11}\sqrt{\left(\frac{ck_2}{2} - e_1\right)^2 - 4H_1} < 0$.

Since one of the two roots is zero, this state is **unstable**.

The solutions of the linearized perturbed equations are

$$u_1 = [u_{10} - L_7]e^{-\left(a_{11}\sqrt{\left(\frac{ck_2}{2}-e_1\right)^2 - 4H_1}\right)t} + L_7 \tag{4.51}$$

where
$$L_7 = \frac{u_{20}ca_{11}\left(\left(\frac{ck_2}{2} - e_1\right) + \sqrt{\left(\frac{ck_2}{2} - e_1\right)^2 - 4H_1}\right)}{2\left(a_{11}\sqrt{\left(\frac{ck_2}{2} - e_1\right)^2 - 4H_1}\right)} \tag{4.51.1}$$

$$u_2 = u_{20} \tag{4.52}$$

Two cases would arise here.

CASE 8A: $u_{10} = L_7$; **CASE 8B:** $u_{10} \neq L_7$

The solution curves of these two cases are illustrated as follows.

CASE 8A: When $u_{10} = L_7$

The solutions (4.51) and (4.52) become

$$u_1 = u_{10} \tag{4.53}$$

$$u_2 = u_{20} \tag{4.54}$$

Case 8A.1: When $u_{10} > u_{20}$

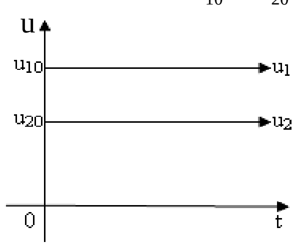


Fig. 38

Case 8A.2: When $u_{10} < u_{20}$

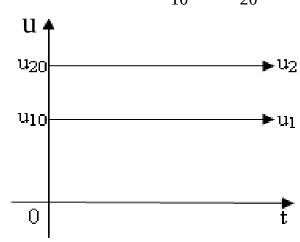


Fig. 39

It can be noted that the dominating species is either commensal or host according as $u_{10} > u_{20}$ or $u_{10} < u_{20}$. Also it is evident that both the species move away from the equilibrium point at a constant distance.

4.8 (a) Trajectories of the Perturbed Species

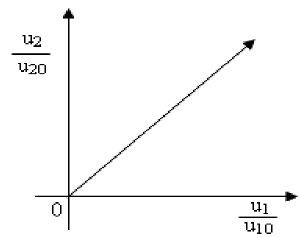


Fig. 40

Eliminating ‘t’ between the equations (4.53) and (4.54), we obtain

$$\frac{u_1}{u_{10}} = \frac{u_2}{u_{20}} \qquad (4.55)$$

and the resulting curve is a straight line as shown in Fig.40.

CASE 8B: $u_{10} \neq L_7$

Case 8B.1: When $u_{10} > u_{20}$

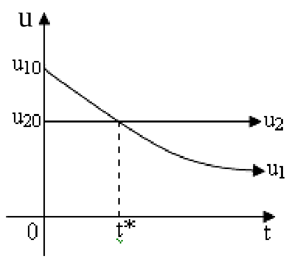


Fig. 41

In this case the commensal out-numbers the host till the time instant $t = t^* = \dfrac{1}{a_{11}\sqrt{\left(\dfrac{ck_2}{2} - e_1\right)^2 - 4H_1}} \log\left(\dfrac{u_{10} - L_7}{u_{20} - L_7}\right)$ and there after the dominance is reversed. This is shown in Fig.41. Further, the commensal is asymptotic to the equilibrium point while the host goes far away from the equilibrium point.

Case 8B.2: When $u_{10} < u_{20}$

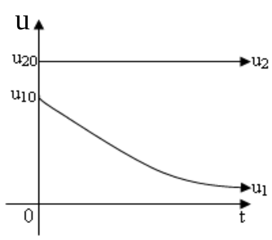


Fig. 42

The host continues to out-number the commensal in natural growth rate as well as in its initial population strength is shown in Fig.42.

4.8 (b) Trajectories of the Perturbed Species

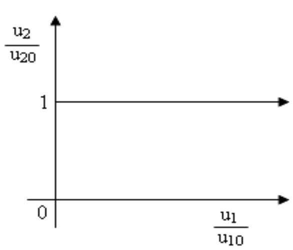


Fig. 43

Eliminating ‘t’ between the equations (4.51) and (4.52), we obtain

$$\frac{u_2}{u_{20}} = 1 \qquad (4.56)$$

and the resulting curve is a straight line as shown in Fig.43.

4.9 Stability of the Equilibrium State E_9:

The corresponding perturbed equations are

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} a_{11}\sqrt{\left(\frac{ck_2}{2}-e_1\right)^2 - 4H_1} & \frac{ca_{11}}{2}\left(\left(\frac{ck_2}{2}-e_1\right) - \sqrt{\left(\frac{ck_2}{2}-e_1\right)^2 - 4H_1}\right) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (4.57)$$

The corresponding characteristic equation is $\lambda\left(\lambda - a_{11}\sqrt{\left(\frac{ck_2}{2}-e_1\right)^2 - 4H_1}\right) = 0$ (4.58)

The roots of which are $\lambda_1 = 0$ and $\lambda_2 = a_{11}\sqrt{\left(\frac{ck_2}{2}-e_1\right)^2 - 4H_1} > 0$. Since one of the two roots is zero and other positive, this state is **unstable**.

The solutions of the linearized perturbed equations in this state are

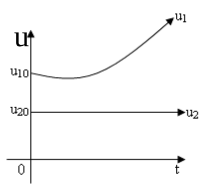
$$u_1 = [u_{10} + L_8]e^{\left(a_{11}\sqrt{\left(\frac{ck_2}{2}-e_1\right)^2 - 4H_1}\right)t} - L_8 \quad (4.59)$$

where $$L_8 = \frac{u_{20}ca_{11}\left(\left(\frac{ck_2}{2}-e_1\right) - \sqrt{\left(\frac{ck_2}{2}-e_1\right)^2 - 4H_1}\right)}{2\left(a_{11}\sqrt{\left(\frac{ck_2}{2}-e_1\right)^2 - 4H_1}\right)} \quad (4.59.1)$$

$$u_2 = u_{20} \quad (4.60)$$

The solution curves of (4.59) and (4.60) are illustrated in the following Figures 44 & 45.

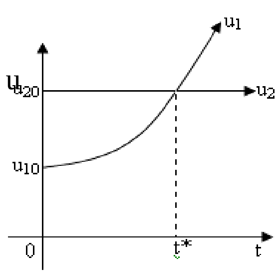
Case (i): When $u_{10} > u_{20}$



The commensal always out-number the host in natural growth rate as well as in its initial population strength. In this case both the species go away from the equilibrium point, the state is **unstable**. This is shown in Fig.44.

Fig. 44

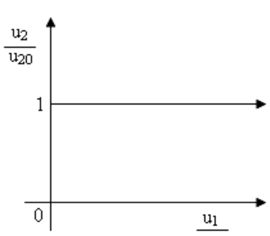
Case (ii): When $u_{10} < u_{20}$



The host species out-number the commensal up to the time instant $t^* = \frac{1}{a_{11}\sqrt{\left(\frac{ck_2}{2}-e_1\right)^2 - 4H_1}}\log\left(\frac{u_{20}+L_8}{u_{10}+L_8}\right)$ after which the commensal out-numbers the host. It is illustrated in Fig.45.

Fig. 45

4.9 (a) Trajectories of the Perturbed Species



Eliminating 't' between the equations (4.59) and (4.60), we obtain

$$\frac{u_2}{u_{20}} = 1 \qquad (4.61)$$

and the resulting curve is a straight line as shown in Fig.46.

Fig. 46

5. Threshold (or) Phase- plane diagram

The conditions $\frac{dN_1}{dt} = 0$ and $\frac{dN_2}{dt} = 0$ imply that neither N_1 nor N_2 changes its density. When we impose these conditions the basic equations give rise to a hyperbola and two straight lines. At the points where $\frac{dN_1}{dt} = 0$; $\frac{dN_2}{dt} = 0$, the resulting curves divide the phase plane in to seven regions in the first quadrant $N_1 \geq 0$, $N_2 \geq 0$ (vide Fig.5.1).

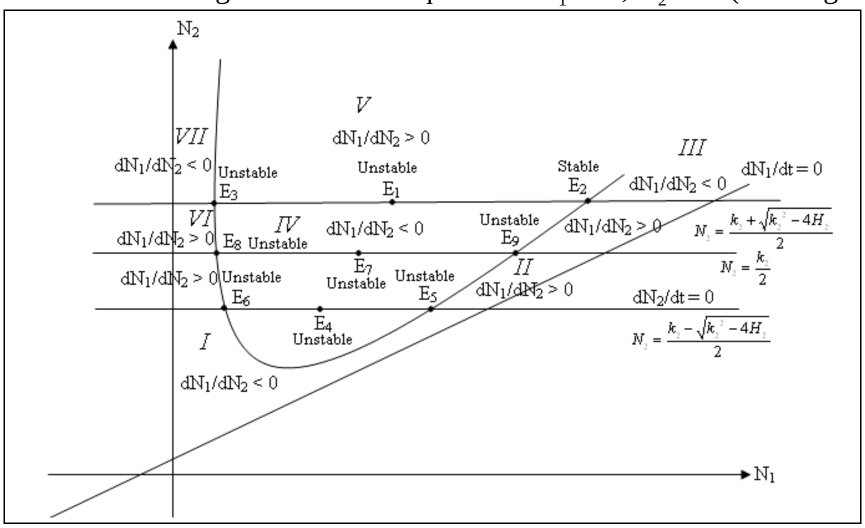


Fig. 5.1

Threshold Regions

0

Region I: The commensal species N_1 flourishes and the host species N_2 declines with time t.

Region II: Both the species N_1 and N_2 flourish with time t.

Region III: The commensal species N_1 flourishes and the host species N_2 declines with time t.

Region IV: The commensal species N_1 declines and the host species N_2 flourishes with time t.

Region V: Both the species N_1 and N_2 decline with time t.

Region VI: Both the species N_1 and N_2 flourish with time t.

Region VII: The commensal species N_1 flourishes and the host species N_2 declines with time t.

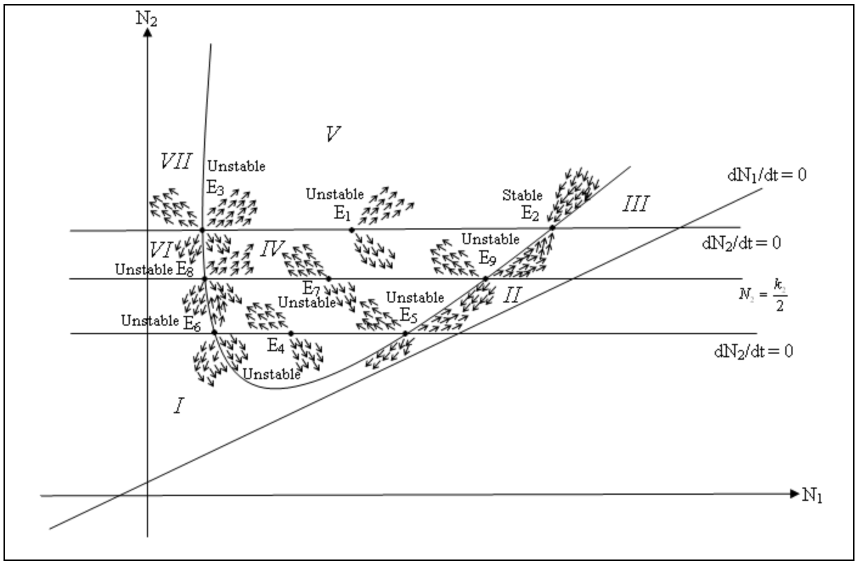


Fig. 5.2

Threshold Diagram

6. Liapunov's Function for Global Stability

In Section 4.2 we have discussed the local stability of the state of co-existence. We now examine the global stability of the dynamical system (2.1) and (2.2). We have already noted that this system has a unique, stable non-trivial co-existent equilibrium state at

$$\overline{N_1} = \frac{\left(c\left(\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)+\sqrt{\left(c\left(\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)^2-4H_1}}{2} \quad ; \overline{N_2} = \frac{k_2+\sqrt{k_2^2-4H_2}}{2}$$

Basic Equations:

$$\frac{dN_1}{dt} = a_{11}\left[-e_1N_1 - N_1^2 + cN_1N_2 - H_1\right] \tag{6.1}$$

$$\frac{dN_2}{dt} = a_{22}\left[k_2N_2 - N_2^2 - H_2\right] \tag{6.2}$$

The linearized perturbed equations over the perturbations (u_1, u_2) are

$$\frac{du_1}{dt} = -2a_{11}\left[\overline{N_1} - \frac{\left(c\left(\frac{k_2+\sqrt{k_2^2-4H_2}}{2}\right)-e_1\right)}{2}\right]u_1 + ca_{11}\overline{N_1}u_2 \tag{6.3}$$

$$\frac{du_2}{dt} = -2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right] u_2 \tag{6.4}$$

The corresponding characteristic equation is of the form

$$\lambda^2 + p\lambda + q = 0 \tag{6.5}$$

where

$$p = 2a_{11}\left[\overline{N_1} - \frac{\left(c\left(\frac{k_2 + \sqrt{k_2^{\,2} - 4H_2}}{2}\right) - e_1\right)}{2}\right] + 2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right] > 0 \tag{6.6}$$

$$q = 4a_{11}a_{22}\left[\overline{N_1} - \frac{\left(c\left(\frac{k_2 + \sqrt{k_2^{\,2} - 4H_2}}{2}\right) - e_1\right)}{2}\right]\left[\overline{N_2} - \frac{k_2}{2}\right] > 0 \tag{6.7}$$

$\therefore$ The conditions for the existence of Liapunov's function are satisfied.

Now we define $\quad E\,(u_1, u_2) = \frac{1}{2}\left(Au_1^{\,2} + 2Bu_1u_2 + Cu_2^{\,2}\right)$ (6.8)

where

$$A = \frac{\left(2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]\right)^2 + \left(4a_{11}a_{22}\left[\overline{N_1} - \frac{\left(c\left(\frac{k_2 + \sqrt{k_2^{\,2} - 4H_2}}{2}\right) - e_1\right)}{2}\right]\left[\overline{N_2} - \frac{k_2}{2}\right]\right)}{D} \tag{6.9}$$

$$B = \frac{2ca_{11}a_{22}\overline{N_1}\left[\overline{N_2} - \frac{k_2}{2}\right]}{D} \tag{6.10}$$

$$C = \frac{\left(2a_{11}\left[\overline{N_1} - \frac{\left(c\left(\frac{k_2 + \sqrt{k_2^{\,2} - 4H_2}}{2}\right) - e_1\right)}{2}\right]\right)^2 + \left(ca_{11}\overline{N_1}\right)^2 + \left(4a_{11}a_{22}\left[\overline{N_1} - \frac{\left(c\left(\frac{k_2 + \sqrt{k_2^{\,2} - 4H_2}}{2}\right) - e_1\right)}{2}\right]\left[\overline{N_2} - \frac{k_2}{2}\right]\right)}{D} \tag{6.11}$$

And

$$D = pq > 0 \tag{6.12}$$

From the equations (6.6) and (6.7) it is clear that D>0 and A>0

Also

$$D^2\left(AC - B^2\right) = D^2\left\{\left(\frac{\left(2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]\right)^2 + 4a_{11}a_{22}\left[\overline{N_1} - \frac{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)}{2}\right]\left[\overline{N_2} - \frac{k_2}{2}\right]}{D}\right.\right.$$

$$\frac{\left(4a_{11}\left[\overline{N_1} - \frac{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)}{2}\right]\right)\left(a_{11}\left[\overline{N_1} - \frac{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)}{2}\right] + a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]\right)}{D}$$

$$\left.+\left(ca_{11}\overline{N_1}\right)^2 - \frac{\left(2ca_{11}a_{22}\overline{N_1}\right)^2\left[\overline{N_2} - \frac{k_2}{2}\right]}{D^2}\right\}$$

$$\Rightarrow D^2\left(AC - B^2\right) > 0 \Rightarrow AC - B^2 > 0 \qquad \text{i.e., } B^2\text{-}AC < 0 \tag{6.13}$$

$\therefore$ The function E (u_1, u_2) is positive definite.

Further

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} = \left\{\left(Au_1 + Bu_2\right)\left(-2a_{11}\left[\overline{N_1} - \frac{\left(c\left(\frac{k_2+\sqrt{k_2^{\,2}-4H_2}}{2}\right)-e_1\right)}{2}\right]u_1 + ca_{11}\overline{N_1}u_2\right)\right.$$

$$\left.+\left(Bu_1 + Cu_2\right)\left(-2a_{22}\left[\overline{N_2} - \frac{k_2}{2}\right]u_2\right)\right\} \tag{6.14}$$

Substituting the values of A, B and C from (6.9) (6.10) and (6.11) in (6.14) we get

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} = -\frac{1}{D}\left[Du_1^2 + Du_2^2\right] \tag{6.15}$$

$$= -\left(u_1^2 + u_2^2\right) \tag{6.16}$$

$$\therefore \frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} = -\left(u_1^2 + u_2^2\right) \tag{6.17}$$

which is clearly negative definite.

So, E (u_1, u_2) is a Liapunov's function for the linear system.

Next we prove that E(u_1, u_2) is also a Liapunov's function for the non-linear system.

Let f_1 and f_2 be two functions of N_1 and N_2 defined by

$$f_1(N_1, N_2) = a_{11}\left(-e_1 N_1 - N_1^{\ 2} + cN_1 N_2 - H_1\right) \tag{6.18}$$

$$f_2(N_1, N_2) = a_{22}\left(k_2 N_2 - N_2^{\ 2} - H_2\right) \tag{6.19}$$

Now we have to show that $\dfrac{\partial E}{\partial u_1} f_1 + \dfrac{\partial E}{\partial u_2} f_2$ is negative definite.

Putting $N_1 = \overline{N_1} + u_1$; $N_2 = \overline{N_2} + u_2$ in (6.1) and (6.2) we get

$$f_1(u_1, u_2) = \frac{du_1}{dt} = a_{11}\left(-e_1\left(\overline{N_1} + u_1\right) - \left(\overline{N_1} + u_1\right)^2 + c\left(\overline{N_1} + u_1\right)\left(\overline{N_2} + u_2\right) - H_1\right)$$

$$= a_{11} u_1\left(-e_1 - 2\overline{N_1} + c\overline{N_2}\right) + ca_{11}\overline{N_1}u_2 - a_{11}u_1^2 + ca_{11}u_1 u_2$$

$$\Rightarrow \quad f_1(u_1, u_2) = \frac{du_1}{dt} = a_{11}u_1\left(-e_1 - 2\overline{N_1} + c\overline{N_2}\right) + ca_{11}\overline{N_1}u_2 + F(u_1, u_2) \tag{6.20}$$

where $F(u_1, u_2) = -a_{11}u_1^{\ 2} + ca_{11}u_1 u_2$ (6.21)

Similarly

$$f_2(u_1, u_2) = \frac{du_2}{dt} = a_{22}\left(k_2\left(\overline{N_2} + u_2\right) - \left(\overline{N_2} + u_2\right)^2 - H_2\right)$$

$$= a_{22}u_2\left(k_2 - 2\overline{N_2}\right) - a_{22}u_2^{\ 2}$$

$$\Rightarrow f_2(u_1, u_2) = \frac{du_2}{dt} = a_{22}u_2\left[k_2 - 2\overline{N_2}\right] + G(u_1, u_2) \tag{6.22}$$

where $G(u_1, u_2) = -a_{22}u_2^{\ 2}$ (6.23)

From (6.8)

$$\frac{\partial E}{\partial u_1} = Au_1 + Bu_2 \tag{6.24}$$

$$\frac{\partial E}{\partial u_2} = Bu_1 + Cu_2 \tag{6.25}$$

Now

$$\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2 = \left(Au_1 + Bu_2\right)\left[a_{11}u_1\left(-e_1 - 2\overline{N_1} + c\overline{N_2}\right) + ca_{11}\overline{N_1}u_2 + F(u_1, u_2)\right]$$

$$+ \left(Bu_1 + Cu_2\right)\left[a_{22}u_2\left[k_2 - 2\overline{N_2}\right] + G(u_1, u_2)\right]$$

$$= \left(Au_1 + Bu_2\right)\left[a_{11}u_1\left(-e_1 - 2\overline{N_1} + c\overline{N_2}\right) + ca_{11}\overline{N_1}u_2\right] + \left(Bu_1 + Cu_2\right)\left[a_{22}u_2\left(k_2 - 2\overline{N_2}\right)\right]$$

$$+ \left(Au_1 + Bu_2\right) F(u_1, u_2) + \left(Bu_1 + Cu_2\right) G(u_1, u_2)$$

$$= \left(Au_1 + Bu_2\right)\left[-2a_{11}\left[\overline{N_1} - \frac{\left(c\overline{N_2} - e_1\right)}{2}\right]u_1 + ca_{11}\overline{N_1}u_2\right] + \left(Bu_1 + Cu_2\right)\left[-2a_{22}u_2\left(\overline{N_2} - \frac{k_2}{2}\right)\right]$$

$$+ \left(Au_1 + Bu_2\right) F(u_1,u_2) + \left(Bu_1 + Cu_2\right) G(u_1,u_2) \qquad (6.26)$$

$$\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2 = -(u_1^{\,2} + u_2^{\,2}) + \left(Au_1 + Bu_2\right) F(u_1,u_2) + \left(Bu_1 + Cu_2\right) G(u_1,u_2) \qquad (6.27)$$

Introducing polar co-ordinates $u_1 = r\cos\theta,\ u_2 = r\sin\theta,$ the equation (6.27) can be written as

$$\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2 = -r^{\,2} + r\left[(A\cos\theta + B\sin\theta)F(u_1,u_2) + (B\cos\theta + C\sin\theta)\,G(u_1,u_2)\right] \qquad (6.28)$$

Let us denote the largest of the numbers $|A|$,$|B|$ and $|C|$ by K.

Our assumptions imply that $\left|F(u_1,u_2)\right| < \frac{r}{6K}$ and $\left|G(u_1,u_2)\right| < \frac{r}{6K}$ for all sufficiently small r > 0.

So, $$\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2 < -r^{\,2} + \frac{4Kr^2}{6K} = -\frac{r^2}{3} < 0 \qquad (6.29)$$

Thus E(u_1, u_2) is a positive definite function with the condition that $\frac{\partial E}{\partial u_1} f_1 + \frac{\partial E}{\partial u_2} f_2$ is negative definite.

$\therefore$ The equilibrium state E_2 is **"asymptotically stable"** globally.

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